

Bekir Karaoglu



Classical Physics

A Two-Semester Coursebook

 Springer

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PREFACE

This book has been prepared as a textbook for use in the Introductory Physics courses given in the first year of the Science, Engineering, Education and Medicine schools of universities. It is calculus-based and can be taught entirely in two semesters.

Over the last decade, many universities around the world have been reducing undergraduate degree credits in favor of Master's degree programs. Introductory Physics courses that once spanned 4 semesters were reduced to 2 semesters. Subsequently, a serious problem arose in the use of standard textbooks. Of course, one cannot find fault with the physics covered in these books, which I, too, have used in the past. However, it is also a fact that they are not fully useful in Physics courses that have been shortened to 2 semesters with reduced hours. The main reason for this is the excessive volume of these books. I have personally observed the panic and despair of students when they first see these books, which run at least 1600 pages. Because of this volume problem, instructors also have difficulty in deciding which parts they will teach, while still informing the students about the omitted parts. Thus, the coherence and flow of the text are lost.

Sometimes, less is more. You may consider this book as a trimmed-down version of standard textbooks for a two-semester course. Whichever topics you choose to teach, you will find the same calculus-based approach and the same essential concepts covered, and all this without worrying about continuity and time limits.

THE BOOK'S APPROACH TO PHYSICS

No single book can be expected to meet the needs of each and every course, with their varied aims and different student bodies. I would like briefly to explain the philosophy of the book and answer other possibly relevant questions.

An important problem that holds back the teaching of physics in many universities is that the calculus courses given in parallel with physics cannot keep abreast of it. Such topics as integrals, vectors, differential equations, complex analysis, etc., come too late to be useful. Hence, it is necessary to teach physics topics either by keeping mathematics at a reasonable level or by developing some mathematics yourself.

In accordance with this viewpoint, the book develops some vital mathematics (integrals and vectors) along the way and omits minor topics that require advanced mathematics. However, in the treatment of each subject, care has been taken to ensure that the conclusions and concepts reached are based on the fundamental laws of physics. Thus, the student will be able to grasp the unifying concepts of physics and retain the results without the need for memorization.

ABOUT USING THE BOOK

In accordance with the above understanding, the book's distinctive features are:

- (1) Each theoretical development in the book should be considered together with the worked examples that follow. The examples extend the theory a little further, so that the volume of the book and the duration of the lecture can be reasonable. For each topic, the number of examples is more than you may need.
- (2) Students' ability to solve problems is an integral part of learning in physics. One can avoid rote learning only by solving problems. For this purpose, problems and multiple-choice questions are given at the end of each chapter, at a level that can be solved by the student, and hence can be assigned as homework. The numerical answer to each problem is given below it so that the student can easily check his/her own solution. In this approach, the steps of solving are more important than the numerical result.

Lastly, I want to address the students: You live in a wonderful World, rich in detail, sometimes enigmatic, but sometimes dangerous as well. The least you can do is to be aware of its basic laws, knowledge that can take you to new heights of understanding and wisdom.

ACKNOWLEDGMENTS

In the long writing phase of this book, many people from all parts of the world have helped. Many colleagues have contributed by using the book in their courses or by reading and correcting it. Students were in touch, informing us of the mistakes they had spotted. Moreover, it was a pleasant surprise to discover that all of the people of whom we asked permission to use their visual materials wanted to contribute to such a book project without expecting any financial compensation.

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1

UNITS AND VECTORS



The Russian Soyuz spaceship and the Earth in the background as viewed from the International Space Station (ISS). These two spaceships docked at 400 km altitude from the ground in January 2011.

The taking of this picture may have lasted one second, but it was only made possible with centuries of development in Physics.

1.1 DIMENSIONS AND UNITS

The natural sciences started with **measurement**. According to historians, measurement may have started in around 3000 BC in Ancient Egypt. The Nile river used to flood and fields would become submerged. There then arose the problem of finding the former boundaries of the fields after the flood withdrew. You may easily guess how they solved this problem: They first identified a fixed rock or a tree, and then they agreed on a unit of length. They measured and recorded the distance of the field boundaries to the origin in terms of this unit.

You can see the overall structure of measurement from this example:

First, we observe **physical quantities** that describe a physical property or condition, such as distance, area, speed, force, pressure

These physical quantities have **dimensions**, i.e., common characteristics with respect to measurements: For example, although width, length, diameter,

perimeter, etc., are different, their common aspect is that they are a kind of length, i.e., they have the *dimension of length*. Let's take a look at some examples:

Physical quantity	Dimension
distance, width, depth, length ...	} length
day, month, year, season, period,...	
	} time

The dimension of certain quantities can be expressed in terms of more basic dimensions. For example:

$$\begin{aligned}\text{Surface area} &= \text{width} \times \text{length} = (\text{length})^2 \\ \text{Volume} &= \text{width} \times \text{length} \times \text{height} = (\text{length})^3\end{aligned}$$

Then, a *dimension standard* or **unit** is established for each dimension. For example, various units such as yards, leagues, meters, feet, etc., have been used throughout history for the dimension of length.

The act of measurement is thus the act of determining the amount that a physical quantity corresponds to in terms of its own dimensional standard, i.e., unit. For example, to measure the width or length of a table, we take a meter ruler and count how many units the length of the table takes: 3.45 meters, 0.86 meters, etc.

We must emphasize one important point here: *The result of each measurement should be expressed in terms of units*. Although, in daily life, one may say, "My height is 1.67", this is actually incorrect, the correct expression being "my height is 1.67 meters".

The International System of Units (SI)

Which units among many should be accepted as the **basic units**? This selection has differed over various countries throughout history. However, today, the internationally accepted **International System of Units (SI = Systeme Internationale)**, formerly known as the MKS System of Units, is used.

The SI System of Units consists of 7 basic units. All physical quantities can thus be completely expressed and measured in terms of these basic units.

The basic units of the SI system		
Dimension	Unit	Abbreviation
Time	second	s
Length	meter	m
Mass	kilogram	kg
Electrical current	ampere	A
Temperature	kelvin	K
Light magnitude	candela	cd
Amount of matter	mole	mol

Three of these units, the Meter, the Kilogram and the Second are sufficient for mechanical topics. The other units shall be defined in due course.

Now let us define these basic units:

- **Meter** : The distance traveled by light in vacuum in $1/299\,792\,458$ seconds.
- **Second** : $9\,192\,631\,770$ times the vibration period of the Cs^{133} atom.

Note that these two units are defined with atomic methods. The units of meter and second were defined differently in the past: The meter was taken as a fraction of the length of the meridian or the length of a special platinum stick stored in France. Likewise, fractions of the Earth's movement around the Sun (year) or around its own axis (day) were used for the second. However, these became insufficient as the precision of measurements in science gradually increased. Atomic definitions are both very precise and are repeatable at anywhere around the world without the necessity of traveling to France.



Figure 1.1: The laser assembly that determines the **meter** unit (left). The cesium atomic clock that provides the **second** unit (center). The platinum-iridium cylinder stored in France as the **kilogram** standard (right).

- **Kilogram**: The mass of a cylindrical prototype of the platinum-iridium alloy manufactured in 1889 and stored at the Bureau International des Poids et Mesures (BIPM) institution in Paris.

Now, this was true until 2019. However, variations up to 50 micrograms were observed between this prototype and the copies sent to other countries over the many years that passed. After long discussion, it was decided to define it in terms of the **Planck constant** h arising in phenomena at the atomic scale. Accordingly, once the Planck constant is defined as

$$h = 6.62607015 \times 10^{-34} \text{ kg.m}^2/\text{s}$$

the kilogram is then defined in terms of meters or seconds. The new atomic definition became valid as of May 2019. But it will take a long time to implement it in laboratories, hence the prototypes will still be in use.

Prefixes for Multiples and Fractions

Exponents are used in the scientific representation of numbers. For example, $1\,350\,000$ is expressed as 1.35×10^6 or 0.000047 as 4.7×10^{-5} . However, it is difficult to express this in daily verbal language. Instead, fractions and multiples may easily be expressed by placing prefixes before each unit. The most commonly used prefixes are shown in the following table:

Multiples			Fractions		
name	symbol	quantity	name	symbol	quantity
kilo	k	10^3	milli	m	10^{-3}
mega	M	10^6	micro	μ	10^{-6}
giga	G	10^9	nano	n	10^{-9}
tera	T	10^{12}	pico	p	10^{-12}
peta	P	10^{15}	femto	f	10^{-15}

These prefixes have entered daily life: Examples include kilometer (km), gigahertz (GHz) and terabyte (TB).

Derived Units

The 3 basic units (meter, kilogram and second) mentioned above are sufficient to derive the units of all other quantities in mechanics. Here are some examples:

Some derived units			
quantity	definition	unit	abbreviation
Area	width×length	(meter) ²	m ²
Volume	width×length×height	(meter) ³	m ³
Speed	distance/time	meters/second	m/s
Acceleration	speed/time	meters/(second) ²	m/s ²
Force	mass×acceleration	kilogram×meter/(second) ²	kg·m/s ²
Work	force×distance	kilogram×meter ² /(second) ²	kg·m ² /s ²

Consistency of Units

It is not sufficient merely to have the same numbers on both sides of the equation in a physical formula. The dimensions and units must also be consistent. In other words, “you cannot compare apples and oranges.”

Only the quantities with the same dimension can appear on both sides of physical formulas.

For example, let us take a look at the accelerated motion formula you know from high school:

$$x = v_0 t + \frac{1}{2} a t^2$$

The left-hand side of this equation has the dimension of length. Thus, each of the terms on the right-hand side must also have the length dimension. Let us plug in the dimensions of each quantity explicitly:

$$\text{meter} = \frac{\text{meter}}{\text{second}} \times \text{second} + \frac{\text{meter}}{\text{second}^2} \times \text{second}^2 = \text{meter}$$

This gives us a clue from the start about whether or not the formula is consistent.

We shall reconsider examples on units in the next section after learning how to carry out numerical calculations.

Other Systems of Units

It will be useful to know two systems of units other than SI.

- In the **CGS system**, the basic units are **cm**, **gram** and **second**. It is easy to convert into the SI system.
- The **Imperial System of Units** is used in the Anglo-Saxon world; and, with slight differences, in the United States. The unit of length is the foot, the unit of mass is the pound and the unit of time is the second. The fraction of the unit of length is the inch and the multiples are the yard and the mile.

These units are converted into the SI system using the following formulas:

$$\begin{aligned} 1 \text{ foot (ft)} &= 12 \text{ inch (in)} = 0.3048 \text{ meter} \\ 1 \text{ pound (lb)} &= 16 \text{ ounce (oz)} = 0.453 \text{ kg} \end{aligned}$$

1.2 PRECISION AND SIGNIFICANT FIGURES

The **degree of precision** is the degree of closeness of measurement of a quantity to that quantity's true value. The precision is limited by the capability of the measuring instruments. The difference between the real value and the measured value of a quantity is called the **margin of error**. Each measurement has a margin of error.

In science and engineering, the results of measurements and calculations should be expressed properly by taking into consideration the margin of error. In other words, they should be given with the correct number of **significant figures**. For example, it would be incorrect to express the result of weighing an item on a grocery store scale as 352.8461 grams or the result of a distance measurement with a ruler as 25.87346 cm.

Absolute Error – Relative Error

The minimum interval measurable by a measuring apparatus is called the **absolute error**. Consider that you are measuring the length of your book using a ruler. The smallest division on the ruler is 1 millimeter. It is not possible to measure lengths smaller than a millimeter using this ruler (Figure 1.2). Therefore, the absolute error of the measurements with this ruler is $\Delta L = 1 \text{ mm}$. (Actually, the absolute error of a ruler is 0.5 mm, as it is possible to see further as to which side the reading is closer. We will take it as 1 mm here for the sake of simplicity.)

For example, let the measured length of the book be $L = 294 \text{ mm}$. Then, the measurement result can be expressed as follows, taking into consideration the margin of error:

$$L \pm \Delta L = 294 \pm 1 \text{ mm}$$

Accordingly, the length will be within the range ($293 \text{ mm} < L < 295 \text{ mm}$).

The ratio $\Delta x/x$ is called the **relative error**, and it is expressed in terms of percentage (%). For example, if the mass of an object is given as $m = (35 \pm 1) \text{ g}$, its relative error is calculated as follows:

$$\frac{\Delta m}{m} = \frac{1}{35} = 0.029 \approx 3 \%$$

Both absolute and relative errors propagate when calculations are made with these measurements. For example, after measuring the width, length and height of the book using the aforementioned ruler, it will be necessary to calculate the volume V of the book and again provide a $V \pm \Delta V$ margin of error.

How do errors propagate in calculations? There are two simple rules for calculating the margin of error in the results:

- **In additions and subtractions, absolute errors are added:**

$$z = a \pm b \quad \Rightarrow \quad \Delta z = \Delta a + \Delta b$$

- **In multiplications and divisions, relative errors are added:**

$$y = \begin{cases} ab \\ a/b \end{cases} \quad \Rightarrow \quad \frac{\Delta y}{y} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

These two rules are sufficient to find the margin of error in complex calculations. For example, the margin of error in the expression $z = a^3 (b + c)^2$ is found as

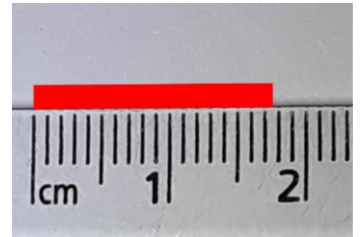


Figure 1.2: The red rectangle measures somewhere between 1.7 and 1.8 cm long. Here, the absolute error is 1 mm since that is the smallest division we can see.

follows:

$$\begin{aligned}
 z &= a^3 (b + c)^2 \\
 \frac{\Delta z}{z} &= \frac{\Delta(a^3)}{a^3} + \frac{\Delta(b + c)^2}{(b + c)^2} \\
 &= \left[\frac{\Delta a}{a} + \frac{\Delta a}{a} + \frac{\Delta a}{a} \right] + \left[\frac{\Delta(b + c)}{b + c} + \frac{\Delta(b + c)}{b + c} \right] \\
 \frac{\Delta z}{z} &= 3 \frac{\Delta a}{a} + 2 \frac{\Delta b + \Delta c}{b + c}
 \end{aligned}$$

It is also possible to guess the margin of error from the expression of data. For example, if two separate mass measurements are given as 45 g and 45.0 g, this means that, in the first measurement, nothing is known after the last digit, in other words, the absolute error is $\Delta m = 1$ g. In the second measurement, the decimal of gram was also measured and found to be 0, in other words, absolute error is $\Delta m = 0.1$ g.

Significant Figures

The precision of physical data is not always given by its absolute error. Sometimes it is understood from the **number of significant figures** expressing the data.

For example, if the mass of an object is given as 76.4 g, the number of significant figures is 3. Nothing changes when we express this as 0.0764 kg; the number of significant figures is still 3. *The number of significant figures of a number is found without taking into consideration the leading zeros.* For example,

1.2398	Number of significant figures: 5
0.00000039	Number of significant figures: 2
3.00007	Number of significant figures: 6
2.70	Number of significant figures: 3

(Notice, on the last row, that the zero on the right-hand side was written explicitly. This means it was measured and found to be zero. Hence, the number of significant figures is 3.) The higher the number of significant figures, the more precisely that quantity is known.

How many significant figures should be kept in the result after arithmetic operations are carried out with two numbers? For example, if a moving object travels 8.0 meters in 3.0 seconds, the speed value goes on as $v = 8.0/3.0 = 2.6666\dots$ Where should we cut this value?

There are, again, two simple rules for keeping the correct number of significant figures in calculation results:

- **In additions and subtractions, the lowest of the number of decimal places is kept:**

$$\begin{aligned}
 3.2339 + 5.4 &= 8.\underline{6339} = 8.6 \\
 9.12 - 5.4917 &= 3.\underline{6283} = 3.63
 \end{aligned}$$

(In the last row above, the **round off rule** was applied when discarding two digits. According to this rule, *if the first number of the discarded part is 5 or greater, the last kept number is rounded up.* Here, the first number of the discarded part is 8, hence the last kept number was rounded up from 2 to 3.)

- In multiplications and divisions, the lowest number of significant figures is kept:

$$3.4567 \times 2.7 = 9.33309 = 9.3$$

$$15.67 \times 0.00012 = 0.0018804 = 0.0019$$

Of course, the **natural numbers** (1, 2, 3, 4 . . .) that are used for counting do not affect the number of significant figures. For example, if the mass of a book is 285.6 g, then the mass of 3 books is $3 \times 285.6 = 856.8$ g, in other words, it does not lose any decimal places.

According to these rules, the speed of the object in the example above should be taken as $v = 8.0/3.0 = 2.666666\dots = 2.7$ m/s. In this course, we shall adopt the following rule: *The number of significant figures shall be taken as 3 in intermediate calculations and 2 in results unless specified otherwise.*

Example 1.1

The dimensions (a, b) of a carpet are given in two different ways:

- (a) $a = (5.2 \pm 0.1)$ m and $b = (8.3 \pm 0.2)$ m. Calculate the surface area of the carpet and give the result with a margin of error.
- (b) Dimensions are given only as $a = 5.2$ m and $b = 8.3$ m. Once again, calculate the surface area and give the result with the correct number of significant figures.

Answer

(a) First, calculate the surface area of the carpet:

$$A = ab = 5.2 \times 8.3 = 43.16 \text{ m}^2$$

The relative error formula for multiplication is used:

$$\frac{\Delta A}{A} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

$$\frac{\Delta A}{43.2} = \frac{0.1}{5.2} + \frac{0.2}{8.3} = 0.043$$

$$\Delta A = 43.2 \times 0.043 = 1.87 \approx 2 \text{ m}^2$$

In the last row, we reduced the surface area error down to one digit, because the width and length margins of error had one digit. Accordingly, the surface area of the carpet, with its margin of error, is written as:

$$A \pm \Delta A = (43 \pm 2) \text{ m}^2$$

(b) The number of significant figures of the width and length is two. Accordingly, only 2 significant figures are kept in the product of the two:

$$A = ab = 5.2 \times 8.3 = 43.16 \approx 43 \text{ m}^2$$

Example 1.2

The speed of an automobile is given as 34 mph (miles/hours). Calculate the speed in terms of km/hour and m/s. (1 mile = 1 609 m).

Answer

Speed is given with 2 significant figures. Thus, the results

shall be kept with 2 significant figures as well. If 1 mil = 1.609 km,

$$34 \text{ mph} = 34 \times 1.609 = 54.706 = 55 \text{ km/hour}$$

Likewise, if 1 hour = 60 minutes and 1 minute = 60 seconds,

$$55 \text{ km/hour} = \frac{55 \times 1000}{3600 \text{ s}} = 15.27 = 15 \text{ m/s}$$

Example 1.3

The conversion of mass into energy is calculated with Einstein's famous formula:

$$E = mc^2$$

Here, $c = 2.997925 \times 10^8$ m/s is the speed of light, E is energy and its unit is joule (J) in the SI system. (1 J = 1 kg·m²/s²).

Calculate the energy of 1.0 gram of material.

Answer

The number of significant figures of the material $m = 1.0$ g is 2, taking into account the zero on the right. Therefore, the result should be given with 2 significant figures. Knowing this, it will be sufficient to round the speed of light down to 3 digits:

$$E = mc^2 = (1 \times 10^{-3} \text{ kg}) \times (3.00 \times 10^8)^2$$

$$E = 9.0 \times 10^{13} \text{ J}$$

1.3 VECTORS

Physical quantities can be divided into two distinct groups with respect to measurement: Some quantities can be completely specified just by giving a numerical value. We call these **scalars**. For example, saying “the temperature of this room is 18° degrees” or “the mass of that table is 25 kg” is sufficient. Quantities of the scalar type include mass, energy, volume, temperature, electrical resistance, refractive index, etc. Scalars can be manipulated by algebraic calculations.

For certain quantities, merely giving a numerical value is not sufficient; one should also specify its direction. For example, when we say, “the speed of this ship is 10 km/hour”, they will ask us: “But 10 km/hour in which direction?” If we reply, “10 km/hour in the north-east direction,” we will have fully specified the velocity of the ship.

Quantities that have both a *magnitude* and a *direction* are called **vectors**. Vectors are manipulated according to certain rules of addition. Velocity, acceleration, force, momentum, electric field, etc., are among vector quantities. Vectors are shown with an arrow ($\vec{}$) over letters, such as \vec{A} , \vec{F} ...

The **magnitude** (or, the norm) of a vector is a positive scalar number shown as $|\vec{A}|$ or, briefly, as A .

In diagrams, a vector is represented by an arrow drawn in the direction of the vector. The length of the arrow is a measure of the magnitude of the vector.

Multiplication of a Vector with a Scalar Number

The product of the number c and the vector \vec{A} is the vector $c\vec{A}$ which has a magnitude of cA .

\vec{A} and $c\vec{A}$ are in the same direction if c is positive,

$c\vec{A}$ is in the opposite direction if c is negative.

For example, the vector $3\vec{A}$ is shown with an arrow that is 3 times as long in the direction of the vector \vec{A} whereas the vector $-2\vec{A}$ is shown with an arrow twice as long in the opposite direction (Figure 1.4).

Addition of Two Vectors

Vectors are not added by rules of arithmetic, because these are quantities with directions. For example, when a ship is displaced 3 km to the east and then 4 km to the north, its total displacement is not $3 + 4 = 7$ km, but only 5 km. Therefore, vector algebra is different from scalar numbers and all algebraic operations must be redefined.

The vector $\vec{A} + \vec{B}$, which is the sum of vectors \vec{A} and \vec{B} is defined with the *triangle rule* or the *parallelogram rule*. These rules are shown in Figures 1.5 and 1.6.

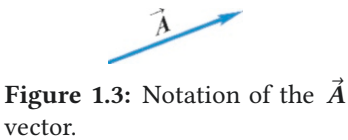


Figure 1.3: Notation of the \vec{A} vector.

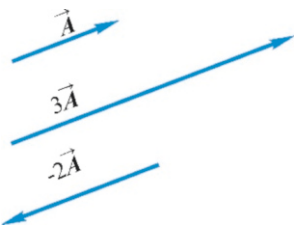
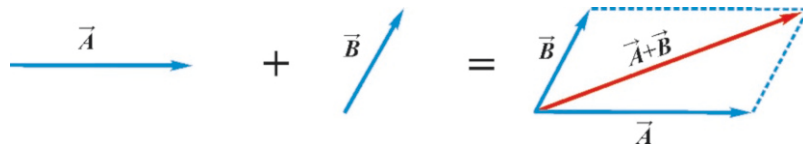


Figure 1.4: Vectors \vec{A} , $3\vec{A}$ and $-2\vec{A}$.

Figure 1.5: The parallelogram rule



In the **parallelogram rule**, the two vectors are first shifted by *preserving their directions* such that both of their tail points are in the same location. Then, a parallelogram is formed by drawing lines from the head points of each vector parallel to the other. The diagonal between the vectors of this parallelogram is the vector $\vec{A} + \vec{B}$.

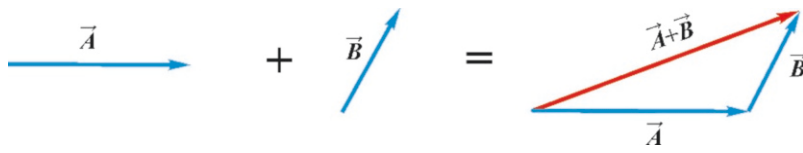


Figure 1.6: The triangle rule.

In the more useful **triangle rule**, one of the vectors (\vec{A} or \vec{B}) is shifted in parallel to itself to the head point of the other vector (Figure 1.6). The vector drawn from the tail of the first vector (\vec{A}) to the head of the second vector (\vec{B}) is $\vec{A} + \vec{B}$.

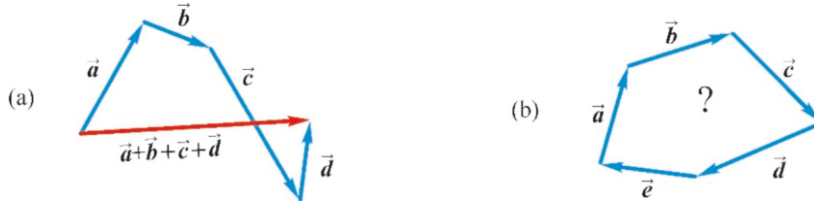


Figure 1.7: (a) Adding many vectors is easier using the triangle rule. (b) What do you think the sum of these five vectors would be?

Both rules give the same result. However, the parallelogram rule is not suitable for adding more than two vectors, as things immediately get complicated. However, the sum of more than two vectors can be drawn immediately using the triangle rule (Figure 1.7a). After the vectors are lined up head-to-tail, it is sufficient to draw a vector from the tail of the first vector to the head of the last vector.

Vector Subtraction

Vector subtraction is performed using the same rule. The difference $\vec{A} - \vec{B}$ is nothing but the sum of the vectors \vec{A} and $-\vec{B}$:

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \tag{1.1}$$

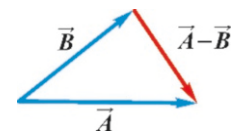
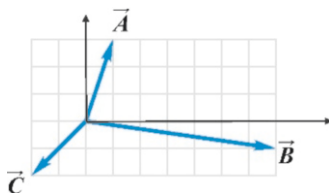


Figure 1.8: Difference of two vectors.

Observing Figure 1.8, you will notice that the head of the vector $\vec{A} - \vec{B}$ ends at the positive-signed vector ($+\vec{A}$).

Example 1.4

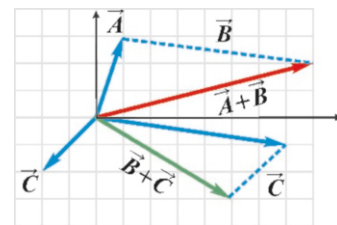


Use the triangle rule to calculate the following for the vectors \vec{A} , \vec{B} , and \vec{C} shown above:

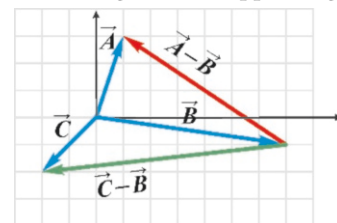
- (a) the sums $\vec{A} + \vec{B}$ and $\vec{B} + \vec{C}$,
- (b) the differences $\vec{A} - \vec{B}$ and $\vec{C} - \vec{B}$.

Answer

According to the triangle rule, one of the two vectors is shifted in parallel to itself until it comes into contact with the tail point of the other vector. The vector drawn from the tail of the fixed vector to the head of the shifted vector is the sum. These sum vectors are shown below:



(b) The difference of two vectors is written as $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ and turned into a sum. Here, the vector $-\vec{B}$ is in opposite direction to \vec{B} . The results shown in the following figure are obtained when the triangle rule is applied again:



Components of a Vector

Vector components are defined in order to perform addition and other vector operations algebraically and not graphically. The components are simply algebraic numbers.

Consider the familiar rectangular coordinate system on the plane. We call this the **cartesian coordinate system**. This system consists of an origin O and two coordinate axes denoted as x - and y -. Positive coordinates are located in the half of the axes between the origin and the direction of the arrow, and the negative coordinates are located in the other half.

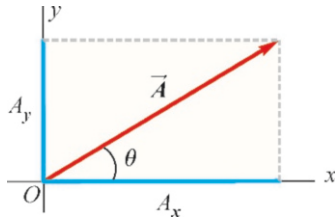


Figure 1.9: Components of a vector \vec{A} .

Let us draw parallel lines from the head points of the vector \vec{A} to the x - and y - axes (Figure 1.9).

The lengths intersected by these parallels are the x - and y -**components** of the vector \vec{A} and are indicated with A_x and A_y respectively. This is shown as,

$$\vec{A} : (A_x, A_y)$$

Now, the components themselves can be used to specify the vector, instead of magnitude and direction.

More generally, in three-dimensional space with three coordinate axes x -, y - and z - the components are defined as follows: A perpendicular is dropped from the head of \vec{A} onto the xy -plane. At the intersection point, two lines are drawn parallel to the x - and y -axes. The lengths that these parallels intersect with the axes constitute the A_x and A_y components of the \vec{A} vector. The perpendicular drawn down to the plane or its projection on the z -axis is the A_z component:

$$\vec{A} : (A_x, A_y, A_z)$$

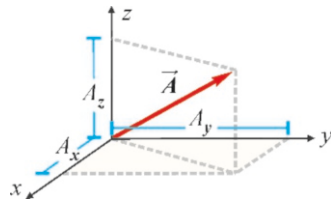


Figure 1.10: Vector components in 3 dimensions.

Now let us find the formula that gives us the components of a vector \vec{A} in a plane. Let the magnitude of this vector be A and its angle with the $+x$ axis be θ . *The positive direction of the angle θ is accepted as the anticlockwise rotation direction.*

Let us remember the trigonometric formulas of a right triangle:

$$\sin \theta = \frac{b}{c}, \quad \cos \theta = \frac{a}{c}, \quad \tan \theta = \frac{b}{a} \quad (1.2)$$

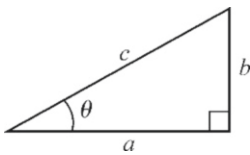


Figure 1.11: Right triangle.

Applying these formulas to the components of \vec{A} (refer to Figure 1.9), we get

$$\begin{aligned} \cos \theta &= \frac{A_x}{A} \quad \longrightarrow \quad A_x = A \cos \theta \\ \sin \theta &= \frac{A_y}{A} \quad \longrightarrow \quad A_y = A \sin \theta \end{aligned}$$

In the opposite case, if the components are given, the magnitude and direction of a vector are found with the following formulas (remember the Pythagorean formula):

$$\begin{aligned} A &= \sqrt{A_x^2 + A_y^2} \\ \tan \theta &= \frac{A_y}{A_x} \end{aligned}$$

We can thus use these formulas to specify a vector either in terms of its components or in terms of its magnitude and direction.

Let us summarize these important formulas together:

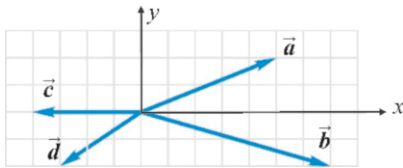
$$\begin{array}{ll} A_x = A \cos \theta & A = \sqrt{A_x^2 + A_y^2} \\ A_y = A \sin \theta & \tan \theta = \frac{A_y}{A_x} \end{array} \quad (1.3)$$

Notice on calculating the angle θ : The expression for $\tan \theta$ in formula (1.3) may not always give the correct result when θ is calculated as an inverse trigonometric function on a calculator. It may sometimes give the complement of that angle, because $(-3)/4$ and $3/(-4)$ have the same value of -0.75 for the calculator.

However, the result of the calculator can be corrected if the signs of the components A_x and A_y are taken into account separately. For example, if $A_x = -3$, $A_y = 4$ then the vector \vec{A} is in the 2nd quadrant, in other words, $[90^\circ < \theta < 180^\circ]$. Likewise, if $B_x = 2$, $B_y = -5$ then the vector \vec{B} is in the fourth quadrant, in other words, $[-90^\circ < \theta < 0^\circ]$. (It is incorrect to say that $[270^\circ < \theta < 360^\circ]$ here; angles greater than 180° are measured from the negative side.)

Example 1.5

Find out the components of the vectors \vec{a} , \vec{b} , \vec{c} and \vec{d} shown in the figure. (Each division has a unit of 1.)



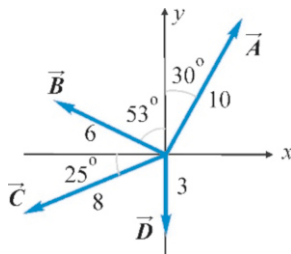
Answer

You can read from the graph by taking note of the signs of the components:

$$\begin{array}{ll} a_x = 5 & a_y = 2 \\ b_x = 7 & b_y = -2 \\ c_x = -4 & c_y = 0 \\ d_x = -3 & d_y = -2 \end{array}$$

Example 1.6

Calculate the components of the vectors \vec{A} , \vec{B} , \vec{C} and \vec{D} whose magnitudes and angles are shown below. (You may use Appendix C at the end of the book for unknown sinus and cosine values.)



Answer

We use formulas (1.3), taking care to measure the angles anti-clockwise from the $+x$ axis:

$$A_x = A \cos(90 - 30)^\circ = 10 \times \cos 60^\circ = 10 \times 0.5 = 5$$

$$A_y = A \sin 60^\circ = 10 \times 0.87 = 8.7$$

$$B_x = B \cos(90 + 53)^\circ = 6 \times (-\cos 37^\circ) = -6 \times 0.8 = 4.8$$

$$B_y = B \sin(90 + 53)^\circ = 6 \times \sin 37^\circ = 6 \times 0.6 = 3.6$$

$$C_x = -C \cos 25^\circ = -8 \times 0.91 = -7.3$$

$$C_y = -C \sin 25^\circ = -8 \times 0.42 = -3.4$$

If a vector is on one of the axes, this means its other component is zero:

$$D_x = 0 \quad \text{and} \quad D_y = -3$$

Example 1.7

The components of the vectors \vec{F} and \vec{G} are given as follows:

$$\begin{array}{ll} F_x = 3, & G_x = -5 \\ F_y = -4, & G_y = -12 \end{array}$$

Find the magnitudes and directions of these vectors.

Answer

We use Eqs. (1.3) that give magnitude and direction,

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{3^2 + (-4)^2} = 5$$

$$\tan \theta = \frac{F_y}{F_x} = \frac{-4}{3} = -1.33$$

Two angles with tangents equal to -1.33, are $+127^\circ$ and -53° . We decide which is correct by taking into account the signs of the components. Since \vec{F} has a negative y -component, this angle must be in the 4th quadrant. Therefore, the answer is

$$\theta = -53^\circ.$$

We use the same method in calculating \vec{G} :

$$G = \sqrt{(-5)^2 + (-12)^2} = 13$$

$$\tan \theta = \frac{-12}{-5} = 2.4$$

Two angles with tangents equal to 2.4 are $+64^\circ$ and -113° . This angle is in the 3rd quadrant, as both components of \vec{G} are negative. Therefore, the answer is $\theta = -113^\circ$.

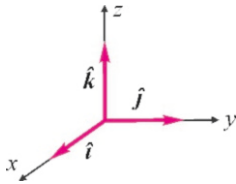


Figure 1.12: Unit vectors.

Unit Vectors

Vectors of unit length (1) are defined along the coordinate axes in order to be able to manipulate vectors easily. First, the \hat{i} vector of unit length is taken along the x -axes in the positive direction. Let us write its components:

$$\hat{i} : (1, 0, 0)$$

Here, the hat ($\hat{\quad}$) sign is used to denote that a vector has a unit length. A unit vector can be defined for every vector. For example, when we write \hat{A} , we understand the unit vector in the direction of \vec{A} .

Likewise, the unit vector in the direction of the y -axis is defined as \hat{j} and the unit vector along the z -axis is defined as \hat{k} . And their components are as follows:

$$\hat{j} : (0, 1, 0), \quad \hat{k} : (0, 0, 1)$$

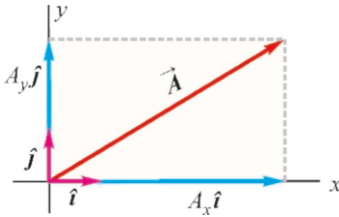


Figure 1.13:

Now let us construct a vector \vec{A} on a plane using what we have learned so far. First, let us try to see what the $A_x \hat{i}$ product is (Figure 1.13). This is a vector of unit length multiplied with the number A_x . Its magnitude is A_x and its direction is \hat{i} , in other words, in the direction of the $+x$ -axis. Likewise, the product $A_y \hat{j}$ is a vector in the $+y$ direction with length A_y .

Now let us look at the sum of these two vectors: Placing these end-to-end starting from the origin, we get the \vec{A} vector according to the triangle rule:

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

This expression can easily be generalized to three-dimensional space:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad (1.4)$$

This expression will be used quite often. When vectors are written in this form, all addition and multiplication operations performed on vectors can be carried out as arithmetic operations on the components. Likewise, the components of the vectors given in such form can be identified immediately. For example, the expression

$$\begin{aligned} \vec{D} &= 3\hat{i} - 5\hat{j} + 6\hat{k} \\ &\quad \downarrow \quad \downarrow \quad \downarrow \\ &\quad D_x \quad D_y \quad D_z \end{aligned}$$

shows us that $D_x = 3$, $D_y = -5$, and $D_z = 6$.

Addition Using Vector Components

Let us write two vectors such as \vec{A} and \vec{B} in terms of their components and unit vectors:

$$\begin{aligned}\vec{A} &= A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \\ \vec{B} &= B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\end{aligned}$$

Using them, let us form the vector $\vec{C} = \vec{A} + \vec{B}$:

$$\begin{aligned}\vec{C} &= \vec{A} + \vec{B} \\ &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k} \\ \vec{C} &= C_x\hat{i} + C_y\hat{j} + C_z\hat{k}\end{aligned}$$

Comparing the last two rows, we see that *the sum of two vectors is a vector whose components are the arithmetic sum of the corresponding components of the two vectors*. In other words,

$$\vec{C} = \vec{A} + \vec{B} \iff \begin{cases} C_x = A_x + B_x \\ C_y = A_y + B_y \\ C_z = A_z + B_z \end{cases} \quad (1.5)$$

Working on components through the use of conventional addition and subtraction is much easier than using graphical methods. This method also applies to vector expressions with more than two vectors or those multiplied with scalars.

Example 1.8

The vectors $\vec{A} = 3\hat{i} - 4\hat{j} + 7\hat{k}$, $\vec{B} = 2\hat{i} - 3\hat{k}$ and $\vec{C} = 8\hat{j}$ are given.

- Find the components of each vector.
- Calculate the sum $\vec{A} + \vec{B}$ and the difference $\vec{B} - \vec{C}$.
- Calculate the expression $3\vec{A} - 8\vec{B} + 9\vec{C}$.

Answer

(a) The coefficients of the unit vectors \hat{i} , \hat{j} , \hat{k} are the x -, y - and z -components respectively. The components are identified from the given expressions:

$$A_x = 3 \quad A_y = -4 \quad A_z = 7$$

$$\begin{array}{lll} B_x = 2 & B_y = 0 & B_z = -3 \\ C_x = 0 & C_y = 8 & C_z = 0 \end{array}$$

(b) When making additions and subtractions using components, the coefficients of unit vectors are algebraically added or subtracted:

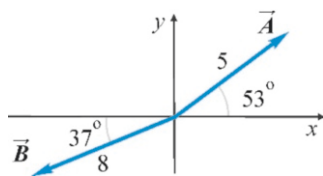
$$\begin{aligned}\vec{A} + \vec{B} &= (3\hat{i} - 4\hat{j} + 7\hat{k}) + (2\hat{i} - 3\hat{k}) \\ &= 5\hat{i} - 4\hat{j} + 4\hat{k}\end{aligned}$$

$$\vec{B} - \vec{C} = (2\hat{i} - 3\hat{k}) - (8\hat{j}) = 2\hat{i} - 8\hat{j} - 3\hat{k}$$

(c) Likewise, the coefficients are calculated using algebraic rules:

$$\begin{aligned}3\vec{A} - 8\vec{B} + 9\vec{C} &= 3(3\hat{i} - 4\hat{j} + 7\hat{k}) - 8(2\hat{i} - 3\hat{k}) + 9(8\hat{j}) \\ &= -7\hat{i} + 60\hat{j} + 45\hat{k}\end{aligned}$$

Example 1.9



The magnitudes and directions of the vectors \vec{A} and \vec{B} are given in the figure.

- Calculate the components and express these two vectors in

terms of unit vectors,

(b) Calculate the vector \vec{R} which is $\vec{R} = 3\vec{A} - 2\vec{B}$.

(c) Find the magnitude and direction of the vector \vec{R} .

Answer

(a) The components:

$$A_x = 5 \cos 53^\circ = 3$$

$$A_y = 5 \sin 53^\circ = 4$$

$$B_x = -8 \cos 37^\circ = -6.4$$

$$B_y = -8 \sin 37^\circ = -4.8$$

Accordingly, in terms of the unit vectors:

$$\vec{A} = 3\hat{i} + 4\hat{j}$$

$$\vec{B} = -6.4\hat{i} - 4.8\hat{j}$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{21.8^2 + 21.6^2} = 31$$

(b) The vector \vec{R} is calculated as follows:

Its direction is calculated as its angle with respect to the $+x$ axis:

$$\vec{R} = 3\vec{A} - 2\vec{B} = 3(4\hat{i} + 3\hat{j}) - 2(-6.4\hat{i} - 4.8\hat{j})$$

$$\tan \theta = \frac{R_y}{R_x} = \frac{21.6}{21.8} \approx 1$$

$$\vec{R} = 21.8\hat{i} + 21.6\hat{j}$$

(c) The magnitude of the vector \vec{R} is calculated as follows:

The angle is $\theta = 45^\circ$ as both components are positive.

Scalar Product

Many formulas in physics can be expressed as products of vectors. Two types of product are defined for this purpose:

- (1) Scalar product,
- (2) Vector product.

As can be understood from their names, the first results in a scalar number and the second in a vector. First, let us consider the scalar product.

The **scalar product** of two vectors \vec{A} and \vec{B} is a scalar number defined as

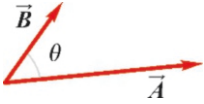


Figure 1.14: Scalar product.

$$\vec{A} \cdot \vec{B} = AB \cos \theta \quad (\text{Scalar product}) \quad (1.6)$$

where A , B are magnitudes, and θ is the angle between the two vectors.

Let us underline the important properties of a scalar product:

- Commutation: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Distribution: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- If two vectors form an angle $\theta = 90^\circ$, in other words, *if the two vectors are perpendicular to each other, the scalar product is zero.* ($\cos 90^\circ = 0$). This feature is often used in calculations as the *condition for perpendicularity*.
- $\vec{A} \cdot \vec{A} = AA \cos 0^\circ = A^2$ or the scalar product of a vector by itself gives the square of its magnitude.
- The sign of the scalar product comes from the $\cos \theta$ term. The product is positive if the angle between the two vectors is less than 90° and negative if greater.

Expression of Scalar Product in Terms of Its Components

First, let us find the scalar products of the unit vectors (\hat{i} , \hat{j} , \hat{k}) with each other:

$$\hat{i} \cdot \hat{i} = 1 \cdot 1 \cdot \cos 0 = 1$$

$$\hat{i} \cdot \hat{j} = 1 \cdot 1 \cdot \cos 90^\circ = 0$$

Likewise, the scalar products of the vectors \hat{j} and \hat{k} with themselves is 1 and with other combinations is zero. Therefore,

$$\begin{aligned} \hat{i} \cdot \hat{i} &= \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\ \hat{i} \cdot \hat{j} &= \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \end{aligned} \quad (1.7)$$

Now let us use these results in the scalar product of two vectors given with components:

$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

As the scalar products in the brackets are expanded, the components, which are numbers, become coefficients, and only the scalar products of the unit vectors remain:

$$\begin{aligned}\vec{A} \cdot \vec{B} = & A_x B_x (\hat{i} \cdot \hat{i}) + A_x B_y (\hat{i} \cdot \hat{j}) + A_x B_z (\hat{i} \cdot \hat{k}) + \\ & + A_y B_x (\hat{j} \cdot \hat{i}) + A_y B_y (\hat{j} \cdot \hat{j}) + A_y B_z (\hat{j} \cdot \hat{k}) + \\ & + A_z B_x (\hat{k} \cdot \hat{i}) + A_z B_y (\hat{k} \cdot \hat{j}) + A_z B_z (\hat{k} \cdot \hat{k})\end{aligned}$$

As the scalar products of the vectors in this expression will either be 1 or 0, the result simplifies as follows:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

In particular, the scalar product of a vector with itself gives the square of its magnitude:

$$A^2 = \vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2 \quad (1.8)$$

Consequently, also recalling the definition (1.6), we can calculate the scalar product in two different ways:

Scalar Product : $\vec{A} \cdot \vec{B} = \begin{cases} A B \cos \theta \\ A_x B_x + A_y B_y + A_z B_z \end{cases}$

(1.9)

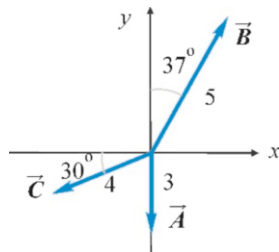
A good application of a scalar product is in finding the angle between two vectors. Combining the two aforementioned expressions for $\cos \theta$,

$$\cos \theta = \frac{A_x B_x + A_y B_y + A_z B_z}{A B} = \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2} \sqrt{B_x^2 + B_y^2 + B_z^2}} \quad (1.10)$$

Scalar product will be used in many definitions in this course, such as work, electric potential, magnetic flux, etc.

Example 1.10

Find the scalar products of the vectors \vec{A} , \vec{B} and \vec{C} with each other.



Answer

Calculate the angles between the vectors from the figure when using the formula $\vec{A} \cdot \vec{B} = AB \cos \theta$:

$$\vec{A} \cdot \vec{B} = AB \cos(180^\circ - 37^\circ) = 3 \times 5 \times (-\cos 37^\circ)$$

$$\vec{A} \cdot \vec{B} = 15 \times (-0.8) = -12$$

$$\vec{A} \cdot \vec{C} = AC \cos(90^\circ - 30^\circ) = 3 \times 4 \times \cos 60^\circ$$

$$\vec{A} \cdot \vec{C} = 12 \times (0.5) = 6$$

$$\vec{B} \cdot \vec{C} = BC \cos(90^\circ + 67^\circ) = 5 \times 4 \times \cos 157^\circ$$

$$\vec{B} \cdot \vec{C} = 20 \times (-\cos 23^\circ) = 20 \times (-0.92) = -18$$

Example 1.11

$$\vec{p} = 3\hat{i} - 8\hat{j}, \quad \vec{q} = 8\hat{i} + 7\hat{j}$$

(a) Find the scalar product $\vec{p} \cdot \vec{q}$,

(b) Find the angle between these two vectors.

Answer

(a) Use the component expression of scalar product:

$$\vec{p} \cdot \vec{q} = p_x q_x + p_y q_y = 3 \times 8 + (-8) \times 7 = -32$$

(b) First, find the magnitudes of the vectors:

$$p = \sqrt{p_x^2 + p_y^2} = \sqrt{3^2 + (-8)^2} = \sqrt{73} \approx 8.5$$

$$q = \sqrt{8^2 + 7^2} = \sqrt{113} \approx 11$$

$$\cos \theta = \frac{\vec{p} \cdot \vec{q}}{p q} = \frac{-32}{8.5 \times 11} = -0.34$$

$$\theta = 110^\circ$$

The result is an obtuse angle, as the scalar product is negative.

This data is used in formula (1.10):

Example 1.12

$$\vec{A} = A_x \hat{i} + 12 \hat{j}, \quad \vec{B} = 4 \hat{i} + 5 \hat{j}$$

What should the value of the unknown component A_x be such that these two vectors are perpendicular?

Answer

The scalar product must be zero if the two vectors are perpendicular. Write this condition in terms of the components,

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = 0$$

$$A_x \times 4 + 12 \times 5 = 0 \quad \rightarrow \quad A_x = -60/4 = -15$$

Vector Product

There are many cases in physics in which operations on two vectors result in a new vector. Hence, it is convenient to define a vector product.

Definition: The **vector product** of the vectors \vec{A} and \vec{B} with an angle θ between them, is a new vector denoted as

$$\vec{C} = \vec{A} \times \vec{B}$$

and its magnitude and direction are:

- **Magnitude:** $C = AB \sin \theta$,
- **Direction:** Perpendicular to the plane formed by \vec{A} and \vec{B} and given by the **right-hand rule**.

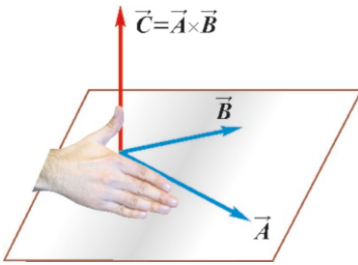


Figure 1.15: The direction of vector product is given with the right-hand rule.

Right-Hand Rule: The right-hand rule needs to be well understood, as it will be used in many topics throughout this course. As in Figure (1.15), four fingers of the right hand are pointed towards the first vector (\vec{A}) and the palm is turned towards the second vector (\vec{B}), thus the thumb gives the direction of the vector \vec{C} .

This rule may be described differently in various textbooks. Although the expression we provide here is quite common, you may continue to use any other technique that suits you.

Let us underline the important properties of a vector product:

- Anticommutativity: $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$.
Hence, *the order matters in a vector product*.
- Distribution: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
- If the two vectors are parallel ($\theta = 0$) or anti-parallel ($\theta = 180^\circ$), then the vector product will be zero, as the sines will have zero value.

In particular, the vector product of a vector with itself is zero: $\vec{A} \times \vec{A} = 0$

Expression of Vector Product in Terms of Components

First, let us find the vector products of the unit vectors (\hat{i} , \hat{j} , \hat{k}) with each other. By definition, the product of each vector with itself will be zero,

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

Now let us look at the product $\hat{i} \times \hat{j}$: As the magnitudes are 1 and the angles are 90° , the result of the vector product will be a vector of magnitude $1 \cdot 1 \cdot \sin 90^\circ = 1$. Its direction will be in the $+z$ direction according to the right-hand rule. Hence, this is just our unit vector \hat{k} .

Also, as the vector product is anticommutative, $\hat{j} \times \hat{i} = -\hat{k}$.

The vector products of unit vectors are thus as follows:

$$\begin{aligned} \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} &= 0 \\ \hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j} \end{aligned} \quad (1.11)$$

Now let us use these results in the vector product of two vectors given with components:

$$\vec{C} = \vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

If the brackets are expanded and the component are taken outside of the vector products,

$$\begin{aligned} \vec{C} = & A_x B_x (\hat{i} \times \hat{i}) + A_x B_y (\hat{i} \times \hat{j}) + A_x B_z (\hat{i} \times \hat{k}) + \\ & + A_y B_x (\hat{j} \times \hat{i}) + A_y B_y (\hat{j} \times \hat{j}) + A_y B_z (\hat{j} \times \hat{k}) + \\ & + A_z B_x (\underbrace{\hat{k} \times \hat{i}}_{\hat{j}}) + A_z B_y (\underbrace{\hat{k} \times \hat{j}}_{-\hat{i}}) + A_z B_z (\underbrace{\hat{k} \times \hat{k}}_0) \end{aligned}$$

The result simplifies as follows when the vector product expressions of unit vectors are used:

$$\vec{C} = \vec{A} \times \vec{B} = \underbrace{(A_y B_z - A_z B_y)}_{C_x} \hat{i} + \underbrace{(A_z B_x - A_x B_z)}_{C_y} \hat{j} + \underbrace{(A_x B_y - A_y B_x)}_{C_z} \hat{i} \quad (1.12)$$

The **circular permutation** technique is used to memorize this formula. Consider that the indexes rotate after each other as follows:

$$x \rightarrow y \rightarrow z, \quad y \rightarrow z \rightarrow x, \quad z \rightarrow x \rightarrow y$$

Accordingly, when writing the C_x component, follow it with the y -component of \vec{A} multiplied by the z -component of \vec{B} and then switch the indexes and subtract. The circular permutation of the indexes is performed for the other components:

$$C_x = \underbrace{A_y B_z - A_z B_y}_{x \rightarrow y \rightarrow z}, \quad C_y = \underbrace{A_z B_x - A_x B_z}_{y \rightarrow z \rightarrow x}, \quad C_z = \underbrace{A_x B_y - A_y B_x}_{z \rightarrow x \rightarrow y}$$

Another way of expressing vector product is to write it as a determinant:

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (1.13)$$

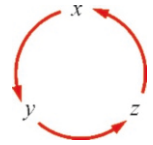


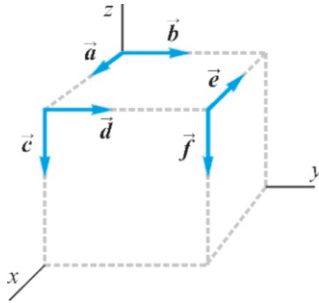
Figure 1.16: Circular permutation.

If you expand this determinant according to the minors of the first row, the expression (1.12) of the vector product follows.

Quantities expressed as vector products in physics include torque, angular momentum, magnetic force, etc.

The limited vector information we provided here is vital to understanding subsequent topics. It would be wrong to continue without understanding these matters.

Example 1.13

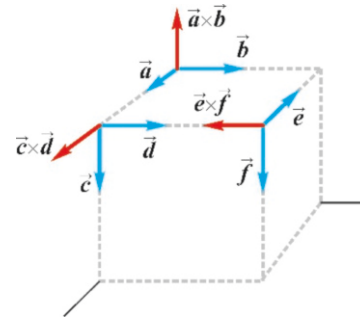


The figure shows vectors drawn at the corners of a cube with unit length (1) edges.

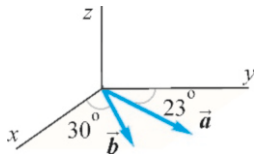
Find only the directions of the vector products $\vec{a} \times \vec{b}$, $\vec{c} \times \vec{d}$ and $\vec{e} \times \vec{f}$ and mark them on a figure.

Answer

According to the right-hand rule, if four fingers are pointed towards the first vector and the palm is turned towards the second vector, then the thumb gives the direction of the product. Accordingly, the results are as follows:



Example 1.14



The magnitudes of vectors \vec{a} and \vec{b} shown on the xy -plane are $a = 15$ and $b = 12$ units respectively. Find the magnitude and direction of the vector \vec{c} which is the product $\vec{c} = \vec{a} \times \vec{b}$.

Answer

Let us first find the magnitude of the vector \vec{c} . It can be seen from the figure that the angle between the vectors is $90 - (23 + 30) = 37^\circ$. Accordingly,

$$c = ab \sin 37 = 15 \times 12 \times 0.6$$

$$c = 108$$

The direction of \vec{c} should be perpendicular to the xy -plane, in other words, along the z -axis. The $-z$ direction is found according to the right-hand rule.

Example 1.15

The vectors $\vec{D} = 3\hat{i} - 5\hat{j}$, $\vec{E} = 7\hat{i} - 3\hat{k}$ are given.

Calculate the components of the vector $\vec{F} = \vec{D} \times \vec{E}$.

Answer

Use the formula (1.12) that gives the components. The circular permutation of the component indexes allows us to easily

remember the components.

$$F_x = D_y E_z - D_z E_y = -5 \times (-3) - 0 \times 0 = 15$$

$$F_y = D_z E_x - D_x E_z = 0 \times 7 - 3 \times (-3) = 9$$

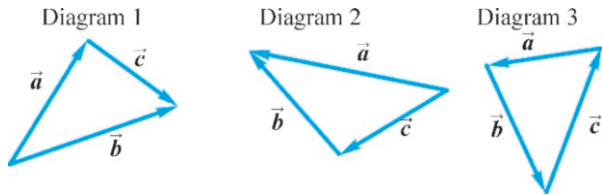
$$F_z = D_x E_y - D_y E_x = 3 \times 0 - 5 \times 7 = -35$$

Accordingly, $\vec{F} = 15\hat{i} + 9\hat{j} - 35\hat{k}$.

Multiple-choice Questions

- How many significant figures does the number 0.003804 have?
 - 2
 - 3
 - 4
 - 5
- How should the result of $1.2 + 0.222 = ?$ be expressed?
 - 1.2
 - 1.22
 - 1.4
 - 1.422

3. The mass of a metal coin is (8.2 ± 0.1) grams. What is the mass of two metal coins?
 (a) 16.4 (b) 16.4 ± 0.2 (c) 16.6 (d) 16 ± 0.1
4. The relative error in the measurement of the edge of a cube is 1%. What is the relative error of the volume of the cube?
 (a) 1% (b) 2% (c) 3% (d) 4%
5. An object has a mass of 8.888 g and volume of 2.0 cm^3 . What is the density of this object in g/cm^3 ?
 (a) 4 (b) 4.4 (c) 4.44 (d) 4.444
6. Which of the following quantities is not a vector?
 (a) Velocity
 (b) Force
 (c) Electric field
 (d) Volume
7. Which of the following is true if two vectors are perpendicular?
 (a) The vectors have the same magnitude.
 (b) Their scalar product is zero.
 (c) Their vector product is zero.
 (d) Their components are the same.



8. Which sum is correct for the three vectors in Diagram 1?
 (a) $\vec{a} = \vec{b} + \vec{c}$ (b) $\vec{b} = \vec{a} + \vec{c}$ (c) $\vec{c} = \vec{a} + \vec{b}$ (d) None.
9. Which difference is correct for the three vectors in Diagram 2?
 (a) $\vec{a} = \vec{b} - \vec{c}$ (b) $\vec{b} = \vec{a} - \vec{c}$ (c) $\vec{c} = \vec{a} - \vec{b}$ (d) None.
10. Which equality is true for the three vectors in Diagram 3?
 (a) $\vec{c} = \vec{a} + \vec{b}$
 (b) $\vec{a} - \vec{b} = \vec{c}$
 (c) $\vec{a} + \vec{b} + \vec{c} = 0$
 (d) $\vec{b} = \vec{a} + \vec{c}$
11. Which of the following is true if the vector product of two vectors is zero?
 (a) The vectors are parallel.
 (b) The vectors are perpendicular.
 (c) The magnitudes of the vectors are equal.
 (d) None of the above.

12. The scalar product of a vector $(\vec{A} \cdot \vec{A})$ with itself is:
 (a) Zero.
 (b) Equal to its magnitude.
 (c) Equal to the square of its magnitude.
 (d) Equal to the square root of its magnitude.
13. Which of the following equalities is true?
 (a) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
 (b) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
 (c) $\vec{A} \times \vec{A} = 0$
 (d) All of the above
14. The magnitude of vector \vec{a} is 5 units and the magnitude of vector \vec{b} is 3 units. Accordingly, which of the following cannot be the magnitude of $\vec{a} + \vec{b}$?
 (a) 1 (b) 3 (c) 5 (d) 7
15. Which of the following propositions is true?
 (a) The sum of two vectors with different magnitudes may be zero.
 (b) The magnitude of a vector is equal or greater than its perpendicular components.
 (c) The sum of the vectors $(\hat{i} + \hat{j})$ is also a unit vector.
 (d) If $\vec{a} = \vec{b} + \vec{c}$ then $a > b$ and $a > c$ are always true.
16. Which of the following vector algebra operations does not comply with the rules?
 (a) $\vec{A} \cdot (\vec{B} \times \vec{C})$
 (b) $\vec{A} \times (\vec{B} + \vec{C})$
 (c) $\vec{A} \times (\vec{B} \cdot \vec{C})$
 (d) $\vec{A} \times (\vec{B} \times \vec{C})$
17. Which of the following vector equalities is false?
 (a) $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
 (b) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
 (c) $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$
 (d) $\vec{A} \cdot (\vec{B} \times \vec{B}) = B^2 \vec{A}$
18. Which of the following is true if the sum of three vectors is zero?
 (a) All three vectors are on the same plane.
 (b) The vectors are perpendicular.
 (c) The vectors have the same magnitude.
 (d) The sum of three vectors cannot be zero.
19. Which of the following is true if the scalar product of two vectors is negative?
 (a) The vectors have equal magnitude.
 (b) The vectors are in opposite directions.
 (c) Their angle is obtuse.
 (d) The vectors are perpendicular.

20. If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then which of the following is always true?

(a) $\vec{b} = \vec{c}$

(b) \vec{b} and \vec{c} are in the same direction.

(c) \vec{b} and \vec{c} are perpendicular.

(d) The projections of \vec{b} and \vec{c} along \vec{a} are equal.

Problems

1.1 Dimensions and Units

1.1 The fuel consumption of an automobile is given as 60.0 mil/gallon. Convert this into the unit km/liter. (1 mil = 1.609 km and 1 gallon = 3.788 liter.)

[Answer: 25.5 km/L.]

1.2 The unit *light-year*, a unit of length used in astronomy, is the distance traveled by light in one year with the speed $c = 2.998 \times 10^8$ m/s. Another unit of length, the *Astronomic Unit* (AU), is the average distance between the Earth and the Sun and $1 \text{ AU} = 1.50 \times 10^8$ km. (a) How many meters is one light-year? (b) How many AUs is one light-year?

[A: (a) 9.45×10^{15} m, (b) 6.3×10^4 AU.]

1.3 A wall of 6.0 m^2 can be completely covered with 1.0 liters of paint. What is the thickness of the paint?

[A: 0.17 mm.]

1.4 Express the following data using prefixes of basic units: 3×10^{-9} m, 8×10^{13} bytes, 5×10^{-6} seconds.

[A: 3 nm, 80 terabytes, 5 μ s.]

1.2 Precision and Significant Figures

1.5 The radius of a sphere is measured as 6.5 ± 0.2 cm. Calculate the surface area and volume of this sphere, and express the results with a margin of error. (The surface area of a sphere is $A = 4\pi r^2$ and its volume is $V = (4/3)\pi r^3$.)

[A: $A = (531 \pm 30) \text{ cm}^2$, $V = (1150 \pm 100) \text{ cm}^3$.]

1.6 A watch brand claims that its watches produce 8 seconds of error in one year. (a) What is the relative error of this watch? (b) How much error is there at the end of a 90-minute football match?

[A: (a) 3×10^{-7} , (b) 0.001 s.]

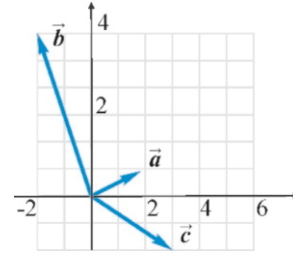
1.7 The length of a rectangular plate is measured with a millimetric ruler and found to be 18 mm. Then, its width is measured more precisely with a micrometer and found to be 3.5 mm. (a) Find the relative errors of the width and length. (b) Calculate the surface area of the plate with a margin of error.

[A: (a) 0.06, 0.03. (b) $(63 \pm 6) \text{ mm}^2$.]

1.8 According to FIFA rules, the width of a football field should be within the range 65 – 75 m and length within the range 100 – 110 m. (a) What are the relative errors for the width and the length? (b) If we wish to cover this field with grass, what would the surface area of the field with a margin of error be?

[A: 7% 5%, (b) $7350 \pm 882 \text{ m}^2$.]

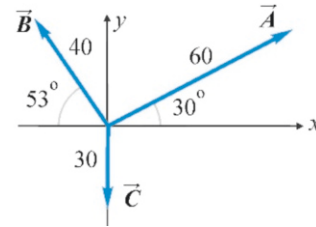
1.3 Vectors



Problem 1.9

1.9 For the vectors shown on a millimeter graph paper in the figure, find the sums $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{a} + \vec{c}$ by drawing them according to the triangle rule.

1.10 The magnitudes and angles to the $+x$ axis of three vectors are given as follows: (a) 33 m/s, 60° , (b) 128 m/s, 150° , (c) 22 m/s, -145° . Calculate the components of these vectors. [A: (a) 17, 29, (b) $-111, 64$, (c) $-18, -13$.]



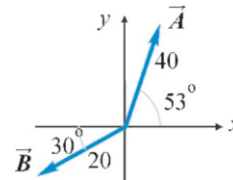
Problem 1.11

1.11 Calculate the components of the vectors \vec{A} , \vec{B} and \vec{C} , whose magnitudes and directions are given in the figure.

[A: $A_x=52, A_y=30, B_x=-24, B_y=32, C_x=0, C_y=-30$.]

1.12 Calculate the magnitudes and directions (angles to the x axis) of the following three vectors specified with their components: $a_x=5, a_y=12, b_x=-4, b_y=-3, c_x=-3, c_y=1$.

[A: $a = 13, 67^\circ, b = 5, -143^\circ, c = \sqrt{10}, 162^\circ$.]



Problem 1.13

1.13 (a) Write the vectors \vec{A} and \vec{B} shown in the figure in terms of the unit vectors (\hat{i}, \hat{j}). (b) Find the vector $\vec{C} = 2\vec{A} - 3\vec{B}$. (c) Calculate the magnitude and direction of the vector \vec{C} .

[A: (a) $\vec{A} = 24\hat{i} + 32\hat{j}, \vec{B} = -17\hat{i} - 10\hat{j}$. (b) $\vec{C} = 99\hat{i} + 94\hat{j}$. (c) $C = 137, \theta = 44^\circ$.]

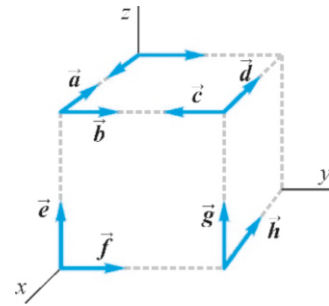
- 1.14 The vectors $\vec{F} = -2\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{G} = 4\hat{i} - 7\hat{j} - 4\hat{k}$ are given. (a) Calculate the magnitudes of the vectors. (b) Calculate the vector $\vec{F} - \vec{G}$.
 [A: (a) 7 and 9, (b) $\vec{F} - \vec{G} = -6\hat{i} + 10\hat{j} + 10\hat{k}$.]

- 1.15 (a) Calculate the magnitudes of the vectors $\vec{A} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{B} = 6\hat{i} + 2\hat{j} - 3\hat{k}$. (b) Calculate their scalar product. (c) Find the angle between the two vectors.
 [A: (a) 3 and 7, (b) 5, (c) 76° .]

- 1.16 The vectors $\vec{a} = 2\hat{i} - 5\hat{k}$, $\vec{b} = 3\hat{j} - 4\hat{k}$ and $\vec{c} = 5\hat{i} + 2\hat{j}$ are given. Calculate the scalar product $\vec{a} \cdot (\vec{b} - \vec{c})$. [A: 10.]

- 1.17 Find a vector \vec{b} perpendicular to the vector $\vec{a} = 3\hat{i} - 5\hat{j}$ such that the component b_x is 4 units. [A: $\vec{b} = 4\hat{i} + 2.4\hat{j}$.]

- 1.18 The magnitudes of the vectors \vec{A} and \vec{B} are known to be $A = 1.2$ and $B = 5$. The vector products of these two vectors is $\vec{A} \times \vec{B} = 3\hat{i} - 4\hat{j}$. Find the angle between these two vectors. [A: 53° .]



Problem 1.19

- 1.19 For the vectors indicated in the figure, show only the directions of the products $\vec{a} \times \vec{b}$, $\vec{c} \times \vec{d}$, $\vec{e} \times \vec{f}$ and $\vec{g} \times \vec{h}$ on the figure.

- 1.20 (a) Calculate the magnitudes of the vectors $\vec{A} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{B} = 4\hat{i} + 4\hat{j} + 7\hat{k}$. (b) Find the angle between them. (c) Find the components of the vector product $\vec{C} = \vec{A} \times \vec{B}$. (d) Calculate the magnitude of \vec{C} .

[A: (a) 3 and 9, (b) 71° , (c) $C_x = 18$, $C_y = -18$, $C_z = 0$, (d) $C = 18\sqrt{2}$.]

2

MOTION IN A STRAIGHT LINE



The masterpiece of Japanese public transportation system, *Shinkansen* (the bullet train) passing by Mount Fuji Yama en route from Tokyo to Osaka. The average speed of Shinkansen can reach up to 320 km/h. How can we estimate the position of this train at a future time?

Motion is one of the main topics of interest in physics. The branch of physics called **Mechanics** studies all moving objects, from tiny pebbles to automobiles, from cannon shells to planets and massive stars.

Mechanics comprises two parts: **Kinematics** is the branch of Mechanics that studies the relations between the positions and speeds of objects with time, regardless of the reasons for the motion. **Dynamics** sets the laws of motion and examines the motion under physical forces. In this chapter, we will examine one-dimensional kinematics. Vector concepts shall not be required here, but the basic concepts that we will develop will later form the basis of vector expressions in two and three-dimensional motions.

2.1 POSITION, DISPLACEMENT, VELOCITY AND ACCELERATION

Position and time are the two most fundamental concepts in physics. **Time** is a uniform quantity that flows at the same rate for everyone. Because of these properties, **time is the independent variable in kinematics**, as it is not possible to alter it. It is indicated in equations with the symbol t .

Position and Displacement

The **position** of an object is its location on a coordinate system. In three-dimensional space, position is specified with the Cartesian coordinates (x, y, z) . These coordinates vary as a function of time during a general motion:

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

Only one coordinate axis is required in one-dimensional space. For this purpose, an infinite line is taken and the positive direction is marked by an arrow at the end. Then, an **origin** (O), in other words, a yardstick, is chosen to mark the place from which the measurements are to be made. This can be a tree or a pole that is known to everyone. The origin separates the positive region from the negative one on the x -axis.

Figure 2.1: The position of an object in a one-dimensional coordinate system.

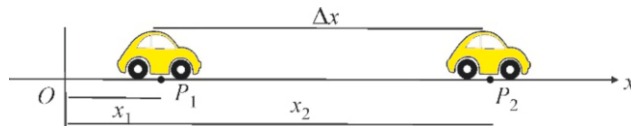


Therefore, the position of an object at point P is its x coordinate at time t :

$$x = x(t) \quad (\text{Position}) \quad (2.1)$$

The property that determines motion is the change in position. Displacement is defined for this purpose.

Figure 2.2: Displacement Δx .



Definition: If the position of an object is x_1 at time t_1 and its position is x_2 at a later time t_2 , the difference

$$\Delta x = x_2 - x_1 \quad (\text{Displacement}) \quad (2.2)$$

is called the **displacement** (Figure 2.2). Its unit is the meter (m). (In physics, the Δ sign before a symbol means the change in that quantity. For example: $\Delta a = a_{\text{last}} - a_{\text{first}}$.)

The sign of displacement determines the direction of motion. If $\Delta x > 0$ then $x_2 > x_1$, in other words, the object moves in the positive direction. Likewise, if $\Delta x < 0$ then $x_2 < x_1$, and the object moves in the negative direction.

Average and Instantaneous Velocity

Anyone can run 100 meters, but some of us run it in a shorter time. It is important to know the amount of displacement in a given time interval. Velocity is defined for this purpose.

Definition: If the position of an object at time t_1 is x_1 and its position at a later time t_2 is x_2 , then the ratio

$$v_{\text{av}} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \quad (\text{average velocity}) \quad (2.3)$$

is called the **average velocity**. Its unit is the meter/second (m/s).

The direction of motion is always determined by the sign of velocity. As Δt is always positive, only the sign of the displacement Δx matters. Accordingly, if displacement $\Delta x > 0$, then the velocity is also positive and the object moves in the $+x$ direction. In contrast, if $\Delta x < 0$, then the velocity is negative and the object moves in the $-x$ direction.

Average velocity is not useful for calculating in physics, because it requires measurement at two points and it is not possible to know the velocity until the object reaches the final point x_2 . Instead, we would like to know the velocity at a given instant. The concept of instantaneous velocity is defined for this purpose.

Let us remember the concept of the derivative in mathematics. Consider placing two sticks at the positions x_1 and x_2 along a path so as to measure average speed. Let us measure the times t_1 and t_2 when the moving object passes through these sticks. This measurement will give us the average speed in the interval $[x_1, x_2]$. Now, let us bring the stick at x_2 closer to x_1 , in other words, let Δx get smaller. In this case, the value Δt will also be smaller. However, the ratio $\Delta x/\Delta t$ may not be small.

Therefore, let us gradually bring x_2 closer to x_1 such that $\Delta x \rightarrow 0$. In such a case, the time interval will also approach zero. Yes, $\Delta x \rightarrow 0$, $\Delta t \rightarrow 0$ but surprisingly, the ratio $\Delta x/\Delta t$ remains at a finite value. The operation of “approaching zero without setting equal to zero” is known in mathematics as the **limit**. And the ratio $\Delta x/\Delta t$ is called “the **derivative** of x with respect to t ”. The first derivative is shown as dx/dt or x' .

Definition: The limit of the average velocity is called the **instantaneous velocity** (or simply the **velocity**):

$$v = \lim_{t_2 \rightarrow t_1} \frac{x_2 - x_1}{t_2 - t_1} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad (\text{velocity}) \quad (2.4)$$

We thus know the velocity at a given time t . The direction of motion again depends on the sign of the velocity v .

Speed

Speed is the absolute value (or, the magnitude) of the velocity, shown as $|v|$.

In one-dimensional motion, the velocity seems merely to be an algebraic quantity, it can be either positive or negative. But, in reality, it is a vector quantity, as will be clear in two- and three-dimensional motion. The speed of a vector velocity has the same definition as the magnitude.

Brief information on derivatives

The concept of derivative and derivation techniques are examined extensively in calculus courses. Here, let us briefly review the derivatives of the most frequently encountered types of functions without proof. The derivatives of most useful functions $y(x)$ with respect to the independent variable x are as follows:

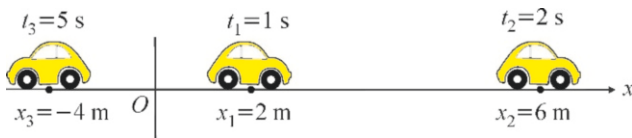
Derivatives of certain functions

function y	derivative $y' = \frac{dy}{dx}$	function y	derivative $y' = \frac{dy}{dx}$
x	1	$\sin x$	$\cos x$
x^3	$3x^2$	$\cos x$	$-\sin x$
x^{-5}	$-5x^{-6}$	$\tan x$	$1/\cos^2 x$
\sqrt{x}	$1/(2\sqrt{x})$	e^x	e^x
x^n	$n x^{n-1}$	$\ln x$	$1/x$

In mechanics, the independent variable should be taken as t when using these formulas.

Example 2.1

The positions of a moving object at various times t are shown on the x axis: (a) Determine the displacements $\Delta x = x_2 - x_1$ and $\Delta x = x_3 - x_2$. (b) Determine the average velocities at the time intervals $\Delta t = t_2 - t_1$ and $\Delta t = t_3 - t_2$.



Answer

(a) We read the positions of the object at various times from

the figure:

During the interval $[t_1, t_2]$: $\Delta x = x_2 - x_1 = 6 - 2 = 4$ m
The object was displaced by 4 m in the positive direction.

During the interval $[t_2, t_3]$: $\Delta x = x_3 - x_2 = -4 - 6 = -10$ m
The object was displaced by 10 m in the negative direction.

(b) The average velocity is the ratio of these displacements to the elapsed time:

$$\text{During the interval } [t_1, t_2]: v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{4}{2-1} = 4 \text{ m/s}$$

$$\text{During the interval } [t_2, t_3]: v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{-10}{5-2} = -3.3 \text{ m/s}$$

Example 2.2

The position of an object is given as a function of time as:

$$x = t^3 - 5t^2 + 8 \quad (\text{meters})$$

- (a) Find the positions of the object at times $t = 1$ s and $t = 5$ s,
 (b) Calculate the average velocity of the object during this time interval,
 (c) Find the velocities of the object at times $t = 1$ s and $t = 5$ s,
 (d) Find the positions at which the velocity of the object is zero.

Answer

(a) The x values of the given function at times $t = 1$ s and $t = 5$ s give the positions:

$$\text{For } t = 1 \text{ s: } x(1) = 1^3 - 5 \times 1^2 + 8 = 4 \text{ m}$$

$$\text{For } t = 5 \text{ s: } x(5) = 5^3 - 5 \times 5^2 + 8 = 8 \text{ m}$$

$$(b) v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{8-4}{5-1} = 1 \text{ m/s}$$

(c) The general expression of velocity is the derivative dx/dt :

$$v = \frac{dx}{dt} = 3t^2 - 10t$$

The values of this expression at the requested times t are the instantaneous velocities:

$$\text{The velocity at } t = 1: v(1) = 3 \times 1^2 - 10 \times 1 = -7 \text{ m/s}$$

$$\text{The velocity at } t = 5: v(5) = 3 \times 5^2 - 10 \times 5 = 25 \text{ m/s}$$

(d) When the velocity is zero, we have $v(t) = 0$:

$$v = 3t^2 - 10t = 0$$

The velocity is zero at the roots of this expression, which are $t = 0$ and $t = 10/3$. We calculate the x positions that correspond to these t values:

$$x(0) = 8 \text{ m} \quad \text{and} \quad x(10/3) = -11 \text{ m}$$

Average and Instantaneous Acceleration

Velocity gives us very important information about the motion of an object. However, we sometimes also need to know the rate of change of velocity with respect to time. We might ask, in regard to two cars, "Which one speeds up more quickly?" The rate of change of velocity is called the acceleration.

Definition: If the velocity of an object is v_1 at time t_1 , and v_2 at a later time t_2 , then the ratio

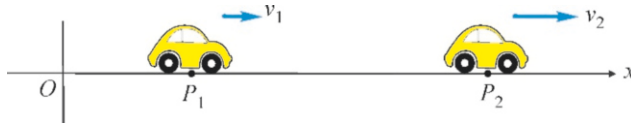


Figure 2.3: Average acceleration.

$$a_{\text{av}} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t} \quad (\text{average velocity}) \quad (2.5)$$

is called the **average acceleration**. Its unit is the meter/second² (m/s^2).

It is important to understand the sign of the acceleration. We sometimes hear definitive statements such as “acceleration is positive if the object speeds up and negative if it slows down.” This is incorrect. The expression for acceleration above shows that *the sign depends on the chosen coordinate system*. For example, if the object moves in the positive direction ($v_1, v_2 > 0$) and its velocity is increasing ($v_2 > v_1$), then surely $\Delta v > 0$ and therefore a_{av} will be positive. However, consider now the case in which the object is moving in the negative direction (v_1, v_2 both negative). If its speed is increasing, for negative numbers, this means that $v_2 < v_1$ (e.g., $-5 < -3$). In this case, $\Delta v = v_2 - v_1 < 0$ and therefore a_{av} is negative!

The correct interpretation of the sign of acceleration is as follows: *Objects that speed up in the positive direction or slow down in the negative direction have positive acceleration*. And, vice versa, objects that slow down in the positive direction or speed up in the negative direction have negative acceleration.

Average acceleration is, again, not a very useful quantity, because it requires measurement at two points and it is not possible to know until the object reaches the final velocity v_2 . Instead, we would like to know the acceleration at a given instant.

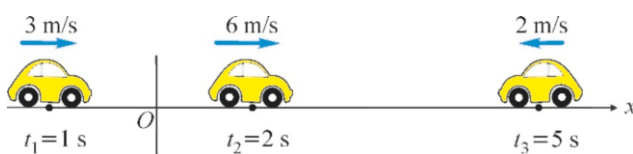
Definition: The limit of the average acceleration is called the **instantaneous acceleration** (or simply the **acceleration**):

$$a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad (\text{acceleration}) \quad (2.6)$$

In other words, acceleration is the derivative of velocity with respect to time. Also, as velocity is the derivative of position, we may regard acceleration as the second derivative of position. In mathematics, the second derivative is shown with d^2x/dt^2 or x'' .

We can thus know the acceleration at a given time t . Acceleration is not often used in daily life. However, we sometimes hear the performance of sports cars expressed as “reaching 100 km/h speed in 6 seconds”, etc. This is, in fact, another expression of acceleration.

Example 2.3



The velocities of a moving object at various times are shown on the x axis.

- Determine the velocity differences $\Delta v = v_2 - v_1$ and $\Delta v = v_3 - v_2$,
- Determine the average accelerations at the time intervals

$$\Delta t = t_2 - t_1 \text{ and } \Delta t = t_3 - t_2.$$

Answer

(a) Reading the velocities of the object at various times from the figure:

$$\text{During the interval } [t_1, t_2]: \Delta v = v_2 - v_1 = 6 - 3 = 3 \text{ m/s}$$

$$\text{During the interval } [t_2, t_3]: \Delta v = v_3 - v_2 = -2 - 6 = -8 \text{ m/s}$$

(b) The average acceleration is the ratio of these differences in velocity to the elapsed time:

$$\text{During the interval } [t_1, t_2]: a_{\text{av}} = \frac{\Delta v}{\Delta t} = \frac{3}{2 - 1} = 3 \text{ m/s}^2$$

$$\text{During the interval } [t_2, t_3]: a_{\text{av}} = \frac{-8}{5 - 2} = -2.7 \text{ m/s}^2$$

Example 2.4

The position of an object as a function of time is given by

$$x = t^3 - 5t^2 + 4 \quad (\text{meters}).$$

- (a) Determine the velocity and acceleration functions using derivative,
 (b) Find the velocities of the object at times $t = 1$ s and $t = 5$ s,
 (c) Calculate the average acceleration of the object during this time interval,
 (d) Find the accelerations of the object at times $t = 1$ s and $t = 5$ s,
 (e) Find the time when the acceleration is zero.

Answer

(a) The first derivative of x is the velocity v and its second derivative is the acceleration a :

$$v = x' = 3t^2 - 10t$$

$$a = v' = 6t - 10$$

(b) For $t = 1$, $v = 3 \times 1^2 - 10 \times 1 = -7$ m/s,

$$\text{for } t = 5, \quad v = 3 \times 5^2 - 10 \times 5 = 25 \text{ m/s}.$$

(c) These velocities are used in the definition of average acceleration:

$$a_{\text{av}} = \frac{\Delta v}{\Delta t} = \frac{25 - (-7)}{5 - 1} = 8 \text{ m/s}^2$$

(d) We use the values $t = 1$ and 5 in the acceleration expression:

$$a(1) = 6 \times 1 - 10 = -4 \text{ m/s}^2$$

$$a(5) = 6 \times 5 - 10 = 20 \text{ m/s}^2$$

(e) We calculate the time when the acceleration expression is zero:

$$0 = 6t - 10 \rightarrow t = 1.7 \text{ s}.$$

2.2 MOTION WITH CONSTANT ACCELERATION

There is accelerated motion whenever the velocity of an object traveling on a straight line changes. Examining the most general accelerated motion is difficult and requires advanced mathematics. However, if the rate of change of velocity is constant, in other words, if the change in velocity is always the same at equal time intervals, then it is a **motion with constant acceleration**. This is the simplest accelerated motion.

Let us remember the formulas for average acceleration and average velocity:

$$a_{\text{av}} = \frac{v_2 - v_1}{t_2 - t_1}, \quad v_{\text{av}} = \frac{x_2 - x_1}{t_2 - t_1}$$

Let us simplify the notation here: Let the object start moving at $t_1 = 0$ from an initial position x_0 with an initial velocity of v_0 and let its final velocity be v at the final time $t_2 = t$ at the final position $x_2 = x$. Since the average of a constant quantity is itself, making these changes in the formulas and rearranging, we get

$$a_{\text{av}} = a = \frac{v - v_0}{t - 0}, \quad v_{\text{av}} = \frac{x - x_0}{t - 0}$$

$$v = v_0 + at, \quad x = x_0 + v_{\text{av}} t$$

We can eliminate the average velocity v_{av} in the last expression as follows: The average of the velocity that changes uniformly from the value v_0 to the value v will be,

$$v_{\text{av}} = \frac{v + v_0}{2}$$

Also, after using this expression, the position formula can finally be written:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

Notice that the position x changes with the square of time (t^2) in motion with constant acceleration.

If the time t in the velocity expression is extracted as $t=(v-v_0)/a$ and used in the position expression, we find a very useful **velocity formula without time**:

$$v^2 - v_0^2 = 2a(x - x_0)$$

These formulas allow us to calculate the position and velocity at time t of the an object with given initial conditions, in other words, initial position x_0 and initial velocity v_0 , moving with acceleration a .

Let us summarize all the formulas for motion with constant acceleration:

$v = v_0 + at$ $x = x_0 + v_0 t + \frac{1}{2} a t^2$ $v^2 - v_0^2 = 2a(x - x_0)$	(motion with constant acceleration) (2.7)
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Motion with Constant Velocity

A special case of accelerated motion is the motion at constant velocity along a line. There is no need to derive separate formulas for this. Acceleration is zero if the velocity is constant. We find the formulas for uniform linear motion if we take $a = 0$ in the aforementioned formulas:

$$x = x_0 + v_0 t \quad (\text{uniform linear motion}) \quad (2.8)$$

If we wish to know the distance traveled by the object rather than its final position, we can write this formula in terms of the distance traveled s using $s = x - x_0$, as follows:

$$s = x - x_0 = v t \quad (\text{distance traveled}) \quad (2.9)$$

Example 2.5

A car starts from rest and accelerates at a constant rate of 2 m/s^2 .

- What will its velocity and distance traveled be after 3 s?
- In how many seconds will it reach the velocity 10 m/s ?
- In how many seconds will it cover a distance of 64 m ?

Answer

If we choose the origin at the initial position of the object, then $x_0 = 0$. Since it starts from rest, $v_0 = 0$. As the acceleration is given as $a = 2 \text{ m/s}^2$, using Eqs. (2.7),

$$v = v_0 + at = 0 + 2t = 2t$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 = 0 + 0 + \frac{1}{2} 2 t^2 = t^2$$

We can solve each item using these equations:

- (a) The velocity and position at time $t=3 \text{ s}$ are found by plugging $t = 3$ in the expressions that we found for v and x :

$$v = 2 \times 3 = 6 \text{ m/s} \quad \text{and} \quad x = (3)^2 = 9 \text{ m.}$$

- (b) t is found by taking $v = 10 \text{ m/s}$ in our velocity expression:

$$10 = 2t \quad \rightarrow \quad t = 5 \text{ s.}$$

- (c) The t value giving $x = 64 \text{ m}$ in the position expression is calculated:

$$64 = t^2 \quad \rightarrow \quad t = 8 \text{ s.}$$

Example 2.6

A car traveling at 72 km/h is slowed down through the pressing of its brakes and its velocity is reduced to 36 km/h in 5 s .

- What is the constant acceleration of the car?
- How much distance does it travel during that time?
- How much time passes from the start until the car fully stops?

Answer First, it is necessary to convert the initial and final

velocities given in terms of km/h into m/s units:

$$v_0 = 72 \text{ km/h} = 72000/3600 \text{ m/s} = 20 \text{ m/s}$$

$$v = 36 \text{ km/h} = 10 \text{ m/s}$$

- (a) We use the time $t = 5 \text{ s}$ that passes between these two velocities to find the acceleration:

$$v = v_0 + at \quad \rightarrow \quad 10 = 20 + a \times 5 \quad \rightarrow \quad a = -2 \text{ m/s}^2$$

As the car traveling in positive direction slows down, its acceleration is negative.

(b) If we take the initial position of the car as the origin, then $x_0 = 0$. Accordingly,

$$x = x_0 + v_0t + \frac{1}{2}at^2 = 0 + 20 \times 5 + \frac{1}{2}(-2) \times 5^2 = 75 \text{ m.}$$

(c) The car coming to a full stop means that its final velocity is $v = 0$. Using this value,

$$v = v_0 + at \rightarrow 0 = 20 - 2t \rightarrow t = 10 \text{ s}$$

Example 2.7

A runner cruising at a velocity of 4 m/s suddenly starts to accelerate and reaches a velocity of 7 m/s over a distance of 11 m.

- (a) What is the constant acceleration of the runner?
 (b) How much time elapsed during this acceleration?

Answer

(a) If we choose the origin as the point at which the runner

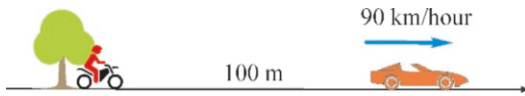
started to accelerate, then $x_0 = 0$ and $v_0 = 4 \text{ m/s}$. It will be suitable to use the velocity formula without time, as the first and final velocities are given. Substituting the position $x = 11 \text{ m}$ and the velocities, we get,

$$v^2 - v_0^2 = 2a(x - x_0) \rightarrow 7^2 - 4^2 = 2a \times 11 \rightarrow a = 1.5 \text{ m/s}^2$$

(b) Both the v and the x formula can be used to find the time t . The velocity formula gives a quicker result:

$$v = v_0 + at \rightarrow 7 = 4 + 1.5t \rightarrow t = 2 \text{ s}$$

Example 2.8



An automobile is traveling at a constant speed of 90 km/h when it passes a police motorcycle parked under a tree. Spotting a traffic violation, the policeman starts the chase on his motorcycle when the automobile is 100 m away and accelerates at a rate of 3 m/s^2 .

- (a) Choose a coordinate system and starting time to write the equations giving the positions and velocities of the automobile and motorcycle.
 (b) How long will it take the policeman to catch the automobile?
 (c) How far will the motorcycle be from its original position?
 (d) What is the velocity of the motorcycle when it catches up to the automobile?

Answer

In problems with two moving objects, a good approach is to choose a single coordinate system and write both of their equations in that same system. This systematic approach will be much more efficient in the long run.

First, let us convert the velocity of the automobile into m/s:

$$90 \text{ km/h} = 90 \, 000 / 3 \, 600 = 25 \text{ m/s.}$$

(a) Let us take the position where the motorcycle sets off as the origin and also start the clock when the motorcycle begins moving. Accordingly, the initial velocity and position of the motorcycle are zero: $x_{M0} = v_{M0} = 0$. If we indicate its

final velocity with v_M and position with x_M , the accelerated motion formulas are as follows:

$$v_M = v_{M0} + at = 0 + 3t \rightarrow v_M = 3t$$

$$x_M = x_{M0} + v_{M0}t + \frac{1}{2}at^2 = 0 + 0 + 1.5t^2 \rightarrow x_M = 1.5t^2$$

The automobile performs uniform linear motion at a constant velocity of 25 m/s, in other words, its acceleration is zero. The initial position of the automobile is 100 m at $t = 0$. If we indicate its final position with x_A ,

$$v_A = v_{A0} = 25 \text{ m/s} = \text{constant}$$

$$x_A = x_{A0} + v_{A0}t \rightarrow x_A = 100 + 25t$$

In this systematic approach, the solution follows easily once the equations of motion are written down:

(b) When the motorcycle catches the automobile, they will both have the same position. The mathematical expression of this is as follows:

$$x_M = x_A$$

Let us substitute the expressions we found for x_M and x_A :

$$1.5t^2 = 100 + 25t$$

From this, we find a quadratic equation:

$$3t^2 - 50t - 200 = 0$$

The roots of these equation are $t = -3.3$ and $t = 20$. As the negative root is not physically valid, the result is $t = 20 \text{ s}$.

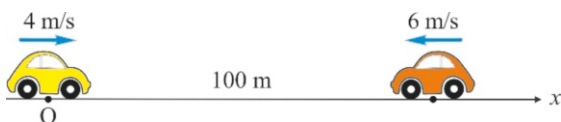
(c) Substituting $t = 20 \text{ s}$ in the expression for x_M , the distance traveled by the motorcycle is found as follows:

$$x_M = 1.5t^2 = 1.5 \times 20^2 = 600 \text{ m}$$

(d) The value $t = 20 \text{ s}$ is used in the expression $v_M = 3t$:

$$v_M = 3 \times 20 = 60 \text{ m/s}$$

Example 2.9



Two cars 100 m apart, start moving toward each other at the same instant. The first car starts with a speed of 4 m/s and increases its speed at a rate of 1 m/s^2 . The second car starts in the opposite direction with a speed of 6 m/s and increases its speed at a rate of 3 m/s^2 .

- (a) Choose a coordinate system for writing the equations of motion for each car.
 (b) When do the cars meet?
 (c) What is the distance of the meeting point to the origin?
 (d) What is the velocity of each car when they meet?

Answer

This problem could be solved by choosing a separate coordinate system for each moving object. In that case, two origins and two positive directions would be necessary. Yet, in the

method we used in the previous problem, it is more consistent to choose only a single coordinate system and write both motions in that same system.

(a) Let us choose the origin at the position where the first car sets off and choose the positive direction towards the other car. Accordingly, we can immediately write the equations of motion for the first car from the given data:

$$v_1 = v_{10} + a_1 t = 4 + 1 \times t$$

$$x_1 = x_{10} + v_{10} t + \frac{1}{2} a_1 t^2 = 0 + 4t + \frac{1}{2} t^2$$

The second car starts off at a distance of 100 m away and its velocity is in the negative direction. Its acceleration is also negative as it gets faster in the negative direction. Therefore, the equations of motion of the second car are as follows:

$$v_2 = v_{20} + a_2 t = -6 - 3t$$

$$x_2 = x_{20} + v_{20} t + \frac{1}{2} a_2 t^2 = 100 - 6t + \frac{1}{2} (-3)t^2$$

It is easier to solve the problem after writing these equations.

(b) The cars are at the same position when they meet:

$$x_1 = x_2$$

This equality does not mean that they traveled the same distance. It is the condition of both of them being at the same coordinate. Therefore, if we substitute the expressions we found for x_1 and x_2 ,

$$4t + \frac{1}{2} t^2 = 100 - 6t - \frac{3}{2} t^2$$

From here, we find a quadratic equation for t :

$$t^2 + 5t - 50 = 0$$

The roots of this equation are -10 and 5 . As negative time is meaningless, the solution is $t = 5$ s.

(c) The position of the meeting place is found by taking $t = 5$ in either of the expressions for x_1 or x_2 . Using x_1 ,

$$x_1 = 4 \times 5 + \frac{1}{2} (5)^2 = 32.5 \text{ m} = x_2.$$

(d) We substitute the value $t = 5$ in the expressions for v_1 and v_2 that we found above:

$$v_1 = 4 + 5 = 9 \text{ m/s}$$

$$v_2 = -6 - 3 \times 5 = -21 \text{ m/s}$$

2.3 FREE FALL

A most familiar example of motion with constant acceleration is falling objects. The fall of a stone dropped from a height, a ball thrown upwards decelerating and falling back down, etc. **Galileo** was the first person to discover that all of these motions occurred under the same constant **gravitational acceleration**. Until the 16th century, everyone believed that “heavier objects fall faster,” as stated in Aristotle’s book. Galileo showed that this was not true by dropping various objects from different heights and showing that all objects fall with the same constant acceleration if air friction is neglected.

Today, Galileo’s experiment can be easily repeated in a vacuum tube in a laboratory (Figure 2.4). A feather and a coin are observed to fall with the same acceleration in a vacuum tube. The motion that occurs under the effect of only gravity is called **free fall**.

The Earth has a constant acceleration near its surface that accelerates all objects towards its center. This is called **gravitational acceleration**, and its absolute value is indicated by g . Although it varies slightly according to geographical location, it is approximately

$$g = 9.81 \text{ m/s}^2.$$

The approximate value of $g \approx 10 \text{ m/s}^2$ shall be used in solving problems in this book. The 2% error arising from this approximation is negligible.

When using this gravitational acceleration g in the formulas of motion with constant acceleration (2.7), what should be used for a ? $a = +g$ or $a = -g$? The answer to this question is: *It depends on how the y -axis is chosen.* If the y -axis is chosen upwards, $a = -g$ regardless of which direction the object is thrown. This is because the gravitational acceleration is towards the center of the earth, in other words, in opposite direction to our y -axis. But, if the y axis is chosen downwards, $a = +g$ regardless of the direction of throwing, because the y -axis is in the same direction as the acceleration.

This may not be important in simple problems. However, if the object moves in both directions or if the motion of two separate objects is examined simultane-

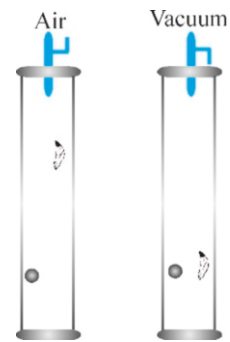


Figure 2.4: All objects fall with the same acceleration in vacuum.

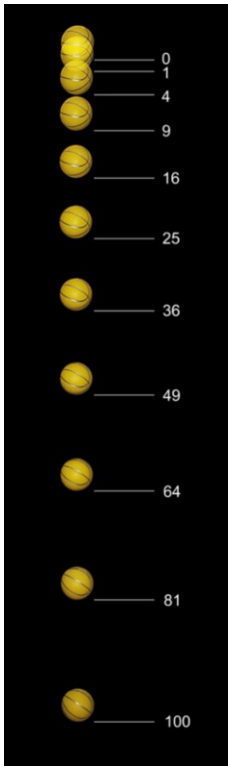


Figure 2.5: The distances traveled by a falling object in equal time intervals.

ously, then working with a single y axis and not changing it provides the correct solutions.

Therefore, the choice of the y -axis must also be specified when applying the constant acceleration formulas (2.7). We summarize them for free fall motion:

<u>y-axis is upwards</u>	<u>y-axis is downwards</u>	
$a = -g$	$a = +g$	(free fall) (2.10)
$v = v_0 - g t$	$v = v_0 + g t$	
$y = y_0 + v_0 t - \frac{1}{2} g t^2$	$y = y_0 + v_0 t - \frac{1}{2} g t^2$	
$v^2 - v_0^2 = -2g(y - y_0)$	$v^2 - v_0^2 = 2g(y - y_0)$	

These formulas apply both when the object is moving upwards or downwards, as long as we do not change the coordinate system. For example, we can use the same equation to examine the motion of a stone thrown upwards until it falls to the ground. We will discuss this in worked examples below.

Maximum height

Equations (2.10) are sufficient to solve all free fall problems; it is not necessary to memorize formulas for special cases. As an example, let us calculate the maximum height reached by an object thrown vertically upwards with an initial speed v_0 . When the object reaches the peak point, it stops for an instant and then starts moving downwards. In other words, the characteristic of the maximum height is that the velocity there is zero:

$$v = 0 \quad (\text{at maximum height})$$

In the last one of Eqs. (2.10, upwards) we take $v = 0$ and then solve for $y - y_0 = h_{\max}$:

$$h_{\max} = \frac{v_0^2}{2g}$$

Example 2.10

A ball is thrown from the ground upwards with a speed of 15 m/s.

- (a) Choose a coordinate system to write the equations giving the velocity and position of the ball.
- (b) What will be its position and velocity 1 s after being thrown?
- (c) In how many seconds does it reach maximum height?
- (d) What is the maximum height?
- (e) At what times does it pass through 10 m above the ground?
- (f) After how many seconds from being thrown will it fall back to the ground?

Answer (a) If we choose the $+y$ direction upwards, the acceleration will be $-g$. Choosing the place where the ball is thrown as the origin ($y_0 = 0$), the equations of motion for a ball thrown in the positive direction with a velocity of $v_0 = 15$ m/s are as follows:

$$v = v_0 - g t = 15 - 10 t$$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 = 15 t - 5 t^2$$

(b) The velocity and position at the value $t = 1$ s can be directly calculated:

$$v = 15 - 10 \times 1 = 5 \text{ m/s}$$

$$y = 15 \times 1 - 5 \times 1^2 = 10 \text{ m}$$

(c) The characteristic of maximum height is that the velocity is zero there: $v = 0$. We use this condition to find the time:

$$0 = 15 - 10 t \rightarrow t = 1.5 \text{ s}$$

(d) We find the maximum height by using the time $t = 1.5$ s in the y equations:

$$y = 15 \times 1.5 - 5 \times 1.5^2 = 11.3 \text{ m}$$

(e) The y equation should satisfy the value $y = 10$ m:

$$10 = 15 t - 5 t^2 \rightarrow t^2 - 3 t + 2 = 0$$

The two solutions to this equation are $t = 1$ s (going upward) and $t = 2$ s (coming downward).

(f) As the time taken to reach the maximum height will be equal to the time it takes to fall back to the ground, we take twice the time found in (c). However, if we had not known this property, we could have directly found the answer by using the y equation above. The equation $y = 15 t - 5 t^2$ is

valid regardless of whether the ball is moving upwards or downwards. In particular, $y = 0$ should be true when the ball is at the ground level. Using this condition,

$$0 = 15t - 5t^2 = t(15 - 5t) \rightarrow t = 0 \text{ and } t = 3 \text{ s.}$$

Example 2.11

A stone is thrown upwards with a velocity of 30 m/s from the roof of a building at a height of 35 m from the ground.

- Choose a coordinate system to write the velocity and position equations of the stone.
- Calculate the maximum height and the time to reach it.
- How many seconds will it take to fall to the ground?
- With what velocity will it hit the ground?

Answer (a) Taking the ground level as the origin and upwards as the positive direction, we get $y_0 = 35$ m and $a = -g$. Accordingly, the equations of motion are as follows:

$$v = v_0 + at = 30 - 10t$$

$$y = y_0 + v_0t + \frac{1}{2}at^2 = 35 + 30t - 5t^2$$

We can answer all of the questions using these two equations.

(b) At maximum height $v = 0$:

$$0 = 30 - 10t \rightarrow t = 3 \text{ s}$$

Plugging this value of t into the y equation,

$$y = 35 + 30 \times 3 - 5 \times 3^2 = 80 \text{ m}$$

Both solutions are valid. The $t = 0$ solution is the starting time and the $t = 3$ s solution is the falling time. In other words, the ball is at $y = 0$ height at both times.

This is the maximum height from the origin, in other words, from the ground. If we wish to find the height from where it was thrown, it is $y - y_0 = 80 - 35 = 45$ m.

(c) The stone falling to the ground means it reaches the position $y = 0$:

$$0 = 35 + 30t - 5t^2 = 0 \rightarrow t = -1 \text{ and } 7 \text{ s}$$

The negative solution is not considered and the answer is $t = 7$ s. Notice here that we used the same equation throughout. Since we remain in the same coordinate system, we are able to solve all of the problems with a single equation without separating the motion into two parts of upwards and downwards.

(d) The velocity of the stone when it falls to the ground is found by taking $t = 7$ in the v equation:

$$v = 30 - 10t = 30 - 10 \times 7 = -40 \text{ m/s.}$$

The negative sign here indicates that the velocity is in the opposite direction to the direction that we chose as positive, in other words, it is downwards.

Example 2.12

The different times technique. Water is dripping in 1 second intervals from the roof of a building at a height of 45 m from the ground. What is the height of the second drop when the first drop hits the ground?

Answer It would be too complicated to try to solve this problem using our familiar methods. However, the solution will be simpler if we write each drop in terms of a separate time t with its own chronometer.

Let us take the origin as being at the roof and the y direction downwards. Let us start the chronometer when the first drop sets off and show the measured time with t_1 . Accordingly, the equation of motion will be:

$$y_1 = \frac{1}{2}gt_1^2$$

When the second drop sets off 1 second later, we start time t_2 in a separate chronometer. Its motion will be $y_2 = \frac{1}{2}gt_2^2$.

As the drops set off in one-second intervals, the second

chronometer will show 1 second less than the first one during each stage. In other words, the relation between the two times will be as follows:

$$t_2 = t_1 - 1$$

Therefore, the expression of the y_2 coordinate will be

$$y_2 = \frac{1}{2}g(t_1 - 1)^2.$$

Now, when the first drop reaches the ground, it will have traveled a distance of $y_1 = 45$ m. We use this value to find the time t_1 :

$$45 = 5t_1^2 \rightarrow t_1 = 3 \text{ s}$$

The distance traveled by the second drop at the end of this time t_1 :

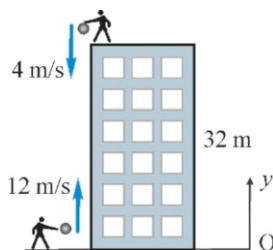
$$y_2 = \frac{1}{2}g(t_1 - 1)^2 = 5 \times 2^2 = 20 \text{ m}$$

The difference between the two distances shows how far behind the second drop is:

$$y_1 - y_2 = 45 - 20 = 25 \text{ m.}$$

The second drop is 25 m above the ground.

Example 2.13



When a ball is thrown upwards with a velocity of 12 m/s from

the ground, a second ball is thrown downwards with a velocity of 4 m/s from the roof of a building at a height of 32 m.

- Using the same coordinate system, write the equations of motion for both balls.
- When will the two balls meet?
- At what height will they meet?
- What are their velocities when they meet?

Answer

In this problem, we will again examine the motion of two

objects in the same coordinate system.

(a) If we choose the origin to be at the ground and the positive direction upwards, the initial position of the first ball thrown from the ground is $y_{10} = 0$ and its initial velocity is $v_{10} = 12 \text{ m/s}$. For the second ball thrown downwards from the top of the building, these values are $y_{20} = 32 \text{ m}$ and $v_{20} = -4 \text{ m/s}$. Accordingly, let us write the equations of motion for both balls side by side:

1st ball	2nd ball
$v_1 = 12 - 10t$	$v_2 = -4 - 10t$
$y_1 = 0 + 12t - 5t^2$	$y_2 = 32 - 4t - 5t^2$

(b) Their y positions will be the same when the balls meet:

$$y_1 = y_2$$

$$12t - 5t^2 = 32 - 4t - 5t^2 \rightarrow 16t = 32$$

The solution to this equation tells us that the balls meet at time $t = 2 \text{ s}$.

(c) Using this value of t in either y_1 or y_2 , we find the height:

$$y_1 = 12 \times 2 - 5 \times 2^2 = 4 \text{ m} = y_2$$

(d) We find the velocities by again using this time t in the velocity equations:

$$v_1 = 12 - 10 \times 2 = -8 \text{ m/s}$$

$$v_2 = -4 - 10 \times 2 = -24 \text{ m/s}$$

Notice that the velocity v_1 is negative at the time of the collision. In other words, the 1st ball went up to its maximum height and returned, and the meeting took place on its way down.

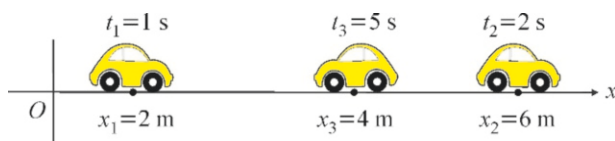
Multiple-choice Questions

1. In a coordinate system, which of the following is incorrect for an object with an acceleration of $a = +2 \text{ m/s}^2$:
 - (a) It is getting faster if it is traveling in the positive direction.
 - (b) It is getting slower if it is traveling in the negative direction.
 - (c) Its velocity is always different from zero.
 - (d) Its velocity changes by equal amounts in equal time intervals.
2. Which of the following is correct if an object traveling in the positive direction has negative acceleration?
 - (a) The velocity of the object increases.
 - (b) The object stops at some point and turns back.
 - (c) The velocity of the object remains constant.
 - (d) None of the above.
3. Which is true for accelerated motion?
 - (a) Acceleration is zero if the velocity is zero.
 - (b) Acceleration is always positive if the velocity is always positive.
 - (c) An object with constant acceleration may stop.
 - (d) An object with constant acceleration cannot stop.
4. How many seconds does it take for an object dropped from a height of 20 m to fall to the ground?
 - (a) 1
 - (b) 2
 - (c) 3
 - (d) 4
5. What distance does an object dropped from a height travels in 5 s?
 - (a) 50
 - (b) 75
 - (c) 100
 - (d) 125
6. The velocity of an object dropped from a height is 40 m/s after 4 s. What is its average velocity in this time interval?
 - (a) 0
 - (b) 10 m/s
 - (c) 20 m/s
 - (d) 40 m/s
7. A stone is thrown upwards from the ground. Which of the following is correct if the $+y$ axis is chosen upwards?
 - (a) Its acceleration is $+g$ when traveling upwards.
 - (b) Its acceleration is zero at maximum height.
 - (c) Its acceleration is $+g$ when moving downwards.
 - (d) Its acceleration is $-g$ throughout the motion.
8. A stone dropped from a height falls to the ground in 5 seconds. What is its average acceleration in m/s^2 ?
 - (a) 0
 - (b) 2
 - (c) 5
 - (d) 10
9. A stone is thrown upwards. Which of the following is correct at maximum height?
 - (a) Velocity and acceleration are zero.
 - (b) Velocity is zero, acceleration is maximum.
 - (c) Velocity is maximum, acceleration is zero.
 - (d) Velocity is zero, acceleration is the gravitational acceleration.
10. A car traveling at speed v can stop at a distance d when the brakes are applied. At what distance will it stop when it travels at a velocity of $2v$?
 - (a) $d/2$
 - (b) d
 - (c) $2d$
 - (d) $4d$
11. The velocity of an object traveling in the positive direction increases by 8 m/s in 5 s. What is its acceleration in m/s^2 ?
 - (a) 1.0
 - (b) 1.2
 - (c) 1.4
 - (d) 1.6
12. The position of an object varies as $x = t^3$ (meters). What is its acceleration at time $t = 1 \text{ s}$ in m/s^2 ?
 - (a) 2
 - (b) 4
 - (c) 6
 - (d) 8

13. The gravitational acceleration of the Moon is less than that of the Earth. Which of the following is true for a stone dropped from a height on the Moon?
 - (a) It falls to the ground in less time than on the Earth.
 - (b) It falls to the ground in more time than on the Earth.
 - (c) It falls to the ground in the same time.
 - (d) None of the above.
14. A bicycle rider traveling at 10 m/s presses the brakes and stops in 5 s. How many meters does the bike travel before stopping?
 - (a) 10
 - (b) 15
 - (c) 20
 - (d) 25
15. The positions of an object at times $t = 1, 2, 3, 4$ s are $x = 2, 7, 9, 15$ m, respectively. What is the average velocity of this object in the $[2, 4]$ s interval in m/s?
 - (a) 1
 - (b) 2
 - (c) 3
 - (d) 4
16. Which of the following propositions is true?
 - (a) The acceleration of an object traveling at constant velocity is zero.
 - (b) The acceleration of an object slowing in the negative direction is positive.
 - (c) The acceleration of an object getting faster in the negative direction is negative.
 - (d) All of the above.
17. What are the ratios of the distances traveled by an object getting faster with constant acceleration in 1, 2, 3 seconds?
 - (a) 1:2:3
 - (b) 1:3:5
 - (c) 1:4:8
 - (d) 1:4:9
18. A stationary object speeds up with constant acceleration. What are the ratios of the distances it travels at the 3rd and 4th seconds?
 - (a) 3/4
 - (b) 5/7
 - (c) 7/9
 - (d) 9/16
19. Two stones are thrown upwards from the ground with velocities v_0 and $2v_0$. What is the ratio of the maximum height of the faster one with respect to the other?
 - (a) 2/1
 - (b) 4/1
 - (c) $\sqrt{2}/1$
 - (d) $1/\sqrt{3}$
20. Two objects start to move from the same position at the same time and in the same direction. The object A travels at constant velocity of 10 m/s. The object B accelerates at a rate of 4 m/s^2 . After how many seconds will the two objects be in the same position again?
 - (a) 2
 - (b) 3
 - (c) 4
 - (d) 5

Problems

2.1 Position, Displacement, Velocity and Acceleration



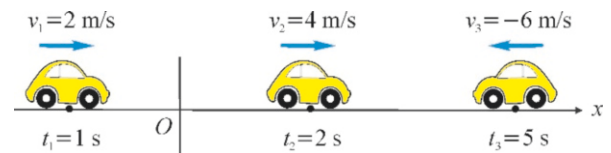
Problem 2.1

2.1 The positions of an object at various times are shown in the figure above. (a) Find the displacements in the $[t_1, t_2]$ and $[t_2, t_3]$ time intervals. (b) Calculate the average velocities in the same intervals.

[A: (a) $\Delta x = 4 \text{ m}$ ve $\Delta x = -2 \text{ m}$, (b) $v_{av} = 4 \text{ m/s}$ and $v_{av} = -0.67 \text{ m/s}$]

2.2 The position of a moving object varies in time as $x = 5t^2$ (meters). (a) What are its positions at times $t = 1$ s and $t = 3$ s? (b) What is its average velocity in the $t : [1, 3]$ time interval? (c) What are its instantaneous velocities at times $t = 1$ s and $t = 3$ s?

[A: (a) 5 and 45m, (b) 20 m/s, (c) 10, 30 m/s.]



Problem 2.3

2.3 The velocities of an object at various times are shown in the figure above. (a) Find the average accelerations in the $[t_1, t_2]$ and $[t_2, t_3]$ time intervals.

[A: $a_{av} = 2 \text{ m/s}^2$ and $a_{av} = -3.3 \text{ m/s}^2$.]

2.4 The velocity of an object varies as $v(t) = 20 - 3t^2$. (a) Calculate its average acceleration in the $t = [1, 2]$ s time interval. (b) Find its instantaneous accelerations at times $t = 1$ and $t = 2$ s.

[A: (a) $a_{av} = -9 \text{ m/s}^2$, (b) $a = -6, -12 \text{ m/s}^2$.]

2.5 A ball hits a wall perpendicularly with a speed of 24 m/s and is rebounded back at the same speed. The ball has contacted the wall for 0.03 s. (a) What is the change in velocity? (b) What is the average acceleration during the collision?

[A: (a) 48 m/s, (b) 1600 m/s^2 .]

2.2 Motion With Constant Acceleration

2.6 In order to make a safe landing, an airplane has to touch the ground at a maximum speed of 120 m/s and slow down with a maximum acceleration of 5 m/s^2 . (a) In how many seconds can the plane stop? (b) What should the minimum length of the runway be? [A: (a) 24 s , (b) 1440 m .]

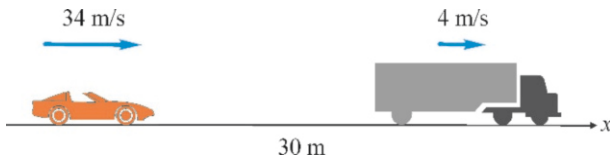
2.7 An object moving with constant acceleration passes through two points 25 m apart at a velocity of 10 m/s from the first and 15 m/s from the second. (a) What is the acceleration of the object? (b) In how many seconds will it travel between the two points? [A: (a) $a = 2.5 \text{ m/s}^2$, (b) $t = 2 \text{ s}$.]

2.8 An object moving with a constant acceleration passes between two points 10 m apart in 2 s . It has a velocity of 8 m/s when passing through the second point. Find the acceleration of the objects and its velocity at the first point. [A: $a = 3 \text{ m/s}^2$, $v_0 = 2 \text{ m/s}$.]

2.9 A train departing from a station accelerates at a rate of 2 m/s^2 for 12 s . It then travels for 5 s at constant velocity. Finally, it decelerates at 4 m/s^2 and stops at the second station. What is the distance between the two station in meters? [A: 336 m .]

2.10 A jet airplane can accelerate at a maximum rate of 4 m/s^2 on the runway and can takeoff once it reaches a velocity of 80 m/s . However, if the pilot changes his/her mind about the takeoff, the plane can be slowed down at a maximum rate of 5 m/s^2 . In an emergency, the pilot changes his/her mind about takeoff when the plane is at the takeoff velocity. What should the minimum total length of the runway be so that the airplane can stop safely? [A: 1440 m .]

2.11 The maximum acceleration of a train is 8 m/s^2 . This train should travel between two stations 1800 m apart in the shortest time possible. For this purpose, it accelerates in the first half of the path and decelerates in the second half, stopping at the station. Calculate the maximum velocity that the train can reach and how many seconds it takes to travel the path. [A: $v = 120 \text{ m/s}$ and $t = 30 \text{ s}$.]



Problem 2.12

2.12 The driver of a sports car traveling at velocity 34 m/s suddenly sees a truck 30 m ahead, traveling in the same direction with a constant velocity of 4 m/s , and applies the brakes. If the car can decelerate at a maximum rate of 6 m/s^2 , will it crash into the truck from behind? (*Hint*: Solve the equivalent problem of a stationary truck and a car with different velocity.)

[A: It crashes. The car needs at least 95 m to slow down before crashing.]

2.13 An automobile is trying to pass a trailer truck that has a length of 20 m . At the start, the automobile and the truck are in the same lane and traveling at the same constant velocity of 15 m/s . Once the opposite lane is empty, the automobile passes to the left lane and starts to accelerate at 5 m/s^2 , managing to pass the truck once it reaches its front end. (a) How long does it take for the automobile to pass the truck? (b) What length would the empty lane need to be for this passing? (*Note*: The length of the automobile is added to that of the truck and can be neglected.) [A: (a) 2.8 s , (b) 62 m .]



Problem 2.14

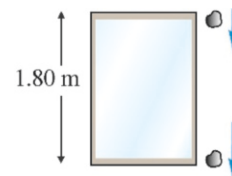
2.14 A motorcycle waiting at a red lights starts to accelerate at 5 m/s^2 once the light changes to green. Just at that moment, a truck traveling at a constant velocity of 8 m/s continues on its way through the green light without changing its velocity. (a) When will the motorcycle catch up with the truck? (b) How much distance has the motorcycle traveled once it catches up? [A: (a) $t = 3.2 \text{ s}$, (b) $x = 25.6 \text{ m}$.]

2.3 Free Fall

2.15 A stone is thrown downwards with a velocity of 20 m/s from the roof of a building at a height of 105 m . (a) How long will it take to reach the ground? (b) With what velocity will it hit the ground? [A: (a) 3 s , (b) 50 m/s .]

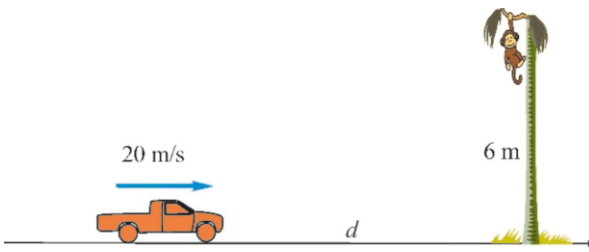
2.16 A stone is dropped into a water well and the sound of the stone hitting the water's surface is heard 4.5 s after it was dropped. Since the propagation speed of sound in air is 340 m/s , how deep down is the surface of the water at the bottom of the well? [A: 90 m .]

2.17 After throwing a ball upwards, a juggler runs to and then back from a door at a distance of 9 m with a velocity of 5 m/s , catching the ball before it falls to the ground. At what minimum velocity should he/she throw the ball upwards to be able to do this? [A: 18 m/s .]



Problem 2.18

2.18 A ball is dropped from the roof of a building. A man standing behind a window on one of the lower floors sees the ball pass his window in 0.2 seconds . The height of the window is 1.80 m . How high is the roof from the upper edge of the window? (*Hint*: The given information is sufficient to find the velocity of the stone at the upper edge of the window.) [A: 3.2 m .]

**Problem 2.19**

2.19 A monkey sitting in a tree by the road on a branch 6 m above the ground wants to drop on a truck approaching with

a velocity of 20 m/s. At what distance d between the truck and the tree should the monkey start to drop in order to land on the truck?
[A: 22 m.]

2.20 A juggler throws a ball such that it barely touches the ceiling of the hall, which is 5 m high. (a) What is the initial velocity of the ball? (b) If he throws a second ball upwards with the same velocity when the first ball is at the level of the ceiling, when and at what height will the two balls meet?

[A: (a) 10 m/s, (b) 0.5 s and 3.75 m.]

3

TWO-DIMENSIONAL MOTION



An acrobatic stunt at a motocross race.

The racer is moving in both the horizontal and vertical directions. Can we examine both of these movements within the same framework?

A bird flying through the air, a fish swimming in water, a free throw in basketball, a spacecraft flying past a planet, etc. How can the motion of all these objects be examined? We cannot apply the method that we discussed in the previous chapter directly because these objects do not move in a straight line. In a three-dimensional Cartesian coordinate system, the coordinates (x, y, z) of these objects simultaneously vary depending on time t :

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

Notice that, when an object is moving in space, each of the coordinates travels along its own axis, in other words, it performs linear motion! Therefore, we can apply the concepts that we developed for linear motion to each of the (x, y, z) coordinates. Then, we can unify them using vectors.

In this course we will only examine two-dimensional motion. The methods that we will develop can easily be extended to three-dimensional motion.

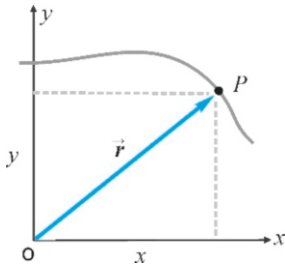


Figure 3.1: Position vector.

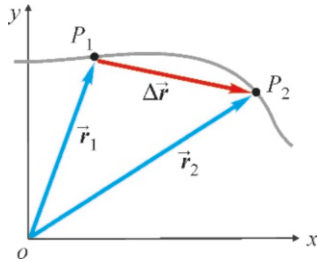


Figure 3.2: Displacement vector.

3.1 POSITION AND DISPLACEMENT VECTORS

Let (x, y) be the coordinates of a point P in the xy coordinate system. The vector drawn from the origin to this point is called the **position vector** and is indicated by \vec{r} . As seen in the figure, the components of this vector \vec{r} are the coordinates of the point P :

$$\vec{r} = x\hat{i} + y\hat{j} \quad (3.1)$$

Displacement Vector

If an object located at point P_1 with position \vec{r}_1 at time t_1 , is located at point P_2 with position \vec{r}_2 at a later time t_2 (Figure 3.2), the vector

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} \quad (3.2)$$

is called the **displacement vector**. As seen in the figure, the vector $\Delta\vec{r} = \overrightarrow{P_1P_2}$ is the secant line that intersects the trajectory at P_1 and P_2 .

The **average velocity vector** \vec{v}_{av} of the object during this time interval Δt is the ratio of the displacement vector to the time difference:

$$\vec{v}_{\text{av}} = \frac{\Delta\vec{r}}{\Delta t} \quad (3.3)$$

This is the application of the definition in one-dimensional motion to both components. If we write more explicitly,

$$\begin{aligned} \vec{v}_{\text{av}} &= \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \left(\frac{x_2 - x_1}{t_2 - t_1} \right)\hat{i} + \left(\frac{y_2 - y_1}{t_2 - t_1} \right)\hat{j} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} \\ v_{\text{av},x} &= \frac{\Delta x}{\Delta t}, \quad v_{\text{av},y} = \frac{\Delta y}{\Delta t} \end{aligned}$$

Likewise, the **instantaneous velocity vector** (or, simply, the velocity vector), is obtained by taking the derivative as in one dimension:

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \quad (3.4)$$

We understand the following from the derivative of a vector: The derivatives of the x and y components of that vector are taken separately, and these become the components of the derivative vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{dx}{dt}}_{v_x}\hat{i} + \underbrace{\frac{dy}{dt}}_{v_y}\hat{j} = v_x\hat{i} + v_y\hat{j}$$

The magnitude (absolute value, norm) of the velocity vector and its angle with the $+x$ axis are found using familiar formulas:

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}, \quad \tan \theta = \frac{v_y}{v_x} \quad (3.5)$$

What is the direction of the velocity vector? In order to determine this, let us consider the secant $\Delta\vec{r} = \overrightarrow{P_1P_2}$. The $\overrightarrow{P_1P_2}$ vector is in the direction of motion (Figure 3.3). Also, as $\Delta t \rightarrow 0$, P_2 will gradually approach the point P_1 , and in the end, the secant P_1P_2 will come to the tangent line. Therefore, in two-dimensional motion, *the velocity vector is always tangent to the trajectory*.

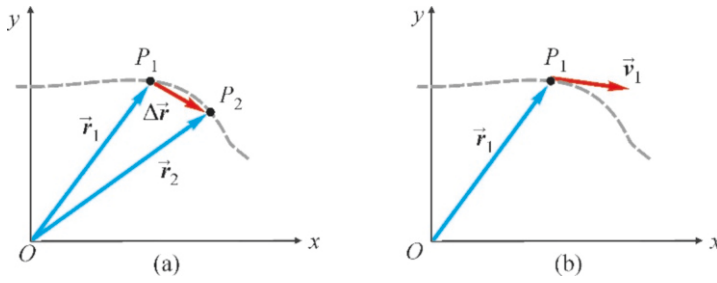


Figure 3.3: (a) Limit direction of the displacement vector, (b) direction of the \vec{v} velocity vector.

Acceleration Vector

Acceleration is the rate of change of velocity with respect to time. In order to extend this to two-dimensional motion, we use the same method for the components.

Definition: If the velocity vector of an object at time t_1 is \vec{v}_1 and its velocity vector at a later time t_2 is \vec{v}_2 (Figure 3.4), then the **average acceleration vector** is

$$\vec{a}_{\text{av}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} \quad (3.6)$$

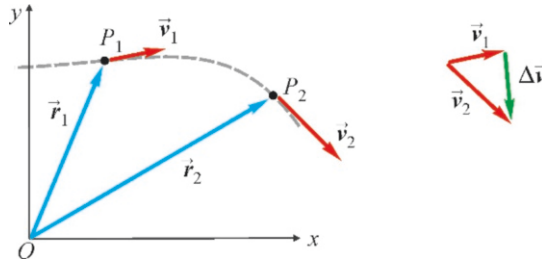


Figure 3.4: $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ vector.

or, in terms of the components,

$$a_{\text{av},x} = \frac{\Delta v_x}{\Delta t} = \frac{v_{2x} - v_{1x}}{t_2 - t_1}, \quad a_{\text{av},y} = \frac{\Delta v_y}{\Delta t} = \frac{v_{2y} - v_{1y}}{t_2 - t_1}$$

Accordingly, the **instantaneous acceleration vector** (or, simply, the acceleration vector) is found by taking the limit of average acceleration:

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \quad (3.7)$$

or, more explicitly in terms of the components,

$$\vec{a} = \frac{d\vec{v}}{dt} = \underbrace{\frac{dv_x}{dt}}_{a_x} \hat{i} + \underbrace{\frac{dv_y}{dt}}_{a_y} \hat{j} = a_x \hat{i} + a_y \hat{j}$$

As the velocity vector is the derivative of position, we can also write the acceleration as the second derivative of position:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \quad (3.8)$$

The magnitude and the angle of the acceleration vector are,

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2}, \quad \tan \theta = \frac{a_y}{a_x} \quad (3.9)$$

In two-dimensional motion, the direction of the acceleration vector does not necessarily have to be along the path and may be in any direction depending on the type of motion.

Example 3.1

The coordinates of an object moving on a plane are given as,

$$x = t^3 - 2t^2 + 5 \quad (\text{m})$$

$$y = 3t^2 - 4t + 4 \quad (\text{m})$$

- (a) Express velocity and acceleration as functions of time.
 (b) Calculate the components of position, velocity and acceleration at time $t = 1$ s.
 (c) Find the magnitudes of velocity and acceleration at time $t = 1$ s.
 (d) Find the angle between velocity and acceleration at time $t = 1$ s.

$$a_x = \frac{dv_x}{dt} = 6t - 4 \quad (\text{m/s}^2)$$

$$a_y = \frac{dv_y}{dt} = 6 \quad (\text{m/s}^2)$$

- (b) We find the components by taking $t = 1$ in these expressions for x, v, a :

$$x = 4 \quad y = 3$$

$$v_x = -1 \quad v_y = 2$$

$$a_x = 2 \quad a_y = 6$$

- (c) We calculate the magnitudes of the vectors with the given components:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{5} = 2.2 \text{ m/s}$$

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{2^2 + 6^2} = \sqrt{40} = 6.3 \text{ m/s}^2$$

- (d) The angle between the two vectors is found using the scalar product method (Eq. 1.10):

$$\cos \theta = \frac{\vec{v} \cdot \vec{a}}{va} = \frac{(-1) \times 2 + 2 \times 6}{\sqrt{5} \sqrt{40}} = \frac{\sqrt{2}}{2}$$

From here, the angle is found to be $\theta = 45^\circ$.

Answer

- (a) Derivatives of the position components with respect to time give the velocity components:

$$v_x = \frac{dx}{dt} = 3t^2 - 4t \quad (\text{m/s})$$

$$v_y = \frac{dy}{dt} = 6t - 4 \quad (\text{m/s})$$

Derivatives of the velocity components give the acceleration components:

3.2 PROJECTILE MOTION

The first example of two-dimensional motion that we examine will be the motion of objects thrown into the air near the Earth's surface under gravitational acceleration. *Neglecting air resistance and other small effects*, the acceleration is always downwards and towards the center of the Earth. Choosing the coordinate system shown in Figure 3.5, its components can be written as,

$$\left. \begin{array}{l} a_x = 0 \\ a_y = -g \end{array} \right\} \rightarrow \vec{a} = -g\hat{j} \quad (3.10)$$

The components of motion can easily be deduced from this: We observe uniform linear motion in the horizontal x direction and motion with constant acceleration in the vertical y direction. We examined both types of motions in Chapter 2. Now, we can write those formulas for each component.

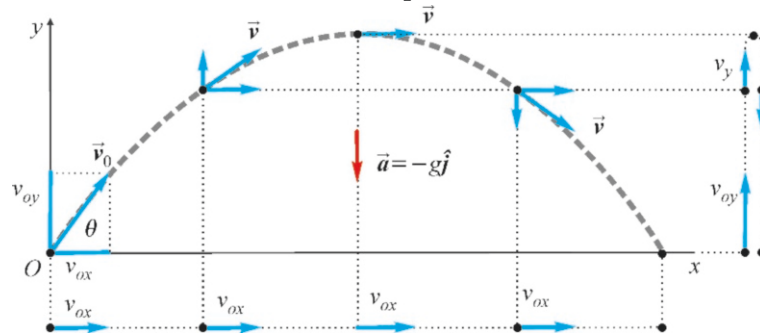


Figure 3.5: Coordinates in projectile motion. As the horizontal x component does uniform linear motion, the vertical y component does free fall motion.

Let us consider an object thrown from the surface of the earth with an initial velocity of v_0 with an angle of θ to the horizontal (Figure 3.5). The components of the initial velocity can be written as follows:

$$v_{0x} = v_0 \cos \theta \quad v_{0y} = v_0 \sin \theta \quad (3.11)$$

Choosing the place where the object was thrown as the origin and the y direction as upwards, we can write the projectile motion formulas:

$$\begin{array}{l} v_x = v_0 \cos \theta \quad x = (v_0 \cos \theta) t \\ v_y = v_0 \sin \theta - gt \quad y = (v_0 \sin \theta) t - \frac{1}{2} g t^2 \end{array} \quad \text{(projectile motion)} \quad (3.12)$$

Trajectory equation: If we wish to find the trajectory, or the curve drawn by the object in the air regardless of its position at any given time, we eliminate t from these equations. Taking $t = x/v_0 \cos \theta$ from the x position formula and substituting it into the y equation and simplifying, we get the trajectory equation:

$$y = (\tan \theta) x - \frac{g}{2v_0^2 \cos^2 \theta} x^2 \quad \text{(Trajectory equation)} \quad (3.13)$$

This is the equation of a parabola. (Figure 3.5).

If the object is thrown not from the origin, but from any initial position (x_0, y_0) , it is sufficient to add x_0 and y_0 to the equations above:

$$\begin{array}{l} x = x_0 + (v_0 \cos \theta) t \\ y = y_0 + (v_0 \sin \theta) t - \frac{1}{2} g t^2 \end{array} \quad (3.14)$$

These formulas are valid for all types of projectile motion in which the air resistance can be neglected. It is not necessary to develop separate formulas for the maximum height or the horizontal range. For example, since $\theta = 0$ for a horizontal projectile, it is sufficient to have $\sin \theta = 0$ and $\cos \theta = 1$ in the formulas:

$$\begin{array}{l} v_x = v_0 \quad v_y = -gt \\ x = x_0 + v_0 t \quad y = y_0 - \frac{1}{2} g t^2 \end{array} \quad \text{(horizontal projectile)} \quad (3.15)$$

Likewise, it is sufficient to know that the y component of velocity is zero ($v_y = 0$) at the maximum height.

Example 3.2

A ball kicked by a football player rises with an initial velocity of 30 m/s at an angle of 37° to the horizontal.

- Write the equations of motion of the ball.
- What are the position and velocity components of the ball at time $t = 1$ s?
- In how many seconds does it reach maximum height?
- What is the maximum height of the ball?
- How far away will the ball land on the ground?

Answer

(a) First, calculate the components of the initial velocity:

$$v_{0x} = v_0 \cos 37^\circ = 30 \times 0.8 = 24 \text{ m/s}$$

$$v_{0y} = v_0 \sin 37^\circ = 30 \times 0.6 = 18 \text{ m/s}$$

We choose the origin at the kicking point, and the equations

of motion (3.12) follow:

$$\begin{array}{l} v_x = v_0 \cos 37^\circ = 24 \quad v_y = v_0 \sin 37^\circ - gt = 18 - 10t \\ x = v_0 \cos 37^\circ t = 24t \quad y = v_0 \sin 37^\circ t - \frac{1}{2} g t^2 = 18t - 5t^2 \end{array}$$

- Plugging $t = 1$ into these expressions,

$$\begin{array}{l} v_x = 24 \text{ m/s} \quad v_y = 18 - 10 \times 1 = 8 \text{ m/s} \\ x = 24 \text{ m} \quad y = 18 - 5 = 13 \text{ m} \end{array}$$
- At maximum height the motion becomes horizontal, in other words, y component of velocity vanishes:

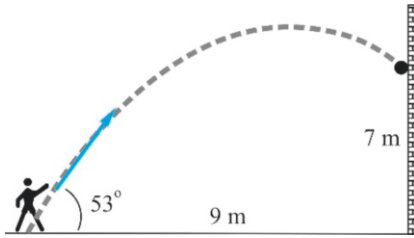
$$v_y = 18 - 10t = 0 \rightarrow t = 1.8 \text{ s}$$
- The y value at this $t = 1.8$ s gives the maximum height:

$$y_{\max} = 18 \times 1.8 - 5 \times 1.8^2 = 16.2 \text{ m}$$
- The range of the ball, in other words, the horizontal distance of the falling point is the value of the x coordinate at $t = 2 \times 1.8 = 3.6$ s. The range is shown as R :

$$R = x_{\max} = 24 \times 3.6 = 86 \text{ m}$$

Example 3.3

A ball is thrown at an angle of 53° towards a wall 9 m away. The ball collides with the wall at height of 7 m. What is the initial velocity of the ball?

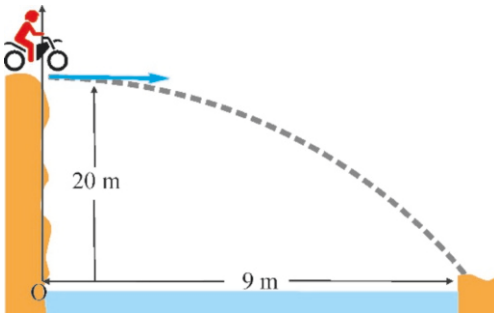
**Answer**

Since no time is involved, the trajectory equation will give the quickest solution. Using the values $x = 9$ m, $y = 7$ m, $\cos 53^\circ = 0.6$ and $\tan 53^\circ = 4/3$ in the trajectory formula (3.13), we get

$$y = (\tan \theta) x - \frac{g}{2v_0^2 \cos^2 \theta} x^2$$

$$7 = \frac{4}{3} \times 9 - \frac{10}{2v_0^2 \times 0.6^2} \times 9^2$$

$$v_0^2 = 225 \rightarrow v_0 = 15 \text{ m/s.}$$

Example 3.4

A motorcycle rider wants to jump off of a cliff at a height of 20 m with a horizontal velocity of v_0 and get to the other side of a river that is 9 m wide.

- What is the time of flight of the motorcycle?
- What should the minimum initial velocity of the motorcycle be in order to cross the river?
- With what velocity will the motorcycle touch the ground?

Answer

Let us choose the origin at the ground level directly below the

jumping point and the y axis upwards. Since the motorcycle jumps horizontally, we take $\theta = 0^\circ$ in the projectile motion formulas (3.12):

$$v_x = v_0 \quad x = v_0 t$$

$$v_y = -gt = -10t \quad y = y_0 - \frac{1}{2}gt^2 = 20 - 5t^2$$

(a) The y coordinate should be zero when the motorcycle falls to the ground:

$$0 = 20 - 5t^2 \rightarrow t = 2 \text{ s}$$

(b) In order to pass the river, its x coordinate should be greater than or equal to 9 m at the time $t = 2$ s when it touches the ground:

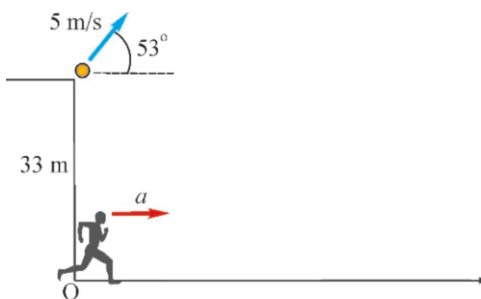
$$x = v_0 t \rightarrow 9 = v_0 \times 2 \rightarrow v_0 = 4.5 \text{ m/s}$$

(c) We calculate the velocity components at time $t = 2$ s:

$$v_x = v_0 = 4.5 \text{ m/s} \quad v_y = -10t = -20 \text{ m/s}$$

The negative sign of the v_y component shows that it is downwards. We calculate the magnitude of the velocity using these components:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(4.5)^2 + (-20)^2} = 21 \text{ m/s}$$

Example 3.5

A ball is thrown with at an angle of 53° and initial velocity 5 m/s from the roof of a building that is 33 m high. Waiting directly below on the ground, a kid starts to run with constant acceleration a at the same time and in the same direction as the ball.

- Choose a coordinate system and write the equations of motion for the ball and for the kid.

(b) What will the time of flight of the ball be until it falls to the ground?

(c) How far away from the building will the ball land?

(d) What should the acceleration of the kid be so that he/she can catch the ball?

Answer

We first calculate the initial velocity components of the ball:

$$v_{0x} = v_0 \cos 53^\circ = 5 \times 0.6 = 3 \text{ m/s}$$

$$v_{0y} = v_0 \sin 53^\circ = 5 \times 0.8 = 4 \text{ m/s}$$

(a) Taking the origin at the place where the kid starts, the x and y coordinates of the ball are as follows:

$$x_1 = v_{0x} t = 3t$$

$$y_1 = y_{10} + v_{0y} t - \frac{1}{2}gt^2 = 33 + 4t - 5t^2$$

The kid accelerates in the x direction and does not move in the y direction:

$$x_2 = \frac{1}{2}at^2 \quad y_2 = 0$$

(b) When the ball reaches the ground level, its y coordinate

should be zero:

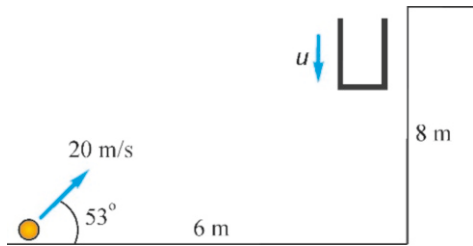
$$y_1 = 33 + 4t - 5t^2 = 0 \rightarrow t = -11/5 \text{ and } +3 \text{ s}$$

The solution is the positive value $t = 3 \text{ s}$.

(c) We find the x position of the ball at this time t :

Example 3.6

The outdoor elevator of a building starts to descend with a constant speed u from a height of 8 m from the ground. At the same time, a kid waiting on the ground at a horizontal distance of 6 m from the building throws a ball towards the elevator with a velocity of 20 m/s and at an angle of 53° .



- Choose a coordinate system and write the equations of motion of the elevator and of the ball.
- After how many seconds will the ball hit the wall?
- At what height will the ball hit the wall?
- What should the velocity u of the elevator be so that the

$$x_1 = 3t = 9 \text{ m}$$

(d) The kid should arrive to the same position $x = 9 \text{ m}$ at the same time $t = 3 \text{ s}$ as the ball:

$$x_2 = \frac{1}{2}at^2 \rightarrow 9 = \frac{1}{2}a \times 3^2 \rightarrow a = 2 \text{ m/s}^2$$

ball will collide with the elevator?

Answer

We first calculate the initial velocity components of the ball:

$$v_{0x} = v_0 \cos 53^\circ = 20 \times 0.6 = 12 \text{ m/s}$$

$$v_{0y} = v_0 \sin 53^\circ = 20 \times 0.8 = 16 \text{ m/s}$$

(a) Taking the origin as the place where the ball is thrown, the equations of the ball are as follows:

$$x_1 = v_{0x}t = 12t \quad y_1 = v_{0y}t - \frac{1}{2}gt^2 = 16t - 5t^2$$

The horizontal distance of the elevator is constant and it is moving in the y direction with a constant downward speed:

$$x_2 = 6 \text{ m} \quad y_2 = y_{20} - ut = 8 - ut$$

(b) The ball will have traveled 6 m in the x direction when it hits the wall:

$$6 = 12t \rightarrow t = 0.5 \text{ s}$$

(c) We calculate the y position of the ball at this time t :

$$y_1 = 16 \times 0.5 - 5 \times (0.5)^2 = 6.75 \text{ m}$$

(d) The elevator should be at the same y position at that time:

$$6.75 = 8 - u \times 0.5 \rightarrow u = 2.5 \text{ m/s}$$

Example 3.7

Maximum range. What is the optimal angle for an object that is always thrown at the same speed v_0 so that its horizontal range is maximum?

Answer

Let us use the trajectory equation to find the range formula. $y = 0$ should be true when the ball is at the ground level. This condition is used in the (3.13) trajectory formula:

$$0 = (\tan \theta)x - \frac{g}{2v_0^2 \cos^2 \theta}x^2$$

$$\text{Solution: } x = 0 \text{ and } x = \frac{2v_0^2 \cos^2 \theta \tan \theta}{g}$$

The solution $x = 0$ is where the object was thrown. The other solution is the range and is indicated by R . Using the trigonometric identities $\tan \theta = \sin \theta / \cos \theta$ and $2 \sin \theta \cos \theta = \sin 2\theta$, we find the range formula as follows:

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

For the same v_0 initial velocities, the sine function should be maximum for R to be maximum. The sine function takes its maximum value 1 at 90° . Therefore,

$$2\theta = 90^\circ \rightarrow \theta = 45^\circ$$

For maximum range, objects should be thrown at an angle of 45° .

3.3 UNIFORM CIRCULAR MOTION

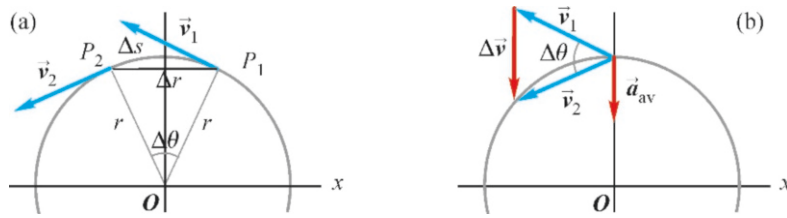
Centripetal Acceleration

The second important example of two-dimensional motion is the motion of an object rotating at constant speed around a circle. Although the speed is constant, the object has an acceleration, because the velocity changes direction. Let us try to determine how this happens.

Let an object rotating with a constant speed v on a circle with radius r have a velocity vector \vec{v}_1 at position P_1 at time t_1 and then a velocity vector \vec{v}_2 at position P_2 at a later time t_2 (Figure 3.6a).

The magnitudes of these two velocities are the same: $|\vec{v}_1| = |\vec{v}_2| = v$. However, they should be considered as two different vectors, as their directions are different.

Figure 3.6: Velocity vectors and $\Delta\vec{v}$ difference vector in circular motion.



Now let us remember the definitions of the average acceleration,

$$\vec{a}_{\text{av}} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

and the acceleration vector \vec{a} as the limit of this expression when $\Delta t \rightarrow 0$:

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{v}}{\Delta t} \quad (3.16)$$

Since this acceleration is a vector, we must determine both its magnitude and its direction. We start with the magnitude.

Now, what is this vector $\Delta\vec{v}$? To get an idea about it, let us shift both vectors to the midpoint of the arc on the circle, keeping their directions unchanged (Figure 3.6b). The equal sides of the isosceles triangle formed are \vec{v}_1 and \vec{v}_2 while the opposite side is $\Delta\vec{v}$.

Let us compare this triangle with the triangle OP_1P_2 formed in the original circle. These are similar triangles, because both are isosceles and both of their apex angles are equal to $\Delta\theta$. (When the object turns by $\Delta\theta$, its velocity vector also turns by the same amount.) Therefore, the ratios of similar sides are equal:

$$\frac{|\Delta\vec{r}|}{r} = \frac{|\Delta\vec{v}|}{v}$$

Here, we shall introduce a small error by using the length of the arc Δs instead of the length of the secant $|\Delta\vec{r}|$, but the difference will approach zero later when we take the limit $\Delta t \rightarrow 0$. Therefore, substituting $|\Delta\vec{r}| \approx \Delta s$,

$$|\Delta\vec{v}| = \frac{v \Delta s}{r}$$

and, taking the limit $\Delta t \rightarrow 0$ we get,

$$a = \lim_{\Delta t \rightarrow 0} \frac{|\Delta\vec{v}|}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{v \Delta s}{r \Delta t} = \frac{v}{r} \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}$$

where we took the constants outside of the limit. The limit in this expression is just the definition of speed v with which we are familiar, because it is the ratio of the distance taken Δs to the elapsed time: $\lim_{\Delta t \rightarrow 0} \Delta s / \Delta t = v$. Therefore, the magnitude of the acceleration is as follows:

$$a = \frac{v^2}{r}$$

This magnitude is constant, because the radius r and the speed v are constant.

Next, we find the direction of the acceleration \vec{a} . In Eq. (3.16), since Δt is just a scalar, the direction of \vec{a} will be the same as the limit direction of $\Delta\vec{v}$. Look at Figure 3.6b again. As $\Delta\theta \rightarrow 0$, or velocity \vec{v}_2 gets closer to \vec{v}_1 , they align

as almost parallel, with $\Delta\vec{v}$ becoming almost perpendicular to both. In terms of angles, as $\Delta\theta \rightarrow 0$ (Figure 3.7), the base angles of the isosceles triangle will approach 90° . The vector $\Delta\vec{v}$, hence \vec{a} , being perpendicular to the velocity \vec{v} , will point towards the center of the circle.

Therefore, in uniform circular motion, a constant acceleration will be formed that is always directed towards the center (Figure 3.8). This is called **centripetal acceleration** and is denoted by a_r :

$$a_r = \frac{v^2}{r} \quad (\text{centripetal acceleration}) \quad (3.17)$$

This formula can also be written in terms of **period** T , the time for one complete cycle. As the distance traveled in one complete cycle is the perimeter length $2\pi r$, the speed and the acceleration can also be expressed in terms of the period:

$$v = \frac{2\pi r}{T}$$

$$a_r = \frac{4\pi^2 r}{T^2} \quad (3.18)$$

Tangential Acceleration

If, along with the direction, the magnitude of the velocity also changes in circular motion, a **tangential acceleration** (Figure 3.9) will arise in addition to the centripetal acceleration. The expression of tangential acceleration is the change in the magnitude of velocity:

$$a_t = \frac{dv}{dt} \quad (3.19)$$

We will address tangential acceleration later in the discussion of rotational motion in Chapter 7.

The acceleration \vec{a} is the resultant of these centripetal and tangential accelerations:

$$\vec{a} = \vec{a}_r + \vec{a}_t \quad (3.20)$$

Therefore, in the most general case, the acceleration \vec{a} does not necessarily point to the center, but may be in any direction.

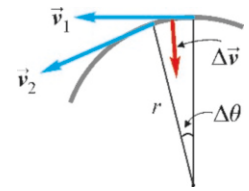


Figure 3.7: As $\Delta\theta \rightarrow 0$, the $\Delta\vec{v}$ vector is directed towards the center of the circle.

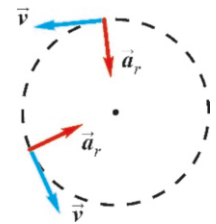


Figure 3.8: In circular motion, the centripetal acceleration \vec{a}_r is always towards the center at every point.

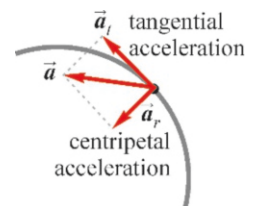


Figure 3.9: If the magnitude of velocity also changes, a tangential acceleration \vec{a}_t is also formed.

Example 3.8

An object revolving in a circular orbit with a radius 80 cm is observed to make 150 cycles per minute.

- What is the velocity of the object?
- What is the centripetal acceleration of the object?

Answer

(a) The constant velocity v is directly calculated as the ratio of the traveled distance to time. As the circumference of the

circle is $2\pi r$, the distance traveled in 150 cycles is divided by 1 minute:

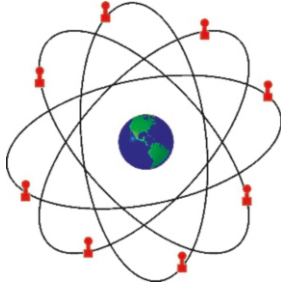
$$v = \frac{\text{distance}}{\text{time}} = \frac{150 \times 2\pi r}{60 \text{ s}} = 5\pi r = 12.5 \text{ m/s}$$

(b) The velocity found above is used in the centripetal acceleration formula ($\pi^2 \approx 10$):

$$a_r = \frac{v^2}{r} = \frac{(5\pi r)^2}{r} = 25\pi^2 r = 200 \text{ m/s}^2$$

Example 3.9

The Global Positioning System (GPS) consists of around 20 satellites revolving at an altitude of 20 000 km from the surface of the Earth. Each satellite circles the Earth twice a day. The Earth's radius is approximately 6 400 km.



(a) Calculate the period of a GPS satellite.

(b) Calculate the centripetal acceleration of a GPS satellite.

Answer

(a) The period of a satellite, in other words, the time it takes to complete one full cycle is half a day. We calculate this in terms of seconds:

$$T = \frac{1}{2} \times \text{day} = \frac{1}{2} \times 24 \times 60 \times 60 = 43\,200 \text{ s}$$

(b) We use the formula that expresses centripetal acceleration in terms of the period T . Here, r is the distance measured from the center of the circle. If the Earth's radius is R and the height of the satellite from the surface is h , then $r = R + h = 6400 + 20000 = 26400 \text{ km} = 26.4 \times 10^6 \text{ m}$. We substitute these values ($\pi^2 \approx 10$):

$$a_r = \frac{4\pi^2 r}{T^2} = \frac{40 \times 26.4 \times 10^6}{(43.2 \times 10^3)^2} = 0.57 \text{ m/s}^2.$$

3.4 RELATIVE MOTION

Consider a rower pulling oars to move a boat along a river. From the shore, we observe that the same rower moves faster when traveling in the same direction as the current and slower when rowing against the current. In fact, the boat may even be observed to go backwards when trying to go against the current. Likewise, on an escalator, even if we stay still, people on the ground see us as moving forward.

These observations remind us that concepts such as velocity and position are relative, in other words, “they depend on the observer making the measurements.” If the observer is moving, the velocity measured by him/her will be different from the velocity measured by an observer at rest. Therefore, it is important to know the relation between such various measurements.

In one dimension, it is easy to see the relation between velocities. When a swimmer with speed v in still water starts to swim in a river with a current speed u , he/she will have a speed of $v + u$ when swimming in the same direction as the current, and $v - u$ when swimming against the current.

To calculate relative velocities in two dimensions, let us consider two objects positioned at points A, B at time t , and an observer standing at rest at the origin O . From Figure 3.10, we can see the following relations between these vectors:

$$\vec{OB} = \vec{OA} + \vec{AB}$$

But these are just position vectors, namely:

$$\vec{OB} = \vec{r}_{BO} : \text{The position of object } B \text{ with respect to origin } O;$$

$$\vec{OA} = \vec{r}_{AO} : \text{The position of object } A \text{ with respect to origin } O;$$

$$\vec{AB} = \vec{r}_{BA} : \text{The position of object } B \text{ with respect to object } A.$$

Hence, while these positions change with time, we will always have:

$$\vec{r}_{BO} = \vec{r}_{AO} + \vec{r}_{BA} = \vec{r}_{BA} + \vec{r}_{AO} \quad (3.21)$$

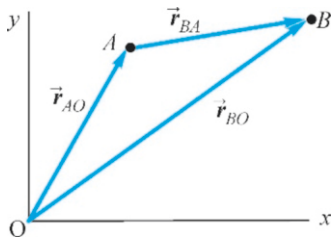


Figure 3.10: The positions of objects A and B with respect to the observer O .

where we switched the order of the last two terms. By taking the derivative of this expression with respect to time, we get velocities:

$$\begin{aligned}\frac{d\vec{r}_{BO}}{dt} &= \frac{d\vec{r}_{BA}}{dt} + \frac{d\vec{r}_{AO}}{dt} \\ \vec{v}_{BO} &= \vec{v}_{BA} + \vec{v}_{AO}\end{aligned}$$

where each term can easily be identified:

$$\begin{aligned}\vec{v}_{BO} &= \text{The velocity of } B \text{ with respect to origin } O; \\ \vec{v}_{AO} &= \text{The velocity of } A \text{ with respect to origin } O; \\ \vec{v}_{BA} &= \text{The velocity of } B \text{ with respect to } A.\end{aligned}$$

Accordingly, the relation between the velocities is as follows:

$$\boxed{\vec{v}_{BO} = \vec{v}_{BA} + \vec{v}_{AO} \quad (\text{relative velocity addition rule})} \quad (3.22)$$

Here, we stress two points that are important for calculations:

- Notice that, when we switch the subscripts, we get a negative sign. For example,

$$\vec{v}_{AB} = \frac{d\vec{r}_{AB}}{dt} = \frac{d(-\vec{r}_{BA})}{dt} = -\vec{v}_{BA}$$

which is true for all velocities, $\vec{v}_{OB} = -\vec{v}_{BO}$, etc.

- It is very easy to remember the velocity addition rule, Eq. (3.22). Notice the ordering of the subscripts. We see that the inner subscripts (A) on the right-hand side are the same. And the outer subscripts (B, O) are just those on the left-hand side.

Hence, any velocity that you wish to obtain can be separated into two terms into which you insert the missing subscript. For example, if we want \vec{v}_{AB} (the velocity of A with respect to B), we write:

$$\vec{v}_{AB} = \vec{v}_{AO} + \vec{v}_{OB} = \vec{v}_{AO} - \vec{v}_{BO}$$

where, in the last term, we reversed subscripts to get the velocity of B with respect to origin, hence we picked up a negative sign.

The **relative velocity addition rule** given in Eq. (3.22) is valid only in classic physics, in which speeds are small compared to the speed of light. When the speed of light is approached, we must use *The Theory of Relativity* developed by Einstein.

Relative Acceleration

Let us reconsider Eq. (3.22) which gives us the velocity addition rule:

$$\vec{v}_{BO} = \vec{v}_{BA} + \vec{v}_{AO}$$

The objects shall have acceleration if their velocities are changing in time. The second derivative of this equation with respect to time will give the relation between the accelerations:

$$\vec{a}_{BO} = \vec{a}_{BA} + \vec{a}_{AO} \quad (3.23)$$

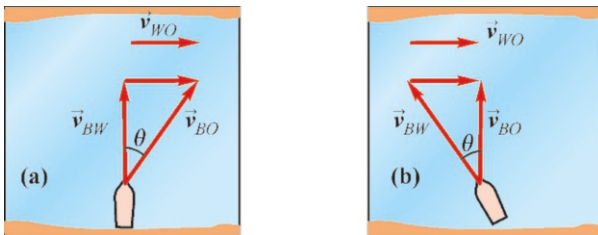
We will not use this general expression in this course. However, in order to draw an important conclusion, let us assume that only the velocity of the object B changes and the object A moves at constant velocity with respect to the origin. In this case, $\vec{a}_{AO} = 0$ and we find that

$$\vec{a}_{BO} = \vec{a}_{BA} \quad (3.24)$$

This is an important results: *Observers moving at constant velocities with respect to each other observe the same acceleration in other objects.* This result will make it easier to understand the laws of dynamics later.

Example 3.10

A boat has a speed of 20 km/h in still water. The captain is trying to cross a river that is 2 km wide. The current speed in the river is 5 km/h.



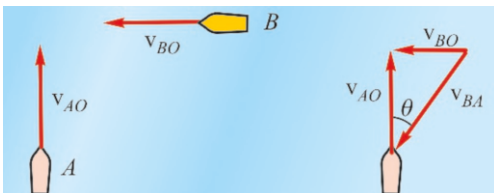
- What will the speed and direction of the boat be with respect to the ground if the captain heads straight across the river?
- The captain wants to cross the river perpendicular to the shore. In which direction should he keep the bow of the boat so that its velocity observed from the ground is perpendicular to the ground? In such a case, what will its velocity be with respect to the ground?
- Calculate the time to cross the river in both cases.

Answer

In order to apply the velocity addition formula, let us designate the river water with W , the boat with B , and the ground with O . Accordingly, the given data are:

The speed of the water with respect to the ground:
 $v_{WO} = 5 \text{ km/h}$

Example 3.11



The velocities of two ships in the Mediterranean Sea are measured from the ground. Ship A is observed to be sailing at 25 km/h in the South-North direction and ship B at 40 km/h in the East-West direction.

What are the magnitude and direction of the velocity of

The speed of the boat with respect to the water:

$$v_{BW} = 20 \text{ km/h}$$

The velocity of the boat with respect to the ground: $v_{BO} = ?$

$$\text{Velocity addition formula: } \vec{v}_{BO} = \vec{v}_{BW} + \vec{v}_{WO}$$

This vector equality is valid for both questions.

(a) We get the situation observed in Figure (a) when the velocity of the boat with respect to the water \vec{v}_{BW} is kept perpendicular to the water. Writing the hypotenuse and angle formulas in this right triangle,

$$v_{BO} = \sqrt{v_{BW}^2 + v_{WO}^2} = \sqrt{20^2 + 5^2} = 21 \text{ km/h}$$

$$\tan \theta = v_{WO}/v_{BW} = 5/20 = 0.25 \rightarrow \theta = 14^\circ.$$

(b) We get the situation observed in Figure (b) when the velocity of the boat with respect to the river \vec{v}_{BW} is kept at some angle, such that the resultant velocity \vec{v}_{BO} is ensured to be perpendicular to the shore. The triangle formulas give:

$$v_{BO} = \sqrt{v_{BW}^2 - v_{WO}^2} = \sqrt{20^2 - 5^2} = 19 \text{ km/h}$$

$$\sin \theta = v_{WO}/v_{BW} = 5/20 = 0.25 \rightarrow \theta = 15^\circ.$$

(c) If we denote the width of the river as L , the diagonal path taken in part (a) will be $L/\cos \theta$. Accordingly,

$$t_a = \frac{L}{v_{BO} \cos 14^\circ} = \frac{2}{21 \times 0.97} = 0.10 \text{ hour} = 6 \text{ minutes.}$$

The path taken in part (b) will be L . The crossing time is calculated accordingly:

$$t_b = \frac{L}{v_{BO}} = \frac{2}{19} = 0.11 \text{ hour} = 6.5 \text{ minutes.}$$

ship B as observed by the captain of ship A ?

Answer

If we denote the velocity of ship B with respect to A as \vec{v}_{BA} , we may write the relative velocity addition formula (by following the subscripts correctly) as follows:

$$\vec{v}_{BA} = \vec{v}_{BO} + \vec{v}_{OA}$$

Here, \vec{v}_{OA} is the velocity of the ground with respect to the ship A and if we reverse it as $\vec{v}_{OA} = -\vec{v}_{AO}$ the formula we seek is:

$$\vec{v}_{BA} = \vec{v}_{BO} - \vec{v}_{AO}$$

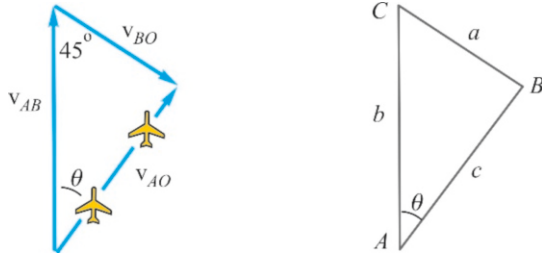
The difference between these two vectors is shown in the figure. If we denote the angle at which the ship A observes

the vector \vec{v}_{BA} as θ , we can calculate the velocity v_{BA} and the angle using the right triangle formulas:

$$v_{BA} = \sqrt{v_{BO}^2 + v_{AO}^2} = \sqrt{40^2 + 25^2} = 47 \text{ km/h}$$

Example 3.12

The pilot of an airplane with a speed of 60 m/s in still air is flying by keeping the nose of the airplane directed towards the North in a region where the wind is 20 m/s in the South-East direction. What is the velocity of the airplane with respect to the ground and its angle with the North?



Answer

Let the airplane be A, the wind W and the ground O. If we add to the velocity of the airplane with respect to the wind

$$\tan \theta = \frac{v_{BO}}{v_{AO}} = \frac{40}{25} = 1.6 \rightarrow \theta = 58^\circ$$

\vec{v}_{AW} , the velocity of the wind with respect to the ground \vec{v}_{WO} , we find the velocity of the airplane with respect to the ground in windy air \vec{v}_{AO} :

$$\vec{v}_{AO} = \vec{v}_{AW} + \vec{v}_{WO}$$

These velocity vectors are shown in the figure.

We use the Sine and Cosine theorems (Appendix B) to solve this triangle:

$$c = \sqrt{a^2 + b^2 - 2ab \cos C} \quad (\text{The cosine theorem})$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (\text{The sine theorem})$$

By comparing these two triangles, we calculate the requested velocity v_{AO} and its direction:

$$\begin{aligned} v_{AO} &= \sqrt{v_{AW}^2 + v_{WO}^2 - 2v_{AW}v_{WO} \cos 45} \\ &= \sqrt{60^2 + 20^2 - 2 \times 60 \times 20 \times \sqrt{2}/2} = 48 \text{ m/s} \end{aligned}$$

$$\frac{\sin \theta}{20} = \frac{\sqrt{2}/2}{48} \rightarrow \theta = 36^\circ$$

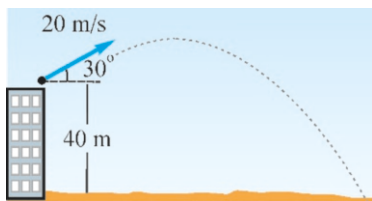
Multiple-choice Questions

- Which of the following is true for an object moving in a plane with a constant magnitude of velocity?
 - The acceleration of the object may not be zero.
 - The velocity of the object is parallel to the trajectory.
 - The object may have centripetal acceleration.
 - All of the above.
- Which is true for a motion on any curved trajectory?
 - Acceleration and velocity are in the same direction.
 - The velocity vector is tangent to the trajectory.
 - The velocity is perpendicular to the trajectory.
 - Acceleration and velocity are perpendicular.
- Two balls are thrown from the same height at the same instant. Ball A is dropped in free fall, while ball B is thrown in a horizontal direction. Which is true?
 - A reaches the ground first.
 - B reaches the ground first.
 - They arrive at the same time.
 - It is impossible to tell.
- Which of the following remains constant in projectile motion?
 - The magnitude of velocity.
 - The horizontal component of velocity.
 - The vertical component of velocity.
 - The angle of velocity.
- Which is correct at the maximum height in projectile motion?
 - Velocity is zero.
 - The horizontal component of velocity is zero.
 - The vertical component of velocity is zero.
 - Velocity is maximum.
- Object A is revolving around a circle with radius r at a constant velocity v . Object B is revolving around a circle with radius $2r$ at velocity $2v$. Which one has higher centripetal acceleration?
 - A
 - B
 - Equal
 - None.
- Object A is revolving around a circle with radius r at a constant velocity v . Object B is revolving around a circle with radius $2r$ at velocity $v/2$. Which one has higher centripetal acceleration?
 - A
 - B
 - Equal
 - None.
- Which of the following is incorrect for an object performing accelerated motion on a plane?
 - The velocity vector may be constant.
 - The velocity magnitude may be constant.
 - The velocity direction may be constant.
 - Velocity may change direction.
- In how many seconds will an object dropped from a height of 80 meters reach the ground?
 - 1
 - 2
 - 3
 - 4
- Two objects are traveling in the same direction along a straight line. Object A has a speed of 10 m/s and object B has a speed of 4 m/s. Which of the following will be observed by object A?
 - Velocity is zero.
 - The horizontal component of velocity is zero.
 - The vertical component of velocity is zero.
 - Velocity is maximum.

- (a) Object B moving away at 6 m/s.
 (b) Object B moving away at 14 m/s.
 (c) Object B approaching at 6 m/s.
 (d) Object B approaching at 14 m/s.
11. Two objects are traveling towards each other in the opposite directions along a straight line. Object A has a speed of 10 m/s and object B has a speed of 4 m/s. Which of the following will be observed by object A?
- (a) B moving away at 6 m/s.
 (b) B moving away at 14 m/s.
 (c) B approaching at 6 m/s.
 (d) B approaching at 14 m/s.
12. What is the cause of the centripetal acceleration in uniform circular motion?
- (a) A change in the magnitude of velocity.
 (b) A change in the direction of velocity.
 (c) A change in position.
 (d) Velocity being tangent to the trajectory.
13. A stone is dropped from the window of a moving train. Which of the following is incorrect?
- (a) Observer on the train sees the stone in free fall.
 (b) Observer on the ground sees the stone in free fall.
 (c) Observer on the ground sees a horizontal projectile motion.
 (d) None of the above.
14. Which of the following is incorrect for projectile motion?
- (a) The time it takes to rise is equal to the time it takes to fall.
 (b) Speeds are the same in two directions at the same height.
 (c) The speed on the ground is maximum.
 (d) The speed at the maximum height is equal to the vertical component of the velocity.
15. In a horizontal projectile motion, the object falls to the ground in t seconds. What will the time of flight be if the height is tripled?
- (a) $3t$ (b) $\sqrt{3}t$ (c) $t/\sqrt{3}$ (d) $9t$
16. Which of the following is incorrect if the direction and magnitude of velocity are changing in circular motion?
- (a) There is centripetal acceleration.
 (b) There is tangential acceleration.
 (c) Acceleration is perpendicular to velocity.
 (d) Acceleration can be in any direction.
17. You want to swim to the point right across you in a river where the current runs from the left to the right. In which direction should you swim?
- (a) Right ahead.
 (b) Towards the left.
 (c) Towards the right.
 (d) Halfway towards left and then towards the right.
18. A boy who can swim at 4 m/s in still water tries to swim perpendicular to the shore in a river where the current is 3 m/s. What will be its velocity and direction be as observed from the ground?
- (a) 4 m/s perpendicular to the shore.
 (b) 5 m/s perpendicular to the shore.
 (c) 5 m/s and 37° wide.
 (d) 5 m/s and 53° wide.
19. A man wants to get upstairs with an escalator. The man walks up the stairs in 6 s when the escalator is still. The man gets upstairs in 3 s when the escalator is moving and he is at rest on the steps. In how many seconds will the man get upstairs by walking when the escalator is moving?
- (a) 1 (b) 1.5 (c) 2 (d) 2.5
20. Two objects are thrown at the same angle from the same point. Object B reaches a maximum height that is 3 times that of object A. What is the ratio of the initial velocity of B with respect to A?
- (a) 3 (b) $\sqrt{3}$ (c) $1/\sqrt{3}$ (d) 9

Problems

3.2 Projectile Motion

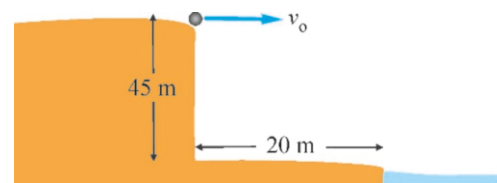


Problem 3.1

- 3.1 A ball is thrown from the roof of a 40 m high building, with a speed of 20 m/s at an angle of 30° to the horizontal.
- (a) What is the time of flight? (b) How far away from the

building will it hit the ground? (c) What are its velocity components when the ball hits the ground?

[A: (a) 4 s, (b) 69 m, (c) $v_x = 17$ $v_y = -30$ m/s.]

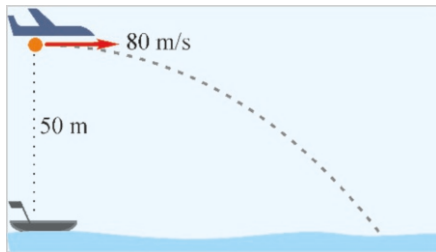


Problem 3.2

- 3.2 A stone is thrown horizontally towards the sea from a

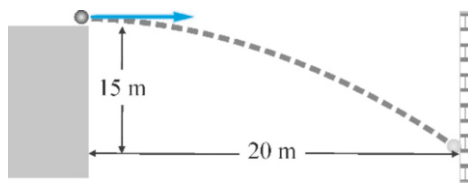
45 m high cliff. There is a 20 m wide beach extending towards the sea under the cliff. (a) What is the time of flight? (b) What should the minimum speed of the stone be in order to land in the sea? [A: (a) 3 s, (b) 6.7 m/s.]

3.3 An athlete jumps up off the ground with a speed of 8 m/s and lands 6.2 m away horizontally. (a) At what angle has he jumped? (b) How long does the jump take in seconds? (*Hint:* Use the trajectory equation.) [A: (a) 37°, (b) 1 s.]



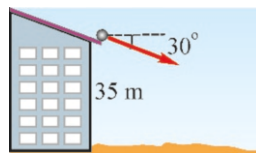
Problem 3.4

3.4 A rescue airplane flying horizontally at a speed of 80 m/s at an altitude of 50 m drops a relief package when a boat on the sea is right underneath it. (a) In how many seconds will the package fall to the sea? (b) How far away from the boat will it fall? [A: (a) 3.2 s, (b) 256 m.]



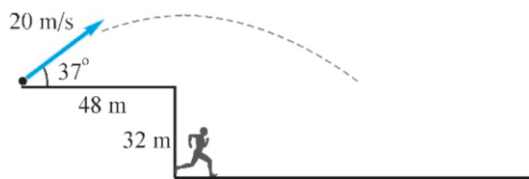
Problem 3.5

3.5 A ball thrown horizontally from a height hits a wall that is 20 m away at a point 15 m below. (a) What is the time of flight? (b) What is the initial speed of the ball? [A: (a) $\sqrt{3} = 1.7$ s, (b) $20/\sqrt{3} = 11.5$ m/s.]



Problem 3.6

3.6 A ball rolls off of the edge of a 30° inclined roof whose edge is 35 m above the ground. The ball falls to the ground in 2 s. (a) What is its initial speed? (b) How far away from the edge of the roof will it fall? [A: (a) 15 m/s, (b) 26 m.]

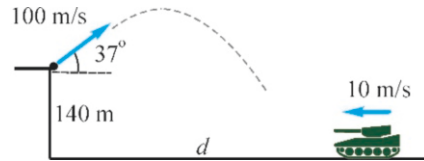


Problem 3.7

3.7 A ball is thrown at an angle of 37° to the horizontal and with a speed of 20 m/s from one end of a horizontal platform, which is 48 m long and 32 m high. At the same time, a kid

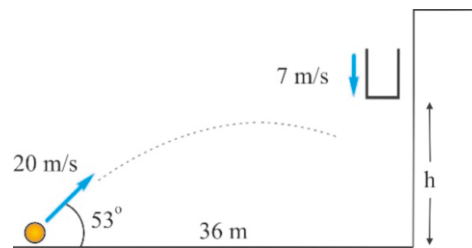
standing next to the platform wall on the ground starts to run with acceleration a . What should the acceleration a be so that the kid can catch the ball? What is the time of flight of the ball? [A: $a = 2$ m/s², $t = 4$ s.]

3.8 A basketball player shoots the ball at the basket, which is 4 m away and 2 m high from the level of his hand. Since the throwing angle is 53°, what should the initial velocity be in order to make a basket? [A: 8.2 m/s.]



Problem 3.9

3.9 A gunner located on a 140 m high hill can fire shells with an initial speed of 100 m/s and at an angle 37° with the horizontal. Observing that a tank is approaching with a constant speed of 10 m/s on a horizontal path on the ground, the gunner fires when the tank is at a distance d . What should the distance d be for the shell to hit the tank? What is its time of flight? [A: $d = 1260$ m and 14 s.]



Problem 3.10

3.10 An outdoor elevator of a building is descending at a constant speed of 7 m/s. As the elevator passes through height h , a boy located on the ground 36 m away from the building throws a ball towards the elevator with an initial speed of 20 m/s and at an angle of 53°. What should the height h be in order for the ball to hit the elevator? What is the time of flight of the ball? [A: $h = 24$ m and 3 s.]

3.3 Uniform Circular Motion

3.11 What is the centripetal acceleration of a point on the Earth's surface due to its rotation about its own axis? The Earth's radius is 6400 km. [A: 0.03 m/s².]

3.12 A racing car must have a maximum centripetal acceleration of 9 m/s² in order to go around a curve without slipping. At what maximum speed can it take a circular curve with a radius of 150 m? [A: 37 m/s.]

3.13 The human body can withstand a maximum of 9g of acceleration. What is the minimum radius of the loop that a pilot can make in the air with a fighter jet having a speed of 900 km/h? [A: 694 m.]

3.14 Protons are accelerated at the European Nuclear Research Center (CERN) and caused to collide at very high

energies. According to the most recent data, protons make 11 000 revolutions in 1 second in a circular trajectory whose perimeter is 27 km long. What is the centripetal acceleration of the protons?
[A: $21 \times 10^{12} \text{ m/s}^2$.]

3.15 The Earth rotates around the Sun in 365 days in an orbit with an approximate radius of $15 \times 10^7 \text{ km}$. The planet Mercury rotates in 88 days around an orbit with an approximate radius of $6 \times 10^7 \text{ km}$. Calculate the ratio of the centripetal acceleration of Mercury with respect to Earth.

[A: $a_r(\text{Mercury})/a_r(\text{Earth}) = 7$.]

3.4 Relative Motion

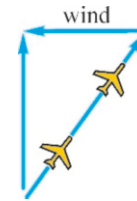
3.16 The velocity of a boat in a river is 10 m/s when it sails in the same direction as the current, and 4 m/s when it sails against the current. Calculate the speeds of the boat and the current.
[A: 7 m/s and 3 m/s.]



Problem 3.17

3.17 The pilot of an airplane with a speed of 60 m/s in still air is flying by keeping the nose of the airplane directed towards the North in a region where the wind is 20 m/s in the North-East direction. What is the velocity of the airplane with respect to the ground and its angle with the North?

[A: 75 m/s and $\theta = 11^\circ$.]



Problem 3.18

3.18 The pilot of an airplane with a speed of 200 km/h in still air wants to fly in the South-North direction on a day when the wind is 60 km/h in the East-West direction. In which direction should he keep the nose of the airplane and what will be the velocity of the airplane with respect to the ground?

[A: 17° with the North towards the East and 191 km/h.]

3.19 The velocities of two airplanes in the air are measured from the ground. Airplane A is observed to be flying at 250 km/hour in the South-North direction and airplane B at 200 km/hour in the East-West direction. What are the magnitude and direction of the velocity of plane A as observed by the pilot of B?

[A: 320 m/s and 39° with the North towards the East.]

3.20 The captain of a ship A sailing at a velocity of 60 km/hour in the South-North direction observes another ship B as sailing with a velocity of 50 km/hour in the West-East direction. What is the velocity and direction of ship B with respect to the ground?

[A: 78 km/hour and 40° with the North towards the East.]

4

NEWTON'S LAWS OF MOTION



The Ariane 5 rocket starts its space journey on April 22, 2011. Developed by the European Space Agency (ESA), this rocket broke a record in carrying 9.6 tons of payload into space by freeing it from the gravitational pull of the Earth.

The fundamental cause that generates motion is the force. But what exactly is the connection between force and motion?

In the previous chapters, we have examined the motion in terms of position, velocity and acceleration, but have not discussed the question of *what causes objects to move?* We know from our daily lives that a force is necessary to set objects into motion. Every object acts in a different manner, depending on its own mass under such forces.

Throughout history, many scientists have examined the effect of forces on the motion of objects. Galileo was the first to establish many of the features of motion. Ultimately, English scientist Isaac Newton (1642–1727), building on the Galileo's results, was able to lay the foundations for the modern science of **Mechanics**, in his book *Principia Mathematica*.

Newtonian mechanics was believed to be the absolute truth for about 200

years. Around the 1900s, these laws were found to give incorrect results both for particles at the atomic scale and for particles traveling close to the speed of light. Two modern theories were later developed, one for microscopic scales (Quantum Mechanics) and the other for large speeds (Relativistic Mechanics). However, Newtonian mechanics continues to remain valid for macroscopic objects.

4.1 NEWTON'S LAWS

Newtonian Mechanics is based on three simple and reasonable laws. By accepting these laws without proof, you can explain the motion of all objects, from stones and balls to cars, from rockets and bullets to stars and planets.

Let us state and discuss Newton's laws in order.

Newton's First Law

Discovered by Galileo, the first law regards objects upon which no net force is acting:

Newton's First Law

If the net force on an object is equal to zero, the object will either remain at rest, or continue in a straight line with the same constant velocity:

$$\vec{F}_{\text{net}} = 0 \iff \vec{a} = 0 \quad (4.1)$$

The force \vec{F}_{net} in this law is the vector sum of many forces and is called the **net force** or **resultant force**:

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots = \sum_i \vec{F}_i$$

Until Galileo, people believed in the following law by Aristotle: "Objects move when a force is applied, and they stop if the force is removed." Even today, many people unaware of physics think in this manner. Indeed, doesn't a book on a tabletop that we push with our hand stop as soon as we withdraw our hand? What is wrong with that?

Galileo answered this question by showing that an invisible friction force was causing the object to stop. He devised very fine experiments by perfectly polishing wood and marble surfaces in order to reduce friction. Then, although they did not immediately stop when the driving force was removed, objects went further, depending on how polished the surface was. He thus discovered the first law.

We emphasize the following points about the first law:

- It is important that the *net* (resultant) force be zero in this law: Many forces may be acting together on the object, but the law shall apply if their resultant is zero.
- It is also easy to understand the "either, or" expression of the law: The acceleration of the object is zero if it is either at rest or in uniform linear motion.



Figure 4.1: Why are these sportswomen sweeping in front of the moving curling stone?

- *The first law is actually the definition of the force.* Notice that the arrow is bidirectional in Eq.(4.1) expression of the law. It means that the reverse statement is also true: If an object is accelerating, there must be a net force acting on it. In other words, acceleration is a sign of the presence of force.

Newton's Second Law

If there is a net force acting on an object, it will perform accelerated motion. The amount of this acceleration gives us the second law.

Newton's Second Law

The acceleration of an object is directly proportional to the net force acting on it, and inversely proportional to the mass of the object. The acceleration is in the same direction as the net force:

$$\vec{F}_{\text{net}} = m \vec{a} \quad (4.2)$$

Galileo again found this law as a result of the fine experiments that he conducted using inclined planes. Galileo was able to show that the acceleration of an object placed on an inclined plane, which he had polished to reduce friction, increased as its slope was increased.

Let us stress some important points about the second law:

- The second law fixes the unit of force. This is a derived unit named the Newton and abbreviated with (N) in the SI system. Writing the units of both sides in the law's expression, we get

$$1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$$

More clearly, a force that gives 1 m/s^2 acceleration to a 1-kg object will have a magnitude of 1 Newton.

- The second law is a vector equation. In other words, this equation must be true for each component in the xy -coordinate system:

$$\vec{F}_{\text{net}} = m \vec{a} \iff \begin{cases} F_{x,\text{net}} = m a_x \\ F_{y,\text{net}} = m a_y \end{cases} \quad (4.3)$$

- At first glance you may think that the first law is a special case of the second law. Indeed, if we set $\vec{a} = 0$ in the second law (4.2), we get $\vec{F}_{\text{net}} = 0$ which gives us the first law, right? This thinking is incorrect, because *the second law is actually the definition of mass*. If the first law did not specify what the force was, the second law could not relate this to mass and acceleration.
- The second law defines mass as a measure of the object's resistance to acceleration. In other words, if the same force is applied to two objects, it will be more difficult to accelerate the one with the larger mass (Figure 4.2). This property of objects "to resist acceleration" is called **inertia**. In other words, mass is the measure of the inertial characteristic of objects.

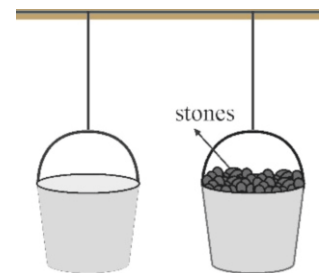


Figure 4.2: Which bucket is harder to push?

- For which observers are Newton's laws valid? For example, an accelerating observer will see a still object on the ground as accelerating in the reverse direction and find an incorrect solution: He/she will observe acceleration although there is net zero force. In order to reply to this question, let us remember the topic of relative speed from Chapter 2. We had seen that (Equation 3.24), observers making uniform linear motion with respect to each other will measure the same acceleration. Therefore, *Newton's laws are valid for observers in uniform linear motion with respect to each other.* We call these **inertial reference frames**.

Newton's Third Law (Principle of Action and Reaction)

Did you ever think about how a person jumps up? The answer is simple: We push the ground down with our feet and the ground pushes us up in the opposite direction. So, we know instinctively that the ground will produce a reaction force.

Isaac Newton was the first to consider that every force always generates a reaction. The third law determines the relation between this action and reaction.

Newton's Third Law

If an object applies a force \vec{F}_{12} on a second object, the second object will always apply a force \vec{F}_{21} equal in magnitude and opposite in direction on the first object:

$$\vec{F}_{21} = -\vec{F}_{12} \quad (4.4)$$

Before rockets were built, some scientists claimed that it would be impossible to fly in the vacuum of space, where there is no air. However, rockets are able to fly in space by utilizing the reaction force generated when they discharge their fuel in the reverse direction with a large thrust. This is possible with the third law.

The points to pay attention to in applying the third law are as follows:

- *Action and reaction are applied on different objects.* There can be a contradiction if this distinction is not made.

As an example, consider a horse-drawn carriage (Figure 4.3). How would you reply to this statement: "If the horse pulls the carriage forward, the carriage will pull the horse back with the same force in the reverse direction. Therefore, both forces will cancel each other out and the carriage will not move." This reasoning is incorrect, because it ignores the fact that the action and reaction are applied on different objects. When examining one of these objects, *only the forces acting on that object should be taken into consideration.* The horse manages to not be pulled back by balancing the reaction force of the carriage with the friction force that it generates on the ground. However, the friction on the wheels of the carriage does not prevent it from going forward, because it is very low.

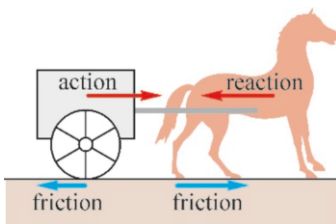


Figure 4.3: Forces on a horse and carriage.

- *Only the external forces acting on an object are taken into consideration when examining its motion.* This is because internal forces mutually cancel each other out according to the third law. Is it possible for a person inside of a boat to make it move by pushing it? No. Let us examine the boat and the

person inside as a whole. In this case, the action-reaction pair applied by the person on the boat and by the boat on the person becomes the **internal forces** of the system (boat+person) that we are examining, and their sum is zero according to the third law.

- It is irrelevant to ask which of these forces between two objects is the action and which is the reaction. Both arise simultaneously, in other words, there is no causality between them. Only an action-reaction pair is mentioned in order to state this fact.

4.2 TYPES OF FORCES IN MECHANICS

There are various types of forces that can lead to motion of objects. These forces include the gravitational force and weight force as its special case, the friction generated on surfaces and the normal reaction force and the tension force on ropes. . . . One or more of these forces can be included in each problem. It is not possible to find the correct solution unless we know how to take them into consideration.

Now let us review the most important forces.

Weight (Gravitational force)

As we discussed in the free fall and projectile motion problems, any object released near the Earth's surface will accelerate with a constant acceleration g towards the center of the Earth.

Interpreting this with the second law, we conclude that there must be a force that causes such an acceleration (Figure 4.4). This special force is called **weight** and its magnitude is indicated by W . As weight will fulfill the $F = ma$ law with the acceleration g ,

$$W = mg \quad (\text{weight force}) \quad (4.5)$$

and it is a vertically downward force. For example, the weight of an object with a mass of 1 kg is $W = 1 \times 9.8 \approx 10 \text{ N}$. This force is present regardless of whether the object is moving or not.

Some students may confuse the concepts of mass and weight. We can prevent such confusion as follows: Weight is the attraction force that Earth applies to objects. Objects will have different weights on other planets and the Moon because the gravitational attraction forces are different there. However, the object has no weight in empty space, where there is no other mass nearby.

Mass, on the other hand, is an intrinsic quality of the object and a measure of its capability for acceleration. The object will have a mass regardless of where it is in space. Here is a striking example that explains this: You can hold a rock with a mass of 1 ton in space on your fingertips, because it has no weight. But if you kick that rock, you may break your foot, because it is very difficult to accelerate, having a very large mass.

Law of Universal Gravitation

Weight is a special case of a much more general attraction force. It is the oldest of the four fundamental forces (gravitation, electromagnetism, nuclear force and weak force). Gravitational force is what makes planets rotate around the sun, keeps our satellite Moon in orbit around Earth and makes an apple fall from a tree. Discovered by Isaac Newton, this **law of universal gravitation** (or gravitational force) is expressed as follows:

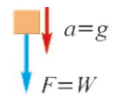


Figure 4.4: Weight is the force that causes gravitational acceleration.

Newton's Law of Universal Gravitation

Every object in the universe attracts every other object with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them:

$$F_g = G \frac{m_1 m_2}{r^2} \quad (4.6)$$

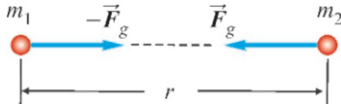


Figure 4.5: Law of gravitation.

The proportionality coefficient here is G , the **universal gravitational constant**, and it is one of the fundamental constants of physics:

$$G = 6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2 \quad (\text{Universal gravitational constant}) \quad (4.7)$$

Now let us apply this law to an object with mass m near the Earth's surface (Figure 4.6). Let one object have mass m and the other object be the Earth, with mass M_E and radius R_E . Accordingly,

$$F_g = G \frac{m M_E}{R_E^2} = m \left(\frac{G M_E}{R_E^2} \right)$$

The expression inside of the brackets depends only on the mass and radius of the Earth. The Earth's mass is $M_E = 5.97 \times 10^{24} \text{ kg}$ and its radius is $R_E = 6.38 \times 10^6 \text{ m}$. Using these values, we find the gravitational force acting on mass m :

$$F_g = m \times (9.81 \text{ m/s}^2)$$

The constant factor has dimensions of acceleration and a value of 9.81 m/s^2 . It is just the gravitational acceleration g that we are familiar with. Hence, this gravitational force is the weight W itself:

$$W = F_g = m g \quad (\text{Weight}) \quad (4.8)$$

$$g = \frac{G M_E}{R_E^2} \quad (\text{The gravitational acceleration on the Earth's surface}) \quad (4.9)$$

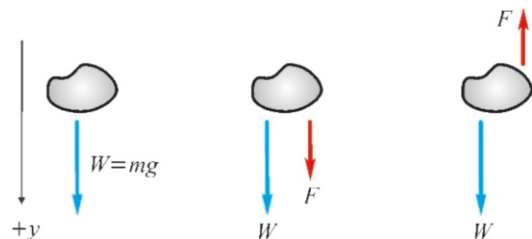
This formula shows us how the gravitational acceleration g decreases with increasing height. To find the value of g at a height h from the surface of the Earth, it is sufficient to substitute $(R_E + h)$ for R_E in this formula. Likewise, the gravitational acceleration at the surface of the Moon or other planets can be calculated using this method.

In calculating the gravitational force between objects with volume, the distance r in the formula is the distance between the centers of mass of the two objects.

Example 4.1

A stone with mass $m = 2 \text{ kg}$ is descending in free fall. A vertical force F is applied on it.

- What is the acceleration of the stone if $F = 0$?
- What is the acceleration of the stone if $F = 5$ downwards?
- What is the acceleration of the stone if $F = 5$ upwards?



Answer

(a) Although there is no force F on the stone, there is still the weight $W = mg$. According to the second law $F_{\text{net}} = ma$, if we take $W = mg$ for F_{net} , we get $a = g$. In other words, the object will continue its free fall with acceleration g .

(b) This time, in addition to the weight W , there is a force F

in the same direction. Using the second law,

$$F_{\text{net}} = W + F = ma$$

$$mg + F = ma \rightarrow a = g + F/m = 10 + 5/2 = 12.5 \text{ m/s}^2$$

(c) Writing the second law for the weight W and the force F in the opposite direction,

$$mg - F = ma \rightarrow a = g - F/m = 10 - 5/2 = 7.5 \text{ m/s}^2.$$

Normal Force On Surfaces (\vec{N})

Consider a book on a desk (Figure 4.7). We know that the book has a weight and is pulled towards the center of the Earth with a force $W=mg$. If this force were acting alone, then the book would have to go through the table and accelerate downwards according to the $F=ma$ rule.

However, the book is motionless, in other words, has zero acceleration. Therefore, there should be another force acting upwards, opposite to the weight, so that the net force can be zero. This force acting perpendicular to the surface and always outwards from the surface is called the **normal force** and is indicated by \vec{N} . It only appears when the object is in contact with the surface and disappears when the objects leaves the surface. The source of the normal force is the complex interactions between the molecules that constitute the table and the book.

Normal force is always perpendicular to the surface and large enough to prevent the object from entering the surface. For example, in the case of the book on the table, we had $N=W$ (Figure 4.7). Now, let us consider an additional force F pressing the book down towards the table. This time, the normal force will increase to a value that counters both the W and F forces, in other words, we have $N = W + F$.

The following question could come to mind here: “Doesn’t the normal force have a reaction force?” Yes, it does. Indeed, there is an opposite force ($-N$) applied by the book on the table. However, as we are only concerned with the forces acting on the book, we did not deal with this ($-N$) force applied on the table.

This ($-N$) force is what is measured when you step on a bathroom scale. If someone presses down on your shoulder at that moment, the scale will add this to your weight.

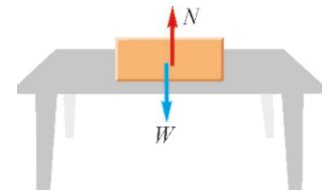
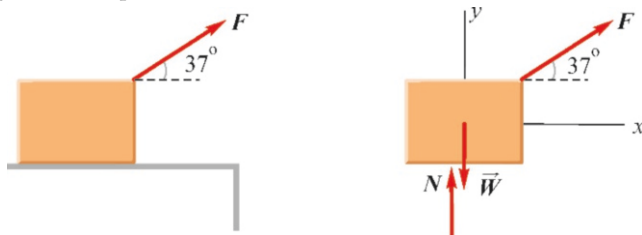


Figure 4.7: The normal reaction force N generated on the surface of the table is always perpendicular to the surface.

Example 4.2

A force of $F = 12 \text{ N}$ is applied at an angle of 37° with the horizontal, on an object with mass $m = 3 \text{ kg}$ located on a frictionless plane.



- (a) Show all of the forces acting on the object.
 (b) What is the acceleration of the object?

(c) What is the normal reaction force of the table?

Answer

(a) In addition to the force \vec{F} applied on the object, there is the downward weight $W = mg$ and the vertical reaction force N applied by the surface. These forces are shown in the figure. (b) If we choose the axes as shown in the figure, the acceleration will be in the x -direction. Since there is no acceleration perpendicular to the table, the net force component is zero in the y -direction. Therefore, the second law can be written as follows for the components:

$$\vec{F}_{\text{net}} = m\vec{a} \rightarrow \begin{cases} F_{x,\text{net}} = ma \\ F_{y,\text{net}} = 0 \end{cases}$$

The first equation gives the acceleration. In the x -direction there is only $F_x = F \cos \theta$:

$$F \cos 37^\circ = ma \rightarrow 12 \times 0.8 = 3a \rightarrow a = 3.2 \text{ m/s}^2$$

F should also be included:
 $N + F \sin 37^\circ - mg = 0 \rightarrow N + 12 \times 0.6 - 3 \times 10 = 0$
 $N = 22.8 \text{ newton}$

(c) The vertical component gives the normal reaction force N .
 Here, in addition to the weight W , the vertical component of F is also included. As you can see, the reaction N may not always equal mg .

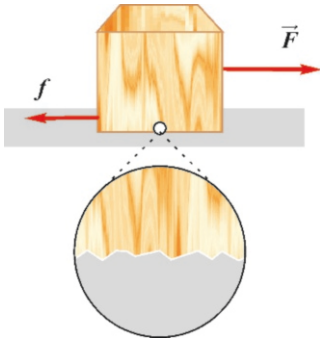


Figure 4.8: Friction force is caused by the fact that surfaces are rough.

Friction Force

Let us again consider a book on a tabletop. Let us push this book with a force F that is parallel to the table. Will the book move along the table? The answer depends on the magnitude of the force. We know from experience that there will be a friction force preventing motion and that it will be necessary to apply a minimum force that is large enough to overcome the friction force.

We experience friction force in every aspect of our daily lives. In fact, we can say that life would be impossible without friction. We would be unable to walk on the ground or hold anything in our hands. On the other hand, many technologies became possible through the reduction of friction. For example, without oil in the engine of an automobile, it would be necessary to replace its cylinders and pistons before it had traveled barely one kilometer.

Similar to the normal force, friction force arises only upon contact with a surface. Both are a type of force called **contact force**. If we look closely at the cross-section of the interface between the book and the table (Figure 4.8), we will observe rough edges at the microscopic scale, even if they are perfectly polished. The atoms and chemical broken bonds at the tips of these rough surfaces immediately interact and make weak bonds. When the object moves, these bonds are broken and then immediately reformed along the path. They thus oppose the motion.

Forces that oppose motion are also formed in liquids and gases. These have different characteristics. We shall only deal with the friction force between solid surfaces.

Experiments have demonstrated the following features of friction force:

- Friction force is approximately the same regardless of which surface of the object is in contact if all surfaces have the same roughness.
- The friction force is approximately constant regardless of the velocity of the object.

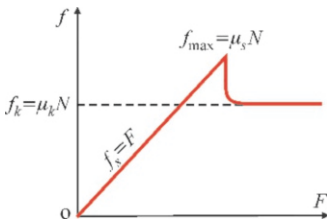


Figure 4.9: Friction force acts differently in static and kinetic cases.

Figure 4.9 shows the behavior of friction force as the applied force F increases. When F is small, the friction force f adjusts itself in the opposite direction so as to fully counter it, and the object does not move. This balancing continues until it reaches a maximum value f_{\max} and it can no longer counter the force F , whereupon motion starts. Experiments show that this maximum value of the friction force is proportional to the normal force on the surface:

$$f_{\max} = \mu N \quad (4.10)$$

where the dimensionless coefficient μ is the **coefficient of friction** and depends on the type and roughness of the surfaces.

The maximum value of the friction force slightly decreases after the object starts moving (Figure 4.9). In order to describe this behavior, a **coefficient of kinetic friction** μ_k is defined for moving objects and a **coefficient of static friction** μ_s for objects at rest:

$$\begin{aligned} 0 \leq f_s \leq \mu_s N & \quad (\text{if the object is at rest}) \\ f_k = \mu_k N & \quad (\text{if the object is moving}) \end{aligned} \quad (4.11)$$

Static and kinetic coefficients of friction measured between certain surfaces are given in the following table.

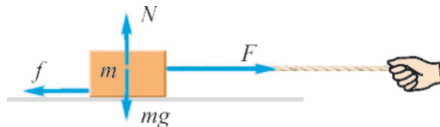
Friction coefficients of certain surfaces		
Surface	Static friction, μ_S	Kinetic friction, μ_K
Wood–wood	0.35	0.30
Steel–steel	0.80	0.50
Steel–ice	0.1	0.05
Rubber–dry asphalt	1.0	0.8
Rubber–wet asphalt	0.7	0.5

These values are approximate and may vary depending on whether the surfaces are rough or wet. For this reason, in the problems in this book, no static-kinetic distinction shall be made and only the coefficient μ shall be used.



Figure 4.10: The inclination angle of coal piles depends on friction force. If they get wet in a rainfall, they may become dangerous.

Example 4.3



A block with mass $m=3\text{ kg}$ is on a horizontal plane. The static and kinetic coefficients of friction between the block and the surface are equal and have the value $\mu = 0.5$.

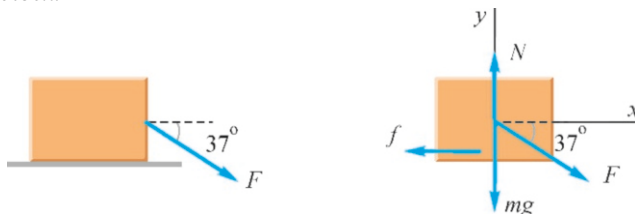
- A horizontal force $F_1 = 12\text{ N}$ is applied on this block. What is the friction force?
- This time, a horizontal force $F_2 = 18\text{ N}$ is applied. What is the acceleration of the block?

Answer

The forces acting on the block are shown in the figure. As

Example 4.4

A block with mass $m = 3\text{ kg}$ is on an inclined plane. The static and kinetic coefficients of friction between the block and the surface are equal and have the value $\mu = 0.3$. A force $F = 20\text{ N}$ with an angle 37° below the horizontal is applied on this block. Calculate the normal reaction force and the acceleration of the block.



Answer

The forces acting on the block are shown in the figure. If we write the second law in terms of the components,

we only have the weight W and the reaction force N in the vertical direction, we get $N = W = mg$. We first calculate the maximum value of the friction force:

$$f_{\max} = \mu N = \mu mg = 0.5 \times 3 \times 10 = 15\text{ N}$$

(a) The force $F_1 = 12\text{ N}$ is less than the maximum value of the friction force. Therefore, the friction force fully counters F and the block does not move:

$$f = 12\text{ N}$$

(b) The block will move because the force $F_2 = 18\text{ N}$ is greater than the maximum value of the friction force. The friction force is at maximum value during motion:

$$f = f_{\max}$$

We can find the acceleration using the second law:

$$F_{\text{net}} = ma \rightarrow F_2 - f = ma$$

$$18 - 15 = 3a \rightarrow a = 1\text{ m/s}^2$$

$$F_{x,\text{net}} = F \cos 37^\circ - f = ma$$

$$F_{y,\text{net}} = N - F \sin 37^\circ - mg = 0$$

The reaction force N is calculated from the second equation:

$$N = mg + F \sin 37^\circ = 30 + 20 \times 0.6 = 42\text{ N}$$

Now, we must determine if there will be motion. For this purpose, it is necessary to compare the $F \cos \theta$ component of the applied force along the direction of motion with the maximum value of the friction force:

$$F \cos 37^\circ = 20 \times 0.8 = 16\text{ N}$$

$$f = \mu N = 0.3 \times 42 = 12.6\text{ N}$$

The block will start moving, because $F \cos \theta > f$. We can find the acceleration by plugging these values into the expression above for the x -component of the second law:

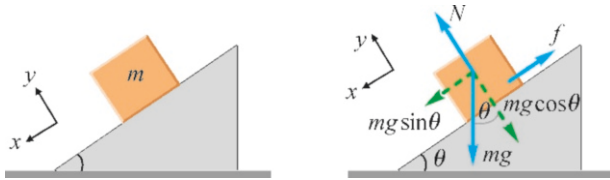
$$F \cos 37^\circ - f = ma \rightarrow 16 - 12.6 = 3a$$

$$a = 1.1\text{ m/s}^2$$

Example 4.5

A block of mass $m = 5 \text{ kg}$ is placed on a plane inclined at 37° . The static and kinetic coefficients of friction between the inclined plane and the object are equal and have the value $\mu = 0.2$.

- (a) Show the forces acting on the object on a figure.
 (b) What is the friction force? Accordingly, will the object move? If so, what is its acceleration?



Answer

(a) The three forces acting on the block are shown in the figure. Normal force N is perpendicular to the inclined plane. As the object will try to slip downwards, the friction force f

must be opposite, hence upward along the inclined plane.

For the coordinate axes, we take the x -axis along the inclined plane and towards the direction of motion, in other words, downwards. The y -axis is taken as perpendicular to the inclined plane.

(b) First, the y -component of the second law is written so as to find the friction force from the normal reaction force:

$$F_{y,\text{net}} = N - mg \cos 37^\circ = 0 \rightarrow N = mg \cos 37^\circ$$

$$f_{\text{max}} = \mu N = \mu mg \cos 37^\circ = 0.2 \times 5 \times 10 \times 0.8 = 8 \text{ N}$$

The force moving the object is the component of weight along the inclined plane:

$$mg \sin 37^\circ = 50 \times 0.6 = 30 \text{ N} > f_{\text{max}}, \text{ hence the object will move.}$$

The x -component of the second law is used to find the acceleration:

$$F_{x,\text{net}} = mg \sin 37^\circ - f = ma \rightarrow 30 - 8 = 5a$$

$$a = 4.4 \text{ m/s}^2$$

Tension Forces on Ropes

The type of force applied by *flexible* objects such as ropes, cordes or cables is called the tension force (or, simply, the **tension**). The most important feature of the tension force is that it can only pull the object. You cannot push objects with a rope, because it will bend quickly.

Let us hang a block with mass m from the ceiling by means of a flexible rope (Figure 4.11). As this block is at rest, this means that the weight mg is balanced by a tension T that is equal and opposite: $T=mg$. When a second mass is added, the tension increases to balance the sum of their weights. Therefore, the tension is an attractive force that always establishes balance, as long as the rope is not broken.

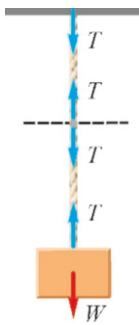


Figure 4.11: Tension on a rope.

Now, let us proceed towards the end of the rope hung from the ceiling. According to the third law, the lower and upper parts in any cross-section of the rope will pull each other with equal tension forces in opposite directions. The tension force at this height is usually not equal to the one at the lower end. This is because the rope also has a mass, and this tension force is always equal to the weight of the sum (object + rope below).

However, if the mass of the rope is small enough to be negligible, each cross-section will keep the same $T=mg$ value, and thus, when we reach the ceiling, the tension on this end will also be T . Therefore, *the tensions on both ends of a rope with zero mass are equal*. The tensions on both ends remain equal even if the rope is moving.

Free-Body Diagrams

In dynamics, it is important to clearly state the examined system. If there are multiple forces acting on multiple masses (Figure 4.12), these forces may be separated into two groups:

1. **Internal forces:** Forces applied by the masses inside of the examined system on each other (T_1, T_2, T_3 in the figure). According to Newton's third law, these forces will always be in pairs. These are not included in the equations, because they are both positive and negative and cancel each other out.

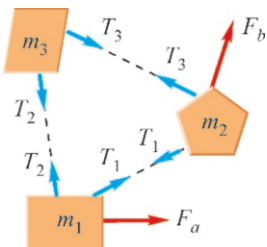


Figure 4.12: Internal and external forces acting on 3 masses.

2. **External forces:** Forces originating from outside of the system (F_a , F_b in the figure).

In order to better understand the issue, let us consider the forces acting on masses m_1, m_2, m_3 separately:

- If we are only considering the mass m_1 as our system, each of the forces acting on it (T_1, T_2, F_a) are external forces (Figure 4.13) and should be included in equations.
- If we are considering the masses together (m_2+m_3) as our system, this time, the external forces are (T_1, T_2, F_b). In this system, T_3 is an internal force and is not taken into consideration, because it will appear twice in the equations in opposite directions, hence canceling itself out.
- If the system is the total mass ($m_1+m_2+m_3$), then only F_a and F_b are external forces and the force pairs (T_1, T_2, T_3) applied by the masses to each other become internal forces and are not included in the equations.

After taking into consideration all of the external forces acting on a system, we can show it as isolated, ignoring all other objects in its environment. This is called a **free-body diagram** (Figure 4.13). For example, after showing the forces T_1, T_2, F_a acting on mass m_1 in the same system with three masses, we no longer need to show the other masses, because we have already taken their effect into consideration.

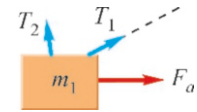


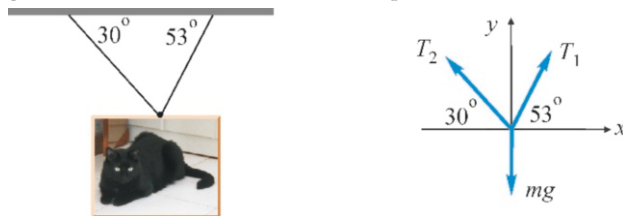
Figure 4.13: The free-body diagram of mass m_1 .

4.3 APPLICATIONS OF NEWTON'S LAWS

In this section, we will use Newton's laws to solve examples of motion that objects may undergo under the action of various forces.

Example 4.6

A painting frame with mass $m = 5$ kg is held in equilibrium, hanging on the wall by two ropes with the angles given in the figure. Calculate the tensions in the ropes.



Answer

Each rope applies an attractive tension along itself. These are shown as T_1 and T_2 in the figure. We apply the first law, as the object is at rest:

$$\vec{F}_{\text{net}} = 0$$

This equality also applies to the components:

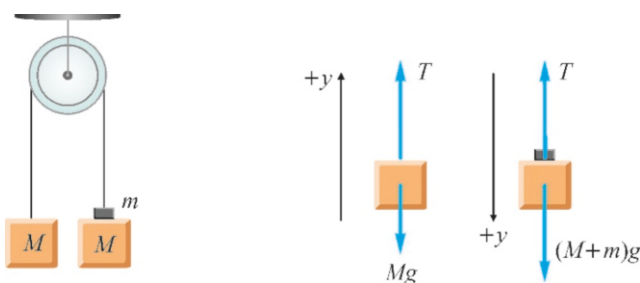
$$F_{x,\text{net}} = 0 \rightarrow T_1 \cos 53^\circ - T_2 \cos 30^\circ = 0$$

$$F_{y,\text{net}} = 0 \rightarrow T_1 \sin 53^\circ + T_2 \sin 30^\circ - mg = 0$$

We find T_1 and T_2 from these two equations:

$$T_1 = 44 \text{ N and } T_2 = 30 \text{ N.}$$

Example 4.7



Atwood's machine. Two equal blocks with masses $M=10$ kg are hung on two ends of a rope passing through a pulley. The pulley is frictionless and the mass of the rope is negligible. An additional small mass $m=1$ kg is placed on one of the blocks.

- Show the forces acting on both blocks in a diagram. (You do not have to show the forces acting on the pulley.)
- For each block, write the equations that give the acceleration and the tension in the rope.

- (c) Solve this system of equations to calculate acceleration a and tension T .

Answer

(a) The tensions on both ends of the rope are equal, because it is assumed that the mass of the rope is negligible and the pulley is frictionless. For the same reasons, both masses will accelerate with the same acceleration a .

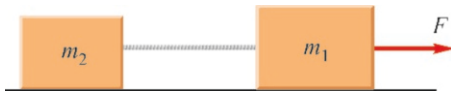
Hence, the forces are as shown in the figure.

(b) The problem is one-dimensional. We apply the second law in the vertical direction to each mass, in their direction of motion:

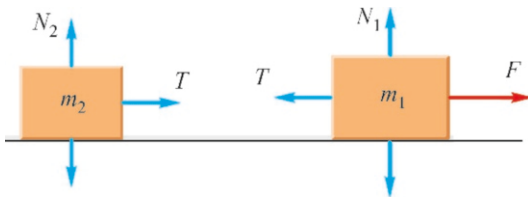
$$F_{\text{net}} = ma$$

Example 4.8

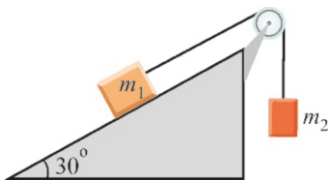
Two blocks with masses $m_1=1$ kg and $m_2=2$ kg on a frictionless horizontal plane are tied to each other with a rope. A horizontal force $F=5$ N is applied on mass m_1 .



- (a) Draw a free-body diagram for each block.
 (b) Find the acceleration and the tension in the rope by separately considering the blocks m_1 and m_2 .
 (c) Consider the blocks together and calculate the acceleration of the $(m_1 + m_2)$ system.



Example 4.9



A block of mass $m_1 = 7$ kg on a frictionless inclined plane with a slope of 30° is tied to the end of a rope that passes through a pulley. A mass $m_2 = 5$ kg is hanging on the other end of the rope.

- (a) Draw a free-body diagram for each block and determine the direction of motion.
 (b) Separately, write the equations of motion for the masses m_1 and m_2 . Calculate the acceleration and the tension force in the rope.

$$\text{For mass } (M + m) : \quad (M + m)g - T = (M + m)a$$

$$110 - T = 11a$$

$$\text{For mass } M : \quad T - Mg = Ma$$

$$T - 100 = 10a$$

- (c) By solving these two equations, we find acceleration a and tension T :

$$a = 0.5 \text{ m/s}^2 \text{ and } T = 105 \text{ N}$$

Note: This setup is used in operating elevators and funiculars. A small mass m can be used to easily accelerate a larger mass M . If we had attempted to pull only the mass M upwards with the same acceleration a , we would have had to use a force F much larger than the force mg here.

Answer (a) The two ends of the rope have equal tension T in opposite directions. The normal reaction forces cancel out the weights in the vertical direction. The forces are as shown in the figure above.

(b) The second law is written in the direction of motion for each block (it is not necessary to write the expressions in the vertical direction, because there is no friction):

$$m_1 \text{ mass : } \quad F - T = m_1 a \quad 5 - T = a$$

$$m_2 \text{ mass : } \quad T = m_2 a \quad T = 2a$$

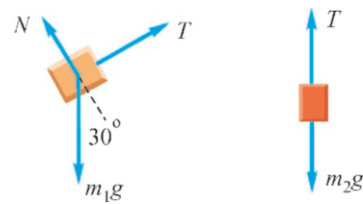
From here, we find the acceleration and the tension:

$$a = 1.7 \text{ m/s}^2 \text{ and } T = 3.3 \text{ N}$$

(c) This problem can also be solved by considering the two masses together. This is possible because the accelerations are equal. Now, the tensions T and $-T$ in the rope in the two-mass system become internal forces, and, according to third law, they cancel each other out. Thus, the acceleration of the whole system is only due to the force F :

$$F = (m_1 + m_2)a \rightarrow a = 5/3 = 1.7 \text{ m/s}^2$$

It is not possible to find the tension T in this system, because T is an internal force.



Answer

- (a) The forces acting on each block are shown above.

Consider first the system as a whole. On one hand, the mass m_2g will try to pull the mass m_2 down and, on the other hand, the component of mass m_1 parallel to the inclined plane, $m_1g \sin \theta$ will try to pull it down.

Let us calculate which of these forces will become dominant:

$$m_1g \sin 30 = 7 \times 10 \times 0.5 = 35 \text{ N}$$

$$m_2g = 5 \times 10 = 50 \text{ N}$$

Since $m_2g > m_1g \sin \theta$, the motion will be downwards for m_2 and upwards along the inclined plane for m_1 .

(b) Since there is no friction, it is sufficient to use the second law only for the component along the motion:

$$\begin{aligned} \text{for } m_1 : \quad T - m_1 g \sin \theta &= m_1 a & T - 35 &= 7a \\ \text{for } m_2 : \quad m_2 g - T &= m_2 a & 50 - T &= 5a \end{aligned}$$

Adding both sides of this equation, we find acceleration a as

follows:

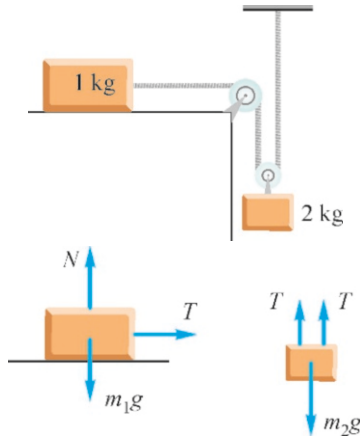
$$50 - 35 = 12a \rightarrow a = 1.25 \text{ m/s}^2$$

Plugging this value into either one of the equations, we get T :

$$T = 44 \text{ N.}$$

Example 4.10

The rope in the figure has zero mass and the surfaces and the pulley are frictionless. (a) Draw a free-body diagram for each block. (b) What is the relation between the accelerations? (c) Calculate the accelerations and the tension in the rope.



Answer

(a) In this problem, the accelerations of the blocks are different. The forces acting on each one are shown in the figure.

(b) The acceleration of the mass m_2 tied to the pulley will be less, because, as it descends, some of the rope will be transferred to the other side. For example, how much does mass m_2 descend when mass m_1 goes to the right by 1 m? The answer is 1/2 m. Therefore, the relationship is as follows:

$$a_2 = a_1/2$$

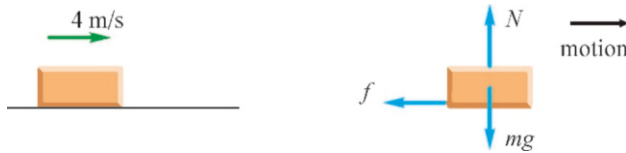
(c) We use the second law for each mass along its direction of motion:

$$\begin{aligned} \text{for } m_1 : \quad T &= m_1 a_1 & \rightarrow & \quad T = a_1 \\ \text{for } m_2 : \quad m_2 g - 2T &= m_2 a_2 & \rightarrow & \quad 20 - 2T = 2a_2 \end{aligned}$$

There are three unknowns in these two equations: a_1, a_2, T . The third equation that we need for the solution is the relation $a_1 = 2a_2$ between the accelerations that we found in item (b). Solving these three equations, the results are as follows:

$$a_1 = 6.7 \text{ m/s}^2, \quad a_2 = 3.3 \text{ m/s}^2 \quad \text{and} \quad T = 6.7 \text{ N.}$$

Example 4.11



A block with mass m is thrown with a speed of 4 m/s along a horizontal plane on which the coefficient of friction is $\mu=0.25$. What distance will the block travel before it stops?

Answer

In this problem, there is no force acting in the direction of motion, rather the block moves due to its initial speed, and then it slows down due to the friction force f acting in the

opposite direction. The forces acting on the object are shown in the figure.

Let us write the equations of motion:

$$\begin{aligned} F_{x,\text{net}} &= ma & \rightarrow & \quad -f = ma \\ F_{y,\text{net}} &= 0 & \rightarrow & \quad N - mg = 0 \end{aligned}$$

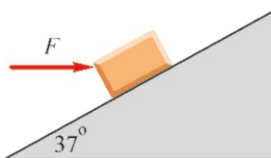
With the friction force $f = \mu N$ and $N = \mu mg$, the acceleration a is found to be negative:

$$a = -\mu g = -0.25 \times 10 = -2.5 \text{ m/s}^2$$

Using the velocity formula without time from kinematics, the distance traveled until the object stops ($v = 0$) is calculated as follows:

$$\begin{aligned} v^2 - v_0^2 &= 2ax & \rightarrow & \quad 0 - 4^2 = 2(-2.5)x \\ x &= 3.2 \text{ m.} \end{aligned}$$

Example 4.12



A block of mass $m = 1 \text{ kg}$ is placed on an inclined plane with a coefficient of friction of $\mu = 0.3$ and incline angle of 37° . A horizontal force $F = 20 \text{ N}$ is applied on the block. (a) In what direction will the block move? Using this information, show the

forces acting on the block. (b) Calculate the acceleration of the block.

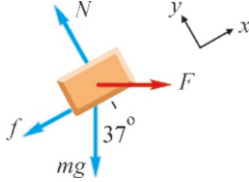
Answer

(a) The friction force will be in the opposite direction to that of the motion, hence we must first determine in which direction the block will move. Two forces come into play along the inclined plane: The component $mg \sin \theta$ of its weight pulling the block downwards, and the component $F \cos \theta$ pushing it upwards. Let us compare these two:

$$mg \sin 37^\circ = 10 \times 0.6 = 6 \text{ N}$$

$$F \cos 37^\circ = 20 \times 0.8 = 16 \text{ N}$$

As $F \cos \theta$ is greater, the block will try to go upward. It may not be able to move, due to the friction force, but, in any case it will not go downwards. The forces acting on the block are thus as follows:



(b) If the x -axis is taken upwards along the inclined plane

and the y -axis is taken as perpendicular to the plane, the equations of motion will be as follows:

$$F_{x,\text{net}} = ma \rightarrow F \cos 37^\circ - mg \sin 37^\circ - f = ma$$

$$F_{y,\text{net}} = 0 \rightarrow N - mg \cos 37^\circ - F \sin 37^\circ = 0$$

and we have the relation between friction and the normal force: $f = \mu N$.

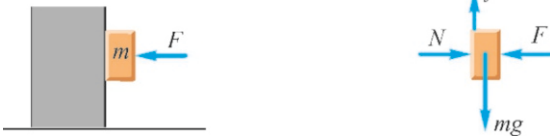
We calculate acceleration a from these three equations:

$$a = \frac{(F \cos 37^\circ - mg \sin 37^\circ) - \mu(mg \cos 37^\circ + F \sin 37^\circ)}{m}$$

$$a = 4 \text{ m/s}^2$$

If the resulting acceleration were not positive here, we would conclude that the block was unable to move.

Example 4.13



A block with a mass of 2 kg can slide downwards when pressed by a horizontal force $F = 30 \text{ N}$ against a wall. The coefficient of friction between the block and the wall is $\mu = 0.5$. Show the forces acting on the block and find its acceleration.

Answer

This is a good example that shows that the normal force is

not always equal to mg . By definition, the normal reaction force is one that counters all of the forces perpendicular to the surface. In this problem, mg has no component perpendicular to the surface; there is only the force F . Therefore, we have

$$N - F = 0 \rightarrow N = F$$

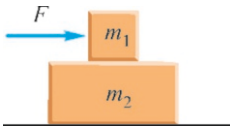
The block can only move downwards. The equations of motion in this direction are as follows:

$$mg - f = ma$$

Again, as the friction force is defined with the formula $f = \mu N$, we get $f = \mu F$. We find the acceleration a from these two equations:

$$a = (mg - \mu F)/m = (20 - 0.5 \times 30)/2 = 2.5 \text{ m/s}^2.$$

Example 4.14



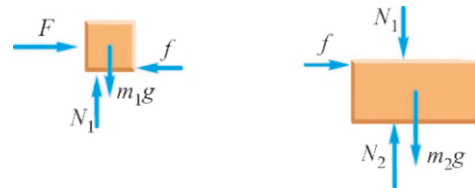
The friction force between the blocks with masses $m_1 = 1 \text{ kg}$ and $m_2 = 2 \text{ kg}$ shown in the figure is $\mu = 0.3$. There is no friction between the lower block m_2 and the horizontal plane underneath. A horizontal force F is applied on the block m_1 .

- (a) Show the forces acting on the blocks in separate diagrams.
 (b) What is the maximum acceleration that the two blocks can travel together without causing the upper block m_1 to slip?
 (c) What is the force F that provides this maximum acceleration?

Answer

(a) This example shows that friction may not always be in the opposite direction to the motion, but may sometimes assist motion. Friction is only between two blocks. As the mass m_1 will try to move to the right compared to the lower mass, the force ($-f$) preventing it will be towards the left. However, according to the third law, a force $+f$ equal and opposite to

this will act on the lower block. This opposite force f is the force that makes m_2 move. Therefore, the forces acting on the objects are as follows:



- (b) Let us write the equations of motion for the lower block:

$$f = m_2 a$$

This friction force f has a maximum value: $f_{\text{max}} = \mu N_1 = \mu m_1 g$. This is the maximum force that will act in the direction of motion on the lower mass. From here, we find acceleration a :

$$\mu m_1 g = m_2 a \rightarrow a = (m_1/m_2)\mu g = 0.5 \times 0.3 \times 10 = 1.5 \text{ m/s}^2$$

Therefore, the maximum acceleration that both masses can travel together will be $a = 1.5 \text{ m/s}^2$.

- (c) We also use this maximum acceleration for the upper block m_1 and find the maximum force F :

$$F - f = m_1 a \rightarrow F = \mu m_1 g + m_1 a = 4.5 \text{ N}.$$

4.4 CIRCULAR MOTION

In Chapter 3, we had examined circular motion and established that an object rotating with a constant speed v around a circle with radius r would get a centripetal acceleration a_r towards the center:

$$a_r = \frac{v^2}{r} \quad (\text{centripetal acceleration}) \quad (4.12)$$

According to the second law, there must be a force F_r acting in the same direction as this acceleration, in other words, towards the center (Figure 4.14):

$$F_r = ma_r = m\frac{v^2}{r} \quad (\text{centripetal force}) \quad (4.13)$$

This **centripetal force** F_r is the *net* force in the radial direction causing the acceleration a_r . It could be achieved in various ways. For example, if we are swirling a stone tied to a rope, it will be the tension T in the rope. If a cart rotates on rails, it will be the normal reaction force of the rails.

You may still wonder whether there is any need for such a force while the object is rotating. Let us consider that the rope suddenly breaks while the object is rotating (Figure 4.15). After that moment, according to the first law, the object will travel along a linear path, since the tension T causing the rotation is no longer present.

As a car is turning around a bend, the centripetal force is caused by the friction force on the tires. Likewise, for an object rotating inside of a railed circle, it will be the normal reaction force N of the rails: $N = mv^2/r$.

Another example is the gravitational force that allows planets to rotate around the Sun. For example, if we use the gravitational force for a satellite with mass m rotating around the Earth on an orbit with radius r , we get

$$F_r = G\frac{mM_E}{r^2} = m\frac{v^2}{r}$$

Using this equation, we can find the velocity of the satellite at any height.

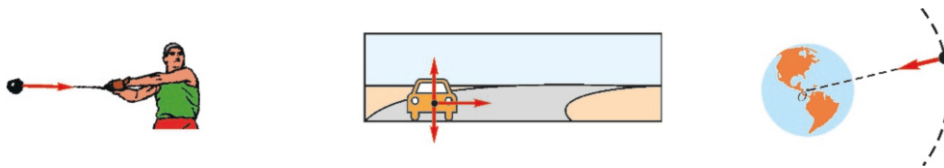


Figure 4.16: Examples of centripetal force.

Let us correct a common misunderstanding here: In many books, a *centrifugal force* is mentioned as the force acting outwards from the center. Students may readily accept this instinctively. Indeed, you may think that, “When the bus enters a curve, there is a force pushing me outwards from the center,” right?

Such a thinking is incorrect. The correct interpretation is as follows: *You are not pushed outwards when the bus enters a circular trajectory. If you do not hold onto anything, you would continue to travel in a straight line. Actually, it is the side of the bus turning inwards that is approaching you.* Consider also the following: When you hold on to a handlebar as the bus turns, in which direction is the handlebar applying a force on you? Towards the center, of course.

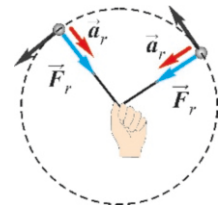


Figure 4.14: There must be a centripetal force causing the centripetal acceleration.

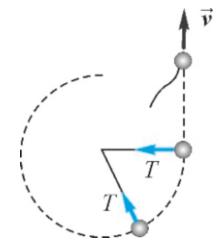


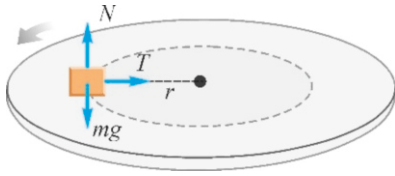
Figure 4.15: What will happen if the force causing centripetal acceleration is removed?

The concept of centrifugal force is a technique developed to solve certain problems as statics. When writing all of the forces acting on a rotating object, an outward force of magnitude mv^2/r is added and then the sum of all forces is set as equal to zero, as if the object is in equilibrium. But this has nothing to do with reality. This is because, *an observer rotating together with the object does not constitute an inertial reference frame*, and thus Newton's laws do not apply there. Therefore, we should refrain from using the term centrifugal force.

Example 4.15

A block with mass $m = 200$ g rotating with constant speed at the end of a rope of length 80 cm on a frictionless horizontal plane makes 500 revolutions in 3 minutes.

Calculate the centripetal acceleration of the block and the tension in the rope.



Answer

The force towards the center causes the circular motion. As seen in the figure, the object's weight and the reaction force

N of the surface have no components towards the center; there is only the force T . Therefore,

$$F_r = ma_r \rightarrow T = m \frac{v^2}{r}.$$

We must first calculate the velocity v . As one rotation around a circle with radius r will have a length of $2\pi r$, we divide the total distance by time:

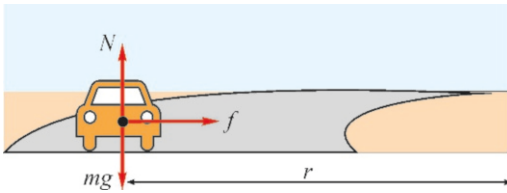
$$v = \frac{\text{distance traveled}}{\text{time duration}} = \frac{500 \times 2\pi \times 0.80 \text{ m}}{3 \times 60 \text{ s}} = 14 \text{ m/s}$$

From here, we calculate the centripetal acceleration and the tension in the rope:

$$a_r = \frac{v^2}{r} = \frac{14^2}{0.8} = 245 \text{ m/s}^2,$$

$$T = ma_r = 0.2 \times 245 = 49 \text{ N}.$$

Example 4.16



(a) An automobile with mass 1200 kg is traveling on a horizontal circular track of radius $r = 200$ m at a speed of 72 km/h. What is the friction force between the wheels and the road?

(b) If the coefficient of friction on this track is $\mu = 0.8$, what will the maximum speed be at which the automobile can turn without sliding?

Answer

(a) As seen in the figure, the friction force f causes the centripetal acceleration:

$$f = m \frac{v^2}{r} = 1200 \times \frac{(72 \times 1000/3600 \text{ m/s})^2}{200} = 2400 \text{ N}$$

(b) The maximum value of the friction force is $f_{\max} = \mu N = \mu mg$, hence it is the maximum force directed towards the center. We can write the centripetal acceleration caused by this force as follows:

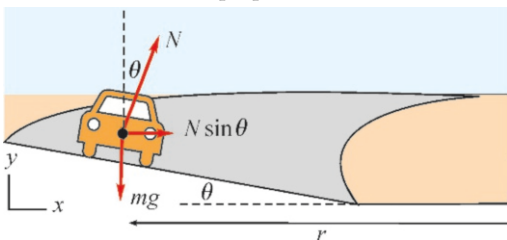
$$\mu mg = m \frac{v^2}{r} \rightarrow v^2 = \mu gr.$$

This maximum velocity is calculated as follows:

$$v = \sqrt{\mu gr} = \sqrt{0.8 \times 10 \times 200} = 40 \text{ m/s} = 144 \text{ km/hour}.$$

Example 4.17

A car rounds a frictionless curve banked at an angle of 30° and with a radius of $r = 100$ m. Calculate the necessary speed v so that the car will not slip up or down the incline.



Answer

This time, there is no friction. So, which is the centripetal force that keeps it on a circular path? As seen in the figure, the normal reaction force N of the road will have a component towards the center. This is the centripetal force that causes rotation.

We write the equations of motion along the x - and y -directions shown in the figure:

$$x\text{-direction: } N \sin \theta = mv^2/r$$

$$y\text{-direction: } N \cos \theta = mg$$

Dividing the two equations on both sides, we get

$$\tan \theta = v^2/gr$$

From this, we calculate the necessary speed:

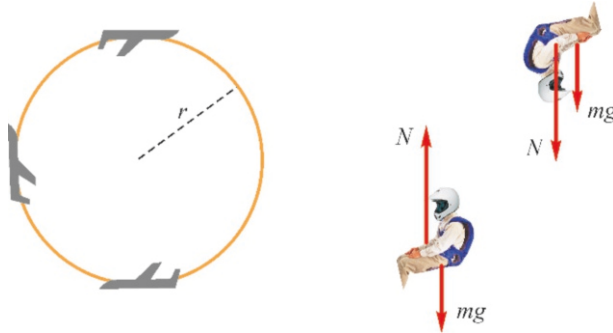
$$v = \sqrt{gr \tan 30^\circ} = \sqrt{10 \times 100 \times 0.58}$$

$$v = 24 \text{ m/s} = 87 \text{ km/hour.}$$

Example 4.18

The pilot of an airplane executes a loop-the-loop stunt at a constant speed of 300 m/s in a vertical circle of radius 2 km.

- (a) Calculate the centripetal acceleration of the airplane.
- (b) Calculate the normal reaction force applied by the seat to the pilot with mass m at the top and bottom of the circle in terms of multiples of the weight mg of the pilot.



In this problem, the centripetal force is the combination of the weight mg of the pilot and the normal force N of the seat. (The centripetal force rotating the airplane is the force generated by the wings, but we only consider the pilot here.)

(a) First, let us calculate the centripetal acceleration in terms of multiples of g :

$$a_r = v^2/r = 300^2/2000 = 45 \text{ m/s}^2 = 4.5g$$

(b) As seen in the figure, the reaction force N of the seat is always towards the center at the top and bottom of the circle. However, the weight mg is towards the center at the top, but outwards from the center at the bottom. Therefore, the resultant force towards the center will be written in both cases:

At the top:

$$N + mg = ma_r \rightarrow N = ma_r - mg = 4.5mg - mg$$

$$N = 3.5mg$$

At the bottom:

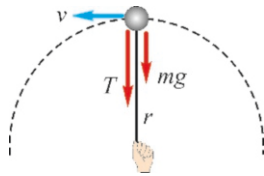
$$N - mg = ma_r \rightarrow N = ma_r + mg = 4.5mg + mg$$

$$N = 5.5mg$$

Clearly, the most difficult part of the pilot's maneuver is at the bottom of the circle.

Answer

Example 4.19



A stone tied to the end of a string with radius r is being rotated around in a vertical circle. What should the speed of the stone be at the top so that it will complete the circle without the string getting loose there?

Answer

Both the weight mg of the stone and the tension T of the string will be towards the center at the top of the circular

motion of the stone.

Therefore, the centripetal force should consist of the sum of these two:

$$mg + T = m \frac{v^2}{r}$$

What will happen if we reduce the stone's speed v ? The right side of this expression will decrease. Since weight is constant on the left side, tension T will also decrease. When a certain speed is reached, we get $T = 0$, in other words, the string will get loose.

Therefore, the minimum velocity v will be the velocity at $T = 0$:

$$mg + 0 = m \frac{v^2}{r} \rightarrow v = \sqrt{gr}$$

Multiple-choice Questions

1. A bicycle and truck collide head-on. Which of the following is correct?
 - (a) The force on the bicycle is larger.
 - (b) The force on the truck is larger.
 - (c) The forces are equal and opposite.
 - (d) The forces are equal and perpendicular.
2. Which is the force that makes a falling ball bounce back up from the ground?
 - (a) Gravitational force
 - (b) Weight
 - (c) Friction force
 - (d) Normal reaction force

3. Which of the following is incorrect?
- The net force on a car at rest is zero.
 - The net force on a car traveling at constant speed is zero.
 - The acceleration of a car that gets faster is zero.
 - The acceleration of a car traveling in a straight line at constant velocity is zero.
4. A car applies brakes and stops. Which force makes the car stop?
- Weight
 - Friction force
 - Normal force on the surface
 - Gravitational force
5. Which is correct at the maximum height of a stone in projectile motion?
- The net force on the stone is zero.
 - There is a horizontal force acting on the stone.
 - There is a gravitational acceleration on the stone.
 - The acceleration of the stone is zero.
6. On a tabletop with a friction force of 6 N, what will the acceleration be of an object with mass 2 kg pulled horizontally with a force of 8 N in units of m/s^2 ?
- 1
 - 2
 - 3
 - 4
7. Passengers on a train with no windows and traveling at constant speed observe that a ball on the floor starts to roll forward. Which of the following could be true?
- The train could be slowing down.
 - The train could be getting faster.
 - The train could be going uphill.
 - The train could be going downhill.
- I & III
 - I & IV
 - II & III
 - II & IV
8. Two people are playing tug of war. Which of the following is correct?
- The one who pulls stronger wins.
 - The one with the higher mass wins.
 - The one with higher friction force on the ground wins.
 - The one who grabs the rope more tightly wins.
9. For which observers do Newton's laws not apply?
- Observers at rest.
 - Observers in uniform linear motion.
 - Accelerating observers.
 - All of the above.
10. An elevator goes into a free fall when its cable breaks. How would a screw that comes off in the ceiling of the elevator move *with respect to an observer in the elevator*?
- It will accelerate with g .
 - It will fall with a constant velocity.
 - It will remain suspended in the air.
 - It will accelerate upward.
11. Which of the following are incorrect?
- The mass of an object in outer space is zero.
 - The weight of an object in outer space is zero.
 - The mass of an object is less on the Moon.
 - The weight of an object is less on the Moon.
- I & II
 - III & IV
 - I & III
 - II & IV
12. What is the force F applied on an object with mass 1 kg moving vertically upwards with an acceleration of 4 m/s^2 ?
- 6 N
 - 10 N
 - 14 N
 - 18 N
13. A man in an elevator releases the suitcase in his hand and the suitcase does not drop to the ground. Which of the following could be correct?
- The elevator is in free fall.
 - The elevator is descending with constant speed.
 - The elevator is moving downwards with acceleration g .
 - The elevator is moving upwards with acceleration g .
- I & II
 - I & III
 - II & III
 - I & IV
14. Which of the following is incorrect for friction force?
- It is always opposite to the direction of motion.
 - It is always proportional to mg .
 - It is always proportional to the normal reaction force N .
 - It is always perpendicular to the surface.
15. How would a ball tied to the end of a string and rotating on a frictionless plane move when the string breaks?
- It will stop.
 - It will continue rotating.
 - It will move along a straight line.
 - It will move towards the center.
16. An object with mass 10 kg is rotating with a speed of 3 m/s on a circular path with radius 2 m. What is the centripetal force acting on the object in newton units?
- 30
 - 35
 - 40
 - 45
17. What could cause an automobile to round a curve?
- Friction force.
 - The mass of the automobile.
 - The speed of the automobile.
 - The normal force of the road.
- I & II
 - II & III
 - III & IV
 - I & IV

18. If a car is able to turn around a curved road with radius r at a speed v without sliding, at what speed can it turn without sliding around a curve with radius $2r$?
- (a) v (b) $\sqrt{2}v$ (c) $2v$ (d) $4v$
19. What is the ratio of the centripetal forces F_B/F_A of an automobile A with mass m turning around a curve with radius r at speed v and an automobile B with mass $2m$ turning around a curve with radius $2r$ at speed $2v$?

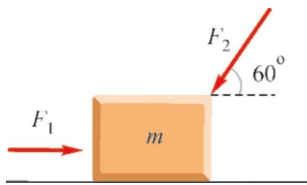
- (a) 1 (b) 2 (c) $\sqrt{2}$ (d) 4
20. When the speed of a rotating stone at the end of a string with length r reaches 10 m/s, the string breaks. If a string with length $4r$ made of the same material is used, at what speed will the rope break (in units of m/s)?
- (a) 10 (b) 20 (c) $10\sqrt{2}$ (d) $10/\sqrt{2}$

Problems

4.3 Applications of Newton's Laws

4.1 A bathroom scale measures the normal reaction force N that it applies on the person standing on it. A boy with a mass of 50 kg is standing on the scale in an elevator. (a) How many newtons (N) will the scale show when the elevator is accelerating upwards at 2 m/s^2 ? (b) What will it show when the elevator is accelerating downwards with the same acceleration?

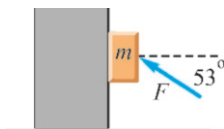
[A: (a) 600 N, (b) 400 N.]



Problem 4.2

4.2 In the figure, the forces $F_1=18 \text{ N}$ and $F_2=20 \text{ N}$ are applied on a block with a mass of $m=1 \text{ kg}$ on a horizontal plane. The coefficient of friction of the plane is $\mu = 0.2$. Calculate the acceleration of the block.

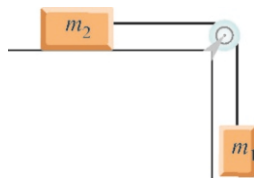
[A: 2.5 m/s^2 .]



Problem 4.3

4.3 A 2 kg block is pressed against a vertical wall with a force $F=30 \text{ N}$ at an angle of 53° with the horizontal. The coefficient of friction between the block and the wall is $\mu=0.2$. In which direction will the block move and what will be its acceleration?

[A: Upwards $a = 0.2 \text{ m/s}^2$.]

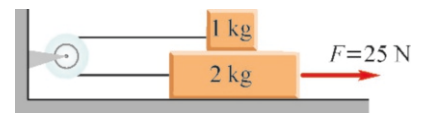


Problem 4.4

4.4 The blocks in the figure with masses $m_1=1 \text{ kg}$ and $m_2=2 \text{ kg}$ have been tied to each other with a rope with negligible mass passing through a frictionless pulley. The coefficient of friction between the horizontal plane and the block m_2

is $\mu=0.3$. Calculate the accelerations of the blocks and the tension in the rope.

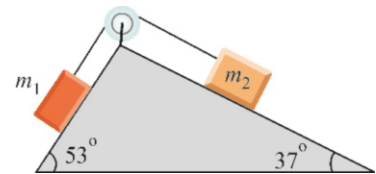
[A: $a = 1.3 \text{ m/s}^2$, $T = 8.7 \text{ N}$.]



Problem 4.5

4.5 The blocks with masses $m_1=1 \text{ kg}$ and $m_2=2 \text{ kg}$ in the figure are tied to each other with a rope passing through a frictionless pulley. The coefficient of friction between the blocks and on the ground is $\mu=0.35$. The lower block is being pulled with a force $F=25 \text{ N}$ parallel to the plane. Draw free-body diagrams for each block and find the acceleration of the blocks and the tension in the rope.

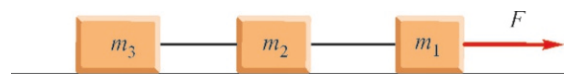
[A: $a = 2.5 \text{ m/s}^2$, $T = 6 \text{ N}$.]



Problem 4.6

4.6 The blocks with masses $m_1=1 \text{ kg}$ and $m_2=2 \text{ kg}$ in the figure are placed on two planes inclined at angles 53° and 37° respectively, and tied to each other with a massless rope passing through a frictionless pulley. The coefficient of friction of the surfaces is $\mu=0.1$. (a) Determine the direction of motion. (b) Calculate the acceleration of the blocks and the tension in the rope.

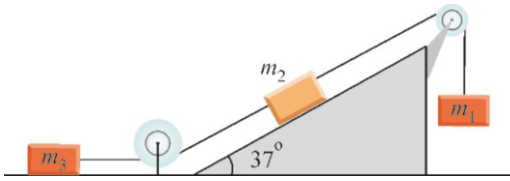
[A: (a) m_2 goes downward, (b) $a = 0.6 \text{ m/s}^2$, $T = 9.2 \text{ N}$.]



Problem 4.7

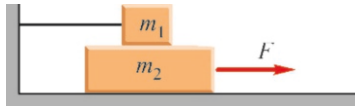
4.7 The masses of three blocks tied to each other on a frictionless horizontal plane are $m_1=1$, $m_2=2$, $m_3=3 \text{ kg}$. The mass m_1 is being pulled with the horizontal force $F=12 \text{ N}$. Draw free-body diagrams for each block and calculate the acceleration of the blocks and the tensions in the ropes.

[A: $a = 2 \text{ m/s}^2$, $T_1 = 10$, $T_2 = 6 \text{ N}$.]

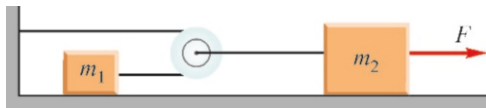
**Problem 4.8**

4.8 The three masses in the figure are equal and $m_1=m_2=m_3=1$ kg. All the surfaces and pulleys can be assumed to be frictionless and the ropes to be massless. Draw free-body diagrams for each block and calculate the acceleration of the blocks and the tensions in the ropes.

[A: $a = 1.7$ m/s², $T_1 = 8.3$ N and $T_2 = 1.7$ N.]

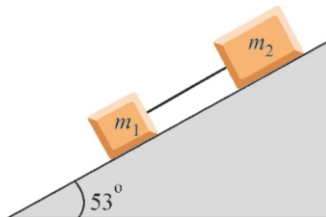
**Problem 4.9**

4.9 The block with mass $m_1=1$ kg in the figure is tied to the wall with a horizontal rope and placed on another block with mass $m_2=3$ kg. The coefficient of friction of all of the surfaces is $\mu=0.4$. The block m_2 is being pulled away from the wall with the horizontal force $F=30$ N. Draw free-body diagrams for each block and calculate the acceleration of m_2 and the tension in the rope. [A: $a = 3.3$ m/s², $T = 4$ N.]

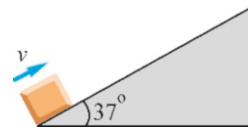
**Problem 4.10**

4.10 The blocks with masses $m_1=1$ kg and $m_2=4$ kg have been tied to each other with a pulley and ropes on a horizontal plane, as seen in the figure. The mass m_2 is being pulled away with the horizontal force $F=20$ N. Draw free-body diagrams for each block and calculate the accelerations of the blocks and the tension in the rope tied to m_1 .

[A: $a_1 = 5$, $a_2 = 2.5$ m/s², $T_1 = 5$ N.]

**Problem 4.11**

4.11 Masses $m_1=1$ kg and $m_2=2$ kg are tied to each other with a rope on an inclined plane with slope angle 53° . The coefficients of friction of the masses are different because they are made of different materials. The coefficient of friction between m_1 and the inclined plane is $\mu_1=0.1$ and it is $\mu_2=0.2$ for m_2 . Calculate the acceleration of the blocks and the tension in the rope. [A: $a = 7$ m/s², $T = 0.4$ N.]

**Problem 4.12**

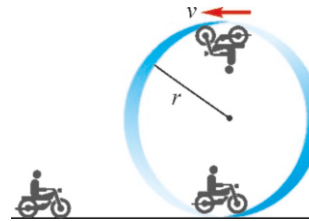
4.12 A block with mass m is thrown with an initial speed of $v_0=8$ m/s upwards along an inclined plane with a slope angle of 37° from the bottom of the plane. The coefficient of friction is $\mu=0.3$. Find the acceleration of the block and the distance it travels along the inclined plane until it stops.

[A: $a = 8.4$ m/s², 3.8 m.]

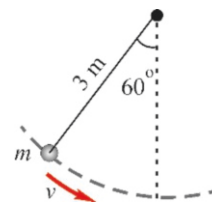
4.4 Circular Motion

4.13 A car is driven with constant speed v on a horizontal circular track with a radius of $r=200$ m. The coefficient of friction between the tires and the road is $\mu=0.8$. What is the maximum speed with which the car can turn without slipping? [A: $v = 40$ m/s.]

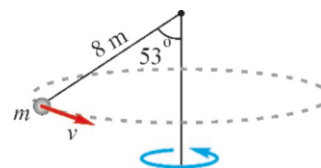
4.14 Car A with mass m goes around a circular path with radius r at speed v . Car B with mass $5m$ goes around another circular path with radius $3r$ at speed $2v$. What will be the ratio of the centripetal force F_B on car B to the centripetal force F_A on car A? [A: $F_B/F_A = 6.7$.]

**Problem 4.15**

4.15 A motorcycle showman with mass m drives through a vertical circular track with radius $r=40$ m. What should his minimum speed v be at the top point so that the motorcycle will not lose contact with the track? [A: 20 m/s.]

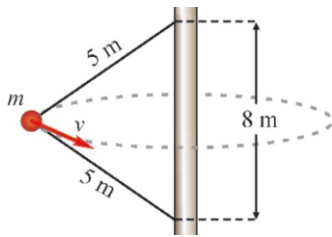
**Problem 4.16**

4.16 The mass $m = 2$ kg tied to the end of a pendulum with length 3 m has a speed of 10 m/s at the position where the rope makes an angle of 60° with the vertical. Calculate the tension in the string. [A: $T = 77$ N.]

**Problem 4.17**

4.17 Conical pendulum. A ball with mass $m = 1$ kg is tied to a string of length 8 m hung from the ceiling. The ball

is rotating on a circular trajectory on the horizontal plane where the rope makes an angle of 53° with the vertical.
 (a) What is the tension in the rope? (b) What is the speed of the ball?
 [A: (a) $T = 16.7\text{ N}$, (b) $v = 9.2\text{ m/s}$.]

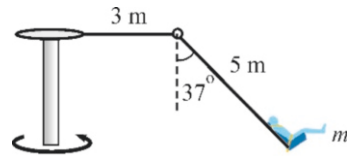


Problem 4.18

4.18 Two strings with equal lengths $L=5\text{ m}$ are tied to a mass of $m=4\text{ kg}$ and their other ends are fixed to a vertical axis at a distance of 8 m from each other. The mass m is rotating with a speed of $v=6\text{ m/s}$ on a horizontal plane around the axis with the strings staying tight. Find the tensions in the strings.
 [A: $T_1 = 65\text{ N}$ and $T_2 = 15\text{ N}$.]

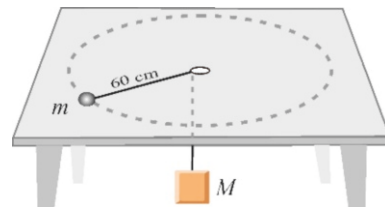
4.19 In an amusement park, a chairplane is able to rotate people at the end of a chain with length 5 m that can freely swing at the end of a horizontal bar of length 3 m (See the figure below). A boy with a mass of $m = 50\text{ kg}$ is rotating on

this chairplane at an angle of 37° with the vertical. Calculate the speed v of the boy and the tension in the chain.
 [A: $v = 6.7\text{ m/s}$, $T = 625\text{ N}$.]



Problem 4.19

4.20 One end of a string is tied to a puck of mass $m=2\text{ kg}$ on a frictionless table and the other end passes through a hole at the center of the table, with another mass $M = 3\text{ kg}$ tied to that end (See the figure below). The suspended mass M is able to stay at rest when the mass m on the table is rotating on a circular trajectory with a radius of 60 cm . What is the speed of the mass m ?
 [A: 3 m/s .]



Problem 4.20

5

WORK AND ENERGY



In an amusement park the speed of a roller coaster increases or decreases as it goes up and down.

What is it that is stored when its speed decreases and regained when its height decreases?

As a principle, all mechanical problems can be solved by using Newton's three laws alone. However, this method may often be complicated or difficult to interpret. On the other hand, using concepts such as work, energy and momentum, which are closer to our daily intuitions, we can both work in an easier manner and also make interpretations that are more easily adoptable for technology.

From this chapter onwards, we shall not set forth new laws but merely define new quantities and seek solutions in terms thereof. The most important feature of these new quantities is that they can be expressed within general conservation laws.

5.1 WORK

The concept of ‘work’ borrowed from daily life is the ability of a force to displace an object. It takes on a precise meaning in physics.

Work Done by a Constant Force

In order to understand the concept of work in an easier manner, let us first define the work done by a constant force.

Definition: The **work** done by a constant force F exerted on an object during displacement d is the scalar quantity

$$W = Fd \cos \theta \quad (\text{Work}) \quad (5.1)$$

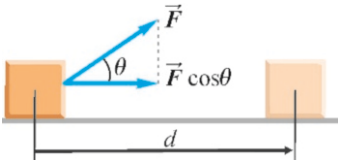


Figure 5.1: Work by a force.

In this equation, θ is the angle between the force and the displacement. As seen in the figure, $F \cos \theta$ is the projection of the force along the path d . Thus, the work is the projection of the force along the path times the displacement.

The unit of work is newton \times meters. This derived unit was named as the ‘Joule’ after English scientist James P. Joule, and is abbreviated as J.

Let us emphasize the most important features of the work:

- The work is zero if the object is not displaced ($d = 0$) despite a force being applied on it.

This may seem contrary to our daily intuitions. For example, a weightlifter standing motionless holding 200-kg dumbbells, does zero work according to our definition. Yet, it would be obvious that the sweating sportsman is getting exhausted, just like someone who performs work. This should not be seen as a contradiction, because the concepts of physics are defined within a precise context.

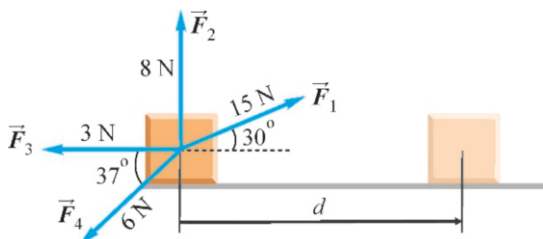
- The work is zero if the force is perpendicular to the displacement ($\cos 90^\circ = 0$).
- If the force forms a wide-angle with the direction of motion, in other words, if the projection of the force along the path is in the opposite direction to the displacement, then the work done is negative.

Expression of Work As a Scalar Product

Our definition of work actually fits the form of the scalar product of two vectors that we defined in Chapter 1. Instead of the vectors \vec{A} and \vec{B} used there, if we take \vec{F} as the force vector and \vec{d} as the displacement vector, then we can write the expression (5.1) as follows:

$$W = Fd \cos \theta = \vec{F} \cdot \vec{d} \quad (5.2)$$

Example 5.1



Calculate the works done by each of the forces on the object in the figure for a displacement $d=2\text{m}$ in the $+x$ direction.

Answer

We apply the definition of work (Eq.5.1) to each force:

$$W_1 = F_1 d \cos 30^\circ = 15 \times 2 \times 0.87 = 26 \text{ J}$$

$$W_2 = F_2 d \cos 90^\circ = 8 \times 2 \times 0 = 0$$

$$W_3 = F_3 d \cos 180^\circ = 3 \times 2 \times (-1) = -6 \text{ J}$$

$$W_4 = F_4 d \cos(180^\circ - 37^\circ) = 6 \times 2 \times (-0.8) = -9.6 \text{ J}$$

Work Done by a Variable Force

As the concept of work will be used extensively, we should also be able to calculate the work done by a force whose magnitude varies along the path. The concept of the integral used in calculus will be useful here. Now, we will show how to build the integral for the work done by a variable force that is parallel to the displacement.

Let the object be moving along the x axis from position $x = a$ to position $x = b$. Thus, the displacement is $d = b - a$. Let us assume that the variable force exerted on the object in this interval varies as a function of position: $F = F(x)$. Such a $F(x)$ function is shown in the figure below.

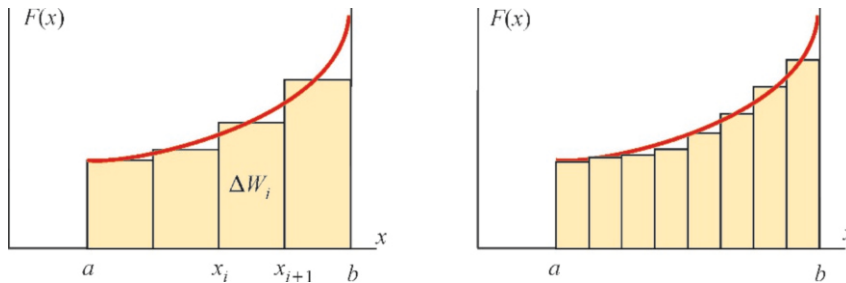


Figure 5.2: The total work done in the interval $[a, b]$ is the sum of the small works ΔW_i done in the small intervals Δx . If the distance Δx decreases gradually ($\Delta x \rightarrow 0$), the difference between the rectangles and the curve vanishes.

Let us divide the path $[a, b]$ into N intervals, each with widths as small as Δx . Assuming that the force F remains approximately constant in each of these intervals, such as between x_i and $x_i + \Delta x$, then the small work performed in this i^{th} interval is

$$\Delta W_i \approx F(x_i) \Delta x \quad i = 1, 2, 3, \dots, N$$

As seen in the figure, ΔW_i is just the area of the thin rectangle drawn in the i^{th} interval. Here, we commit a small error by assuming that the force has a constant value of $F(x_i)$ in this interval. However, this error will vanish when the $\Delta x \rightarrow 0$ limit is taken later.

We may thus write the total work as the sum of the works in these small intervals:

$$W = \sum_{i=1}^N W_i \approx \sum_{i=1}^N F(x_i) \Delta x$$

If we now use the concept of limit in calculus, in other words, if we gradually approach the value of Δx to zero without setting it equal to zero, this sum expression becomes the definition of the **definite integral** of the function $F(x)$ in the interval $[a, b]$:

$$\lim_{\Delta x \rightarrow 0} \sum_{i=1}^N F(x_i) \Delta x = \int_a^b F(x) dx$$

As a result, the integral expression of the work done by a variable force $F(x)$ in the interval $[a, b]$ becomes as follows:

$$W = \int_a^b F(x) dx \quad (\text{Work done by a variable force}) \quad (5.3)$$

Brief Information on Integrals

The concept of the integral and integrating techniques are examined comprehensively in calculus courses. Here, let us briefly review the integrals of the most frequently encountered functions without proof.

The integral expression that we found for work above has the following meaning: Let us find such a function $\Phi(x)$ such that its derivative is the integrated function $F(x)$:

$$\Phi(x) = \int F(x) dx \quad \text{or} \quad \frac{d\Phi}{dx} = F(x)$$

The expression here without the limits is called the **indefinite integral**. If this function Φ is known, the integral can be calculated as follows:

$$\int_a^b F(x) dx = \Phi(x) \Big|_{x=a}^{x=b} = \Phi(b) - \Phi(a)$$

The result is the value of Φ at the higher limit less its value at the lower limit.

Indefinite integrals of certain functions (c a constant).			
function (y)	$\int y(x) dx$	function (y)	$\int y(x) dx$
1	$x + c$	$\cos x$	$\sin x + c$
x	$\frac{1}{2}x^2 + c$	$\sin x$	$-\cos x + c$
x^2	$\frac{1}{3}x^3 + c$	e^x	$e^x + c$
$\sqrt{x} = x^{1/2}$	$\frac{2}{3}x^{3/2} + c$	$\frac{1}{x}$	$\ln x + c$
x^n	$\frac{x^{n+1}}{n+1} + c$	$\ln x$	$x \ln x - x + c$

You do not need to remember this table if you know derivatives. The integral that you are looking for should accept the given function $F(x)$ as its derivative.

Example 5.2

Using integrals, calculate the work done by a force varying as $F(x) = x^3$ along the x axis from the position $a = 1$ to position $b = 5$.

Answer We use the integral definition of work (Eq. 5.3):

$$W = \int_a^b F(x) dx = \int_1^5 x^3 dx = \frac{x^4}{4} \Big|_1^5 = \frac{5^4}{4} - \frac{1^4}{4}$$

$$W = 156 \text{ J}$$

Work Done by a Spring Force

We all are familiar with the spiral springs widely used in technology. Steel springs are used in ball-point pens and clothespins, inside mattresses, the shock absorbers of cars, etc. When we try to stretch out or compress a spring, it resists with an opposing force.

Let us fix one end of a spring with a normal length L_0 to a wall and then let us stretch out or compress the other end along the x axis (Figure 5.3). Let the new length of the spring be L . Let us show the amount of extension of the spring as

$$x = L - L_0 \quad (5.4)$$

The value x is positive for extension and negative for compression.

The expression of the spring force is known as **Hooke's law**, in memory of the English scientist Robert Hooke:

The force exerted by a spring is proportional to the extension and is in the opposite direction to the extension.

$$F = -kx \quad (\text{Hooke's law}) \quad (5.5)$$

The coefficient k in this formula is referred to as the **spring constant** or the **force constant**. Its unit is newton/meters and depends on the type of material used in the spring.

Let us emphasize the important aspects of the Hooke's law:

- The spring force is proportional not to the length of the spring, but to $x=L-L_0$, the amount of extension or compression.
- The negative sign in the formula indicates that the force is in the opposite direction to the extension. If the spring is extended ($x>0$), then the force is in the opposite direction, in other words, $F < 0$. In the reverse case, if the spring is compressed ($x < 0$), then the force is in the positive direction: $F > 0$.
- The flexibility of the spring has a certain limit. When extended for more than a certain x value, the spring gets deformed and loses its flexibility.

We wish to calculate the work done by the spring force when we extend a spring from its normal length ($x = 0$) until it reaches the value $x = d$. The spring would, of course, not extend by itself; we will have to apply a force of at least $(-F)$. But we are only interested in the work performed by the spring force. Therefore, if we take the boundaries of the integral as $a = 0$ and $b = d$, we may write the work performed by the spring force as follows:

$$W_s = \int_0^d F dx = \int_0^d (-kx) dx = -k \int_0^d x dx = -k \left[\frac{1}{2} x^2 \right]_0^d$$

$$W_s = -\frac{1}{2} k d^2$$

As expected, the work performed by the spring force is negative, because it is in the opposite direction to the extension. We need to perform a positive work against this in order to extend the spring.

We will use this result later in the topic of potential energy.

Example 5.3

A spring is observed to extend by 10 cm when a force of 30 N is applied. How much work should be performed to extend this spring by 40 cm?

Answer

We first calculate the spring constant k using the given data.

Hooke's law gives us (the negative sign is not important):

$$F = kx \rightarrow 30 = k \times 0.10$$

$$k = 300 \text{ N/m}$$

Using this value of constant k , we can calculate the work done against the spring force to extend the spring by $x=0.40$ m:

$$W = -W_s = \frac{1}{2} k x^2 = \frac{1}{2} \times 300 \times 0.4^2 = 24 \text{ J}$$

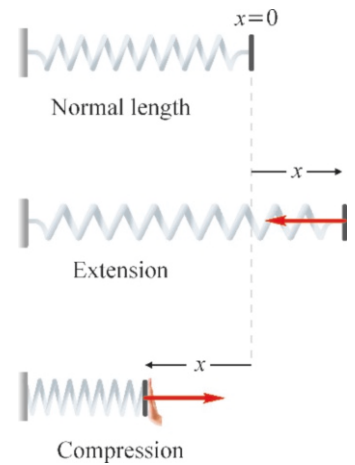


Figure 5.3: Spring force x is in opposite direction to its extension.

5.2 POWER

Anyone can carry a load for a distance of 100 meters, but some of us can do it faster. In daily life, and also in technology, it is important to know the amount of work performed per unit time. We describe this with the concept of power.

Definition: If the amount of work performed in a time interval Δt is ΔW , then the **average power** is defined as,

$$P_{\text{ave}} = \frac{\Delta W}{\Delta t} \quad (\text{average power}) \quad (5.6)$$

The **instantaneous power** at a given instant t is defined as,

$$P = \lim_{\Delta t \rightarrow 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \quad (\text{instantaneous power}) \quad (5.7)$$

which is just the derivative of work with respect to time t .

Units of Power

In the SI system, the unit of power is joule/seconds = watt and is abbreviated as (W). The use of **kilowatt** (1kW = 1000W) is more common in the industry.

The watt is one of the rare units used both in science and in technology. If we rewrite the definition above as $dW = P dt$, we see that the unit of work (joule) can also be expressed as watt \times hours or kilowatt-hours (kWh). When we talk about our electricity consumption as being in “kilowatts,” we actually mean kilowatt-hours, as it refers to the energy that we consume.

Horsepower is another unit of power used in the automotive industry and is abbreviated as (HP):

$$1 \text{ HP} = 746 \text{ watts} = 0.746 \text{ kW}$$

The horsepower unit's first historical use was in extracting coal from the mines of Britain. Back in those days, when Scottish scientist James Watt (1736–1819) invented the steam engine, he proposed this unit in order to compare the power of the steam engine with the power of horses.

There is another useful formula for expressing power in terms of force and velocity in mechanics. In the definition above, let us write the work ΔW as the work performed by a force \vec{F} over a small displacement of $\Delta \vec{r}$. Then,

$$P = \lim_{\Delta t \rightarrow 0} \frac{\vec{F} \cdot \Delta \vec{r}}{\Delta t} = \vec{F} \cdot \left(\lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} \right)$$

The limit in the brackets is just the velocity \vec{v} . Then, we can write the power generated by the force \vec{F} at time t as follows:

$$P = \vec{F} \cdot \vec{v} \quad (\text{Power}) \quad (5.8)$$

Example 5.4

A crane is lifting a mass of 2 tons of coal from under the ground to the surface at a speed of 12 km/hour. Calculate the power of the crane engine in units of watts and horsepower.

Answer Since the crane is pulling the mass m at constant

speed, it is exerting a force $F = mg$ equal and opposite to the weight mg . We use Eq. (5.8) that we found for power:

$$P = Fv = mgv = 2000 \times 10 \times (12000/3600) \\ = 67000 \text{ watt} = 67 \text{ kW}$$

and use the conversion formula 1 HP = 0.746 kW:

$$P = 67/0.746 = 90 \text{ HP}$$

Example 5.5

An air-conditioner is operating at a power of 3 kW. How much energy does it consume in one month? What is the cost for one month if the price is 20 cents per kilowatt-hours?

Answer The consumed energy can be calculated as $\Delta W =$

$P \Delta t$:

$$\Delta W = 3 \times (30 \text{ days} \times 24 \text{ hours}) = 2160 \text{ kWh.}$$

The monthly cost is found by multiplying this by the energy unit price:

$$2160 \times 0.2 = 432 \$.$$



Figure 5.4: Horsepower was used in the past as a measure of the amount of coal extracted from a coal mine.

5.3 KINETIC ENERGY

What really happens when a force performs some work? For example, when we apply a force to an object, it may gain speed or, be lifted to some height or, generate electricity, etc. These are different aspects of the **energy** concept. As the most important concept defined in physics, energy may roughly be regarded as “the ability to do work.” Working with the concept of energy both provides convenience and may also be used outside of mechanics, in fields such as electricity, chemistry and biology.

There are various types of energy in physics, such as kinetic energy, potential energy, electrostatic energy, magnetic energy, chemical energy, etc. Only kinetic and potential energy are used in mechanics.

Definition: The **kinetic energy** of an object with mass m traveling at speed v is,

$$K = \frac{1}{2}mv^2 \quad (\text{kinetic energy}) \quad (5.9)$$

Kinetic energy is always positive. An object at rest has zero kinetic energy. Notice that kinetic energy is also expressed in terms of the work unit:

$$\text{kg} \times (\text{m/s})^2 = \frac{\text{kg} \times \text{m}}{\text{s}^2} \times \text{m} = \text{newton} \times \text{m} = \text{joule}$$

Work-Energy Theorem

To show the relation between work and kinetic energy, let us calculate the work performed by the net force exerted on an object with mass m . However, rather than the most general cases of three-dimensional and variable force, we will consider the work performed by a one-dimensional constant force.

An object with mass m has a speed of v_0 at the location x_0 , and, under the influence of a constant force F_{net} , reaches the speed v when it arrives at the position x (Figure 5.6). Let us write the net work performed by the constant force:

$$W_{\text{net}} = F_{\text{net}} d = F_{\text{net}} (x - x_0)$$

We write the force as the second law $F_{\text{net}} = ma$ and use the formula 2.7 relating the position to speed [$v^2 - v_0^2 = 2a(x - x_0)$], finding that,

$$\begin{aligned} W_{\text{net}} &= ma(x - x_0) = m \underbrace{a(x - x_0)}_{(v^2 - v_0^2)/2} \\ W_{\text{net}} &= \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = K - K_0 \end{aligned} \quad (5.10)$$

This result is known as the **Work-Energy Theorem**:

The net work done on an object is equal to the change in the kinetic energy of the object.

This is a very general result and is always true, even in the cases of three-dimensional motion or with variable forces:

$$W_{\text{net}} = \int_1^2 \vec{F}_{\text{net}} \cdot d\vec{r} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad (\text{Work-energy theorem}) \quad (5.11)$$

Here, $d\vec{r}$ refers to a small displacement vector with components (dx, dy, dz) .



Figure 5.5: Where does the work performed by these racers in pushing the sled go?

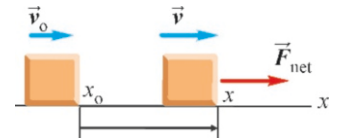


Figure 5.6: Work-energy theorem.

The work-energy theorem is actually another way of expressing Newton's second law, in terms of work and kinetic energy. The theorem shows clearly that, if the work done by the net force is positive, then the object's kinetic energy increases. If the net work is negative, then the kinetic energy decreases. Another advantage of this theorem is the convenience of working with only scalar quantities, instead of the vector law $\vec{F} = m\vec{a}$. Some types of problems can be solved very quickly using this theorem.

Example 5.6

When a net force F is applied on an object with mass 3 kg traveling at a speed of 5 m/s, its speed is increased to 10 m/s at the end of a distance of 9 m.

Calculate the magnitude of the force.

Answer We could have solved this problem the long way,

by first calculating acceleration. But it is easier to use the work-energy theorem: the work performed by the unknown force F produces the change in kinetic energy:

$$Fd = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$$F = \frac{m(v^2 - v_0^2)}{2d} = \frac{3 \times (10^2 - 5^2)}{2 \times 9} = 12.5 \text{ N.}$$

Example 5.7

An object with mass m is thrown along a horizontal surface at a speed of 6 m/s. The coefficient of friction is $\mu = 0.4$. How far will it travel before coming to rest?

Answer

Among the forces exerted on the object, the weight and the normal force are in the vertical direction and balance each

other. Hence, only the friction force remains, and it is opposite to the motion. The work it performs is negative, and it makes the final kinetic energy zero ($v = 0$):

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$$-fd = -(\mu mg)d = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$$d = v_0^2 / (2\mu g) = 6^2 / (2 \times 0.4 \times 10) = 4.5 \text{ m.}$$

5.4 POTENTIAL ENERGY

Conservative and Nonconservative Forces

Forces in mechanics are divided into two groups in terms of one specific feature. The distinction of conservative and nonconservative forces is crucial in defining potential energy.

To understand this, let us first consider throwing a stone upwards from the ground with a speed of v (Figure 5.7). Neglecting air friction, the stone is under the influence of gravity alone when in the air. The kinetic energy of the stone on the ground is $K_1 = \frac{1}{2}mv^2$. When the stone rises to reach its maximum height, its speed is zero for a short instant: $K_2 = 0$. Now, can we say that the kinetic energy of the object has disappeared? No, because, immediately afterwards, the stone resumes its downward motion and recuperates its former kinetic energy when it reaches the ground. In this example, we say that the gravitation force is *conservative*, because it stores the kinetic energy somehow and later returns it.

On the other hand, consider a mass m thrown with a speed of v along a horizontal surface with friction. The net force exerted on this object along the path is the frictional force. The kinetic energy of the object at the start is again $K = \frac{1}{2}mv^2$. The object gradually slows down and its kinetic energy vanishes when it comes to a stop. Now, we can wait for ages, but we will never see this object re-accelerate. In this example, we say that the frictional force is *nonconservative*, because it irreversibly takes the kinetic energy of the object and never returns it. (Actually, it is converted into another type of energy: heat.)

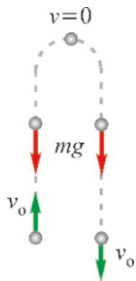


Figure 5.7: Kinetic energy may be reduced with the effect of gravity but it does not disappear. It can be regained.

Potential energy is that type of energy that can store and return the work. According to the above example, potential energy can be defined only for conservative forces.

The following criterion determines whether or not a force is conservative:

A force is conservative if the work done is independent of the path taken between any two points.

Let us test this criterion in the example of gravitational force. We want to calculate the work performed by gravity on different paths taken by the mass m in Figure 5.9 from A to B . Since the work is zero when the force is perpendicular to the path, there follows:

$$\begin{aligned} \text{path } ACB: \quad W_{ACB} &= W_{AC} + W_{CB} = -mg \times AC + 0 = -mgh \\ \text{path } ADB: \quad W_{ADB} &= W_{AD} + W_{DB} = 0 - mg \times DB = -mgh \\ \text{diagonal } AB: \quad W_{AB} &= (-mg \sin \theta) \times AB = -mg \times (AB \sin \theta) = -mgh \end{aligned}$$

The work along all three paths is the same. This shows that gravity is a conservative force. If we had made the same calculation for friction force, we would have found that the work varied according to the traveled path .

General Definition of Potential Energy

The work done against a conservative force is equal to the change in potential energy.

If the conservative force is \vec{F}_c then the force doing work against it will be $(-\vec{F}_c)$. If we denote potential energy as U , then we may write this definition as follows:

$$-\int_1^2 \vec{F}_c \cdot d\vec{r} = U_2 - U_1 \quad (\text{potential energy definition}) \quad (5.12)$$

It seems reasonable here to define the increase in potential energy as negative work. Since the negative work decreases kinetic energy, the potential energy should be increasing.

Now let us define gravitational and elastic potential energies in accordance with this general definition.

Gravitational Potential Energy

Consider an object near the Earth's surface, with mass m traveling from point A at position (x_1, y_1) to a point B with position (x_2, y_2) along some curvilinear path (Figure 5.10). Let us denote the weight of the object with the vector $m\vec{g}$ and a small displacement along the path with $d\vec{r}$. The angle between $d\vec{r}$ and the y -axis is θ . We now write the work done against the conservative force, in other words, the work done by the vector $(-m\vec{g})$, as a scalar product:

$$-W = \int_A^B (-m\vec{g}) \cdot d\vec{r} = \int_A^B (-mg) dr \cos(180^\circ - \theta)$$

As can be observed in the figure, $dr \cos(180^\circ - \theta) = -dr \cos \theta = -dy$. Accordingly,

$$-W = \int_A^B mg dy = mg \int_{y_1}^{y_2} dy = mg(y_2 - y_1)$$

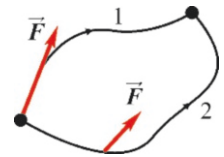


Figure 5.8: The work performed by a conservative force is independent of the traveled path.

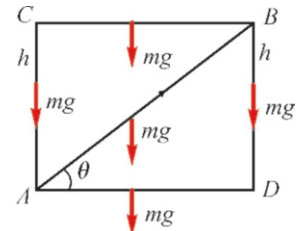


Figure 5.9: Work done by gravity along three different paths.

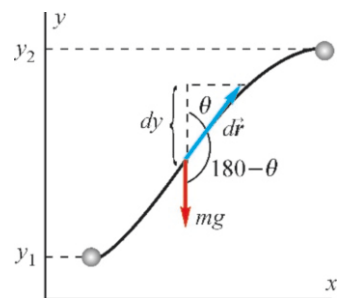


Figure 5.10: Coordinates in a gravitational potential energy.

The result shows that the work done against the gravitational force is independent of the path taken and depends only on the heights (y -coordinates) of the two points. Now, we can define the gravitational potential energy using the general definition of Eq. (5.12):

$$U_2 - U_1 = (\text{Work done against gravity}) = -W = mg(y_2 - y_1)$$

Comparing both sides of this equation, the potential energy at any place with coordinate y can be written as,

$$U(y) = mgy + C$$

The constant C in this equation is determined after some zero-reference point of the potential energy is chosen. For example, if we want $U = 0$ at the level $y = 0$, then $C = 0$. The potential energy takes negative values below this level. As a result, we may write gravitational potential energy as follows:

$$U(y) = mgy \quad (\text{gravitational potential energy}) \quad (5.13)$$

The choice of the zero level of the gravitational potential energy is arbitrary and can be taken at any height. This has no importance, because, as we shall see later, only the potential energy difference between two heights will appear in equations. This difference does not change, regardless of where the zero level is chosen.

Elastic Potential Energy

Next, we find the elastic potential energy due to spring force. As seen in Figure 5.11, the spring force corresponding to an extension of x along the x -axis is $F = -kx$ and in the opposite direction. And the force ($-\vec{F}$) performing work against this force becomes $+kx$, hence it is in the same direction as x . Thus, the work done during the extension from the value x_1 to the final value x_2 is:

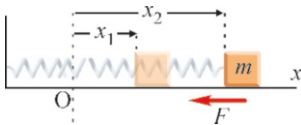


Figure 5.11: Coordinates in elastic potential energy.

$$-W = \int_{x_1}^{x_2} (-F) dx = \int_{x_1}^{x_2} (+kx) dx = k \int_{x_1}^{x_2} x dx = k \left. \frac{1}{2}x^2 \right|_{x_1}^{x_2} = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

Using the general definition of potential energy (Eq. 5.12), we find that

$$U_2 - U_1 = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

$$U(x) = \frac{1}{2}kx^2 + C$$

The constant C is again determined by arbitrarily choosing the place where the potential energy is zero. If we take it to be zero at the normal length of the spring, in other words, at $x = 0$, then $C = 0$. As a result, we can write the elastic potential energy as follows:

$$U(x) = \frac{1}{2}kx^2 \quad (\text{elastic potential energy}) \quad (5.14)$$

Gravitational Potential Energy (General Case)

If the height continuously increases on the surface of the Earth, then the gravitational acceleration g ceases to be a constant and the expression $U = mgy$ becomes invalid. Instead, we should return to Newton's gravitational law, which we had introduced in Chapter 4, and calculate the gravitational potential energy therefrom.

Let us rewrite the gravitational force between the Earth (with mass M_E) and an object of mass m at a distance of r from the center:

$$F = G \frac{mM_E}{r^2} \quad (5.15)$$

We again calculate the potential energy as the work performed against the gravitational force. For this purpose, we need to apply the force $-\vec{F}$ against this force \vec{F} that is exerted when the mass m travels from a distance of r_1 to a distance of r_2 (Figure 5.12). Accordingly,

$$-W = \int_{r_1}^{r_2} (-\vec{F}) \cdot d\vec{r} = - \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = - \int_{r_1}^{r_2} F dr \cos 180^\circ$$

It can be seen from the figure that the angle between the attractive force \vec{F} and the displacement vector $d\vec{r}$ is 180° . Since $\cos 180^\circ = -1$, and if the expression for F is substituted, then

$$\begin{aligned} -W &= GmM_E \int_{r_1}^{r_2} \frac{dr}{r^2} = GmM_E \left[-\frac{1}{r} \right]_{r_1}^{r_2} \\ &= -GmM_E \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \end{aligned}$$

Again, if we write that the change in the potential energy is equal to this opposing work, then

$$-W = U_2 - U_1 = -GmM_E \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

From here, the potential energy at any distance r can be written as:

$$U(r) = -\frac{GmM_E}{r} + C$$

How do we choose the constant C ? It is impossible to choose the origin ($r = 0$), as the expression diverges ($U \rightarrow \infty$). Instead, most naturally, $U = 0$ is chosen at the point where objects are too far apart to interact, in other words, when $r \rightarrow \infty$. Then, $C = 0$ and the gravitational potential energy is obtained:

$$\boxed{U(r) = -\frac{GmM_E}{r} \quad (\text{gravitational potential energy})} \quad (5.16)$$

It is seen from this formula that potential energy is zero at infinity.

5.5 LAW OF CONSERVATION OF ENERGY

Combining the concepts of work, kinetic and potential energies, we can now formulate the most important conservation law in physics.

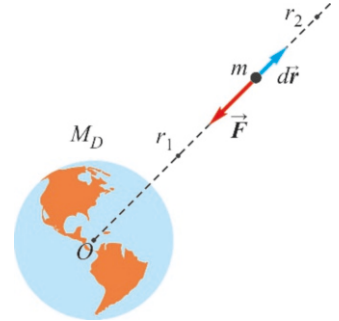


Figure 5.12: Coordinates in a gravitational potential energy.

According to the Work-Energy theorem, the work performed by a net force acting on an object is equal to the increase in kinetic energy (Equation 5.11):

$$\int_1^2 \vec{F}_{\text{net}} \cdot d\vec{r} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = K_2 - K_1$$

We have also learned that we could separate the forces exerted on an object into two groups as conservative and nonconservative. Then, we can divide the work they perform into the following two terms:

$$\begin{aligned} \vec{F}_{\text{net}} &= \vec{F}_c + \vec{F}_{\text{nc}} \\ \int_1^2 (\vec{F}_c + \vec{F}_{\text{nc}}) \cdot d\vec{r} &= K_2 - K_1 \end{aligned}$$

Now let us recall the definition of potential energy: The work performed against conservative forces is equal to the increase in potential energy:

$$\underbrace{\int_1^2 \vec{F}_c \cdot d\vec{r}}_{-(U_2 - U_1)} + \underbrace{\int_1^2 \vec{F}_{\text{nc}} \cdot d\vec{r}}_{W_{\text{nc}}} = K_2 - K_1$$

In this expression, the work performed by nonconservative forces is denoted as W_{nc} . Rearranging the terms on both sides, we arrive at the **Law of Conservation of Energy**:

$$(K_1 + U_1) + W_{\text{nc}} = K_2 + U_2 \quad (\text{Law of Conservation of Energy}) \quad (5.17)$$

Here, $K = \frac{1}{2}mv^2$ is the kinetic energy and U can be any of the gravitational, elastic or other potential energies.

The law of conservation of energy states that, after we subtract the work of nonconservative forces from the total initial energy (kinetic+potential), whatever remains will be equal to the total final energy (kinetic+potential).

Here, we only took into consideration the conservation of mechanical energy. Actually, the conservation of energy is still valid when all other types of energy (electrical, magnetic, nuclear, etc.) are taken into consideration; it is a universal law of physics.

Special case: If there are no nonconservative forces exerted on an object or if their work is somehow zero, then,

$$W_{\text{nc}} = 0 \longrightarrow K_1 + U_1 = K_2 + U_2 \quad (\text{Special Energy Conservation}) \quad (5.18)$$

In such a case, no mechanical energy is lost and the sum of the energies (kinetic+potential) remains constant throughout the motion.

The sum of kinetic and potential energies is called **total mechanical energy** and is denoted with E :

$$E = K + U = \frac{1}{2}mv^2 + U \quad (\text{total mechanical energy}) \quad (5.19)$$

The law of conservation of energy shows us that total mechanical energy is conserved even if the kinetic or potential energies vary. In the equation (5.17), if we state that $K_1 + U_1 = E_1$ and $K_2 + U_2 = E_2$, then



Figure 5.13: When a roller coaster rushes downward, its potential energy decreases to be converted into kinetic energy.

General : $E_1 + W_{nc} = E_2$

If $W_{nc} = 0$: $E_1 = E_2$

The terms in the law of conservation of energy are only scalar quantities. Considering the difficulty of working with vector equations such as Newton's second law $\vec{F} = m\vec{a}$, energy conservation formulas provide great convenience.

Example 5.8

A block with a mass of 4 kg travels at a speed of 3 m/s on a frictionless horizontal plane and hits a spring with one end fixed to a wall. The spring constant is $k = 100 \text{ N/m}$. How much does the block compress the spring?



Answer

Since no friction occurs, the initial and final total energies are the same:

$$E_1 = E_2$$

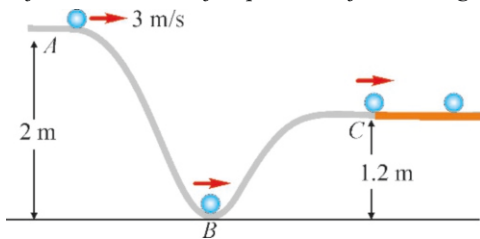
$$\frac{1}{2}mv^2 + 0 = 0 + \frac{1}{2}kx^2$$

$$x = \sqrt{\frac{m}{k}} v = \sqrt{\frac{4}{100}} 3 = 0.60 \text{ m}$$

Example 5.9

The ABC section of the path in the figure is frictionless, while the coefficient of friction is $\mu = 0.5$ beyond C. A ball with a mass of $m = 2 \text{ kg}$ is thrown along the path with a speed of 3 m/s from a point A at a height of 2 m above the ground.

- (a) Find the speeds of the ball at points B and C.
 (b) How far will it travel after point C before coming to a stop?



Answer

(a) Total mechanical energy is conserved at points A, B and C, as there is no friction:

$$\frac{1}{2}mv_A^2 + mgy_A = \frac{1}{2}mv_B^2 + mgy_B = \frac{1}{2}mv_C^2 + mgy_C$$

From these equalities, we find the speeds v_B and v_C using the values $y_A = 2$, $y_B = 0$, $y_C = 1.2 \text{ m}$:

$$v_B = \sqrt{v_A^2 + 2gy_A} = \sqrt{3^2 + 2 \times 10 \times 2} = 7 \text{ m/s}$$

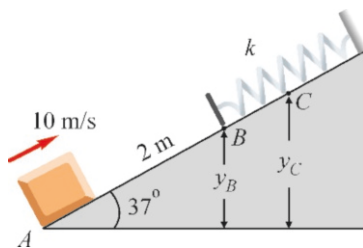
$$v_C = \sqrt{v_B^2 - 2gy_C} = \sqrt{7^2 - 2 \times 10 \times 1.2} = 5 \text{ m/s}$$

(b) The kinetic energy of the object decreases from the point C onward, due to the work done by friction:

$$\frac{1}{2}mv_C^2 - fd = \frac{1}{2}mv_C^2 - (\mu mg) d = 0$$

$$d = v_C^2 / (2\mu g) = 5^2 / (2 \times 0.5 \times 10) = 2.5 \text{ m}.$$

Example 5.10



A spring with constant $k = 100 \text{ N/m}$ is fixed at the top of a plane inclined at an angle of 37° . The coefficient of friction of the plane is $\mu = 0.5$. A block with a mass of 1 kg is thrown with an initial speed of 10 m/s along the plane from the point A located at the lower end and at a distance of 2 m from the free end of the spring.

- (a) What is the speed of the block when it reaches point B?
 (b) How much does the block compress the spring?

Answer (a) We write the general energy conservation between points A and B:

$$E_A + W_f = E_B$$

Here, $W_f = -fd$ is the work performed by the friction force and is $f = \mu mg \cos 37^\circ$. Thus

$$\frac{1}{2}mv_A^2 + mgy_A - (\mu mg \cos 37^\circ) d = \frac{1}{2}mv_B^2 + mgy_B$$

If we choose the zero level of potential energy at point A, then $y_A = 0$ and the height of point B at distance d becomes $y_B = d \sin 37^\circ$. The speed v_B is found when the other values are also substituted:

$$10^2 - 16 = v_B^2 + 24 \rightarrow v_B = 7.7 \text{ m/s}$$

(b) The block stops when it comes to some point C where the compression of the spring is at a maximum: $v_C = 0$. We write the conservation of energy between A and C. This time, we have the elastic potential energy in addition to the gravitational potential energy. If we denote the maximum compression of the spring as x , then

$$\frac{1}{2}mv_A^2 - \mu mg(d + x) = 0 + mgy_C + \frac{1}{2}kx^2$$

The height of point C is $y_C = (d + x) \sin 37^\circ$ and the numerical values are substituted:

$$100 - 20(2 + x) = 100x^2 \rightarrow 5x^2 + x - 3 = 0$$

The positive root of this 2nd degree equation is the result:

$$x = 0.68 \text{ m}$$

Example 5.11

The speed of a meteor approaching the Earth is measured as 100 m/s at a distance of 900 km from the sea level. At what speed does this meteor crash on the Earth's surface?

(For the Earth, $GM_E=4 \times 10^{14} \text{ m}^3/\text{kg}\cdot\text{s}^2$ and $R_E=6400 \text{ km}$.)

Answer

The energy conservation is written as:

$$\frac{1}{2}mv_1^2 - \frac{GM_E m}{R_1} = \frac{1}{2}mv_2^2 - \frac{GM_E m}{r_2}$$

We simplify as follows:

$$v_2 = \sqrt{v_1^2 + GM_E \left(\frac{1}{r_2} - \frac{1}{r_1} \right)}$$

(The product GM_E will appear very often in problems with gravitational potential energy.) Here, the distances r_1 and r_2 must be measured from the center of the Earth, while the data is given as measured from the surface of the Earth. We calculate them first:

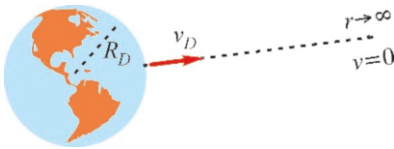
$$r_1 = 6400 + 900 = 7300 \text{ km} = 7.3 \times 10^6 \text{ m}$$

$$r_2 = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}$$

From these values, we find the crash speed of the meteor:

$$v_2 = 2780 \text{ m/s}$$

Example 5.12



Escape speed from the Earth. A rocket or stone thrown from the surface of the Earth must have a minimum speed to move infinitely far away and to not fall back to the Earth. This is called the escape speed. Calculate the value of this speed. ($GM_E = 4 \times 10^{14} \text{ m}^3/\text{kg}\cdot\text{s}^2$ and $R_E = 6400 \text{ km}$.)

Answer What should the condition be for the object to not

fall back to Earth? When the object is slowing down, if its speed becomes zero at some finite distance, it will fall back. Therefore, it should reach infinity when its speed becomes zero: $v_\infty = 0$. Accordingly, we write the energy conservation for an object with mass m between the Earth's surface and infinity:

$$\frac{1}{2}mv_{\text{esc}}^2 - \frac{GM_E m}{R_E} = \frac{1}{2}m \times 0^2 - \frac{GM_E m}{\infty}$$

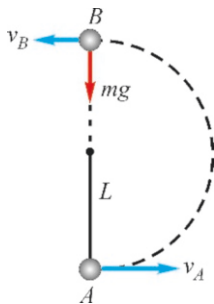
As $1/\infty = 0$, the escape speed formula is,

$$v_{\text{esc}} = \sqrt{\frac{2GM_E}{R_E}}$$

Substituting the numerical values, we get:

$$v_{\text{esc}} = 11.2 \times 10^3 \text{ m/s} = 11.2 \text{ km/s}$$

Example 5.13



An object with mass m is tied to the end of a rope with length $L = 50 \text{ cm}$ and is thrown with a horizontal speed v_A from the point A at the bottom. With what minimum speed v_A should it be thrown to ensure that the rope does not get loose when passing through the point B at the top?

Answer In example 4.19, we saw that the tension T at point B should be at a minimum ($T=0$) so that the rope would not get loose. Accordingly, the centripetal acceleration at point B is only due to the weight mg . We thus find the minimum speed v_B :

$$F_r = ma_r \rightarrow mg = m \frac{v_B^2}{L}$$

$$v_B^2 = gL$$

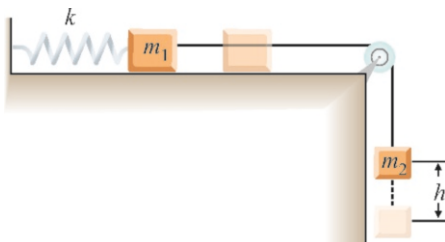
Now, we write the energy conservation between points A and B. As the height of point B is $2L$,

$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}mv_B^2 + mg(2L)$$

Using the expression v_B^2 that we found above, the speed v_A is found as follows:

$$v_A^2 = gL + 4gL = 5gL \quad v_A = \sqrt{5gL} = 5 \text{ m/s}$$

Example 5.14



Blocks $m_1=1 \text{ kg}$ and $m_2=2 \text{ kg}$ are tied to each other with a weightless rope passing through a frictionless pulley. The mass m_1 is tied to a spring with $k=100 \text{ N/m}$ on a surface with a coefficient of friction $\mu=0.4$ and the mass m_2 hangs freely. Both blocks are released when the spring is at normal length. Calculate the maximum distance h that the mass m_2 will descend.

Answer We could have solved this problem the long way

again by formally writing the conservation of energy formula. However, a quick thinking simplifies the problem: Since the blocks are at rest both at the beginning and at the end, the elastic potential energy gained by the spring and the negative work done by the friction are solely due to the potential energy loss m_2gh of the mass m_2 . Hence, we write:

$$m_2gh = \frac{1}{2}kh^2 + \mu m_1gh$$

(When the mass m_2 descends a distance of h , the spring also extends for h and the mass m_1 travels a distance of h .)

The height h is calculated from there:

$$h = \frac{2(m_2 - \mu m_1)g}{k} = \frac{2(2 - 0.4 \times 1) \times 10}{100} = 0.32 \text{ m}$$

Example 5.15



A block with mass $m = 1 \text{ kg}$ is connected to two springs with spring constants $k_1 = 30 \text{ N/m}$ and $k_2 = 70 \text{ N/m}$ on a frictionless plane. The springs are initially at normal length when the block is initially at rest. The block is pulled a distance of $x_0 = 50 \text{ cm}$ to the right from its equilibrium position and released.

- (a) What is the speed of the block when it is at a distance of 40 cm from the equilibrium position?
- (b) What is the maximum speed of the block and at what point is it reached?

Answer

(a) When one of the springs is extended by x , the other is compressed by the same amount, and the total elastic po-

tential energy is $\frac{1}{2}(k_1 + k_2)x^2$. Hence, we may consider this system as a single spring with a spring constant of $(k_1 + k_2)$. If the block is released from rest with springs extended by x_0 at the start, then the speed v at a later extension x is found from the conservation of energy:

$$\frac{1}{2}(k_1 + k_2)x_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(k_1 + k_2)x^2$$

$$v = \sqrt{\frac{(k_1 + k_2)(x_0^2 - x^2)}{m}}$$

Calculating for $x = 0.4 \text{ m}$, we find that $v = 3 \text{ m/s}$.

(b) For the speed to be maximum, according to the expression above, the factor $(x_0^2 - x^2)$ in the square root must be maximum. As x_0 was given at the beginning, x must be minimum, in other words, $x = 0$. Thus, the block reaches maximum speed when passing through the equilibrium position. The maximum speed is found when we set $x = 0$ in this formula:

$$v_{\text{max}} = \sqrt{\frac{k_1 + k_2}{m}} x_0 = 5 \text{ m/s}$$

Multiple-choice Questions

1. What is the kinetic energy of an object with mass 2 kg and speed 2 m/s?
 - (a) 2 J (b) 4 J (c) 6 J (d) 8 J
2. Which is incorrect?
 - (a) Force perpendicular to the displacement performs no work.
 - (b) Force opposite to the displacement performs negative work.
 - (c) Work is the product of the parallel component of the force to the displacement with the displacement.
 - (d) Work is the product of the perpendicular component of the force with the displacement.
3. What is the speed of an object with mass 1 kg and kinetic energy 8 J?
 - (a) 3 m/s (b) 4 m/s (c) 5 m/s (d) 6 m/s
4. If the work performed by the net force on an object is zero, then which of the following is correct?
 - (a) Its kinetic energy remains constant.
 - (b) Its potential energy remains constant.
 - (c) Its kinetic energy increases, while its potential energy decreases.
 - (d) Its kinetic energy decreases, while its potential energy increases.
5. Which of the following is correct for a simple pendulum?
 - (a) Potential is minimum at the highest point.
 - (b) Kinetic energy is maximum at the highest point.
 - (c) Potential energy is minimum at the lowest point.
 - (d) Total energy is minimum at the lowest point.
6. What is the power of a crane if it is able to pull a mass of 100 kg upwards at a speed of 2 m/s?
 - (a) 1 kW (b) 2 kW (c) 3 kW (d) 4 kW
7. Which of the following performs the most work?
 - (a) A man carrying a stone with a force of 10 N for 2 m.
 - (b) A man carrying a stone with a force of 5 N for 3 m.
 - (c) A weight-lifter keeping dumbbells of 200 kg fixed.
 - (d) A kid who raises a mass of 20 kg to a height of 1 m.

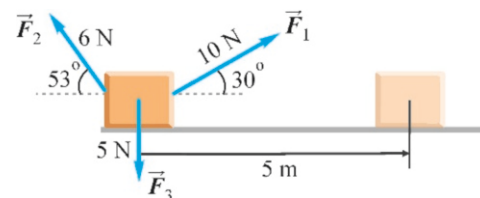
8. Which of the following is incorrect?
- The work performed by a conservative force is independent of the path.
 - The work performed by a nonconservative force is dependent on the path.
 - Potential energy can be defined for a conservative force.
 - Potential energy can be defined for a nonconservative force.
9. An elevator is being pulled upwards at a constant speed. The work performed on the elevator is:
- Zero.
 - Positive.
 - Negative.
 - Constant.
10. If the work done by a net force on an object at rest is doubled, its final speed increases by how many times?
- 1
 - $\sqrt{2}$
 - 2
 - 4
11. How much should the net force acting on an object at rest be increased such that its final speed increases 4 times?
- 2
 - 4
 - 8
 - 16
12. If an amount of work 5 J is needed to extend the length of a spring by 1 cm, then how much work should be done to extend it by 2 cm?
- 6 J
 - 10 J
 - 15 J
 - 20 J
13. Which of the following is correct for the gravitational potential energy of the Earth?
- It is zero at the center of the earth.
 - It is zero at infinity.
 - It is zero on the surface of the earth.
 - It is zero at the center of the Sun.
14. Which of the following is correct for kinetic energy?
- It depends on the direction of motion of the object.
 - It may be negative.
 - It is always positive.
 - It is the same for all observers.
15. Which of the following is correct for the gravitational potential energy?
- It is inversely proportional to the square of the distance.
 - It is proportional to the total mass of the two objects.
 - It is inversely proportional to the distance.
 - It is inversely proportional to the mass.
16. The rotation radius is doubled for an artificial satellite rotating on an orbit around the Earth. How does its potential energy change?
- It halves.
 - It doubles.
 - It decreases to one fourth.
 - It increases by 4 times.
17. An car applies the breaks to stop on a horizontal road. Where is the kinetic energy that it lost spent?
- In the work performed by the friction force.
 - To increase its potential energy.
 - To the heating of the engine.
 - All of the above.
18. Which of the following is correct for an object sliding along a path?
- The friction force does not perform any work.
 - The normal force performs positive work.
 - The normal force performs negative work.
 - The friction force performs negative work.
19. Which of the following is incorrect?
- Friction force is conservative.
 - Gravitational force is conservative.
 - Spring force is conservative.
 - Friction force is nonconservative.
20. Two satellites are rotating in orbits around the Earth at radii $r_1 < r_2$. Which of the following is correct for the gravitational potential energy?
- $U_1 < U_2$
 - $U_1 = U_2$
 - $U_1 > U_2$
 - None.

Problems

5.1–3 Work, Kinetic Energy, Power

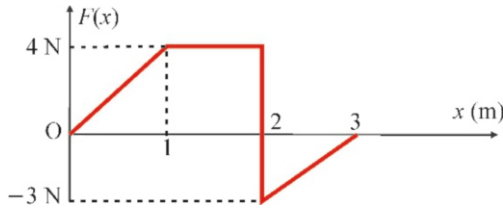
5.1 Calculate the work performed by the forces in the figure on the right, along the 5 m horizontal displacement.

[A: 43 J, -18 J, 0.]



Problem 5.1

5.2 A block with weight $W=10\text{ N}$ is pulled upward on a 37° inclined plane with a force $F=9\text{ N}$ that is parallel to the inclined plane at a constant speed over a distance of 2 m . (a) What is the work performed by the force F ? (b) The work performed by the gravitational force? (c) The work performed by the friction force? [A: (a) 18 J , (b) -12 J , (c) -6 J .]



Problem 5.3

5.3 Find the work performed by the variable force $F(x)$ in the figure. [A: 4.5 J .]

5.4 A force exerted on an object is given as:
 $F(x) = 6x^2 - 2x$ (newton)

Calculate the work performed by this force within the range $x : [0.5\text{ m}]$. [A: 225 J .]

5.5 5 J work is performed to extend a spring by 10 cm . How much work should be done to further extend it by 30 cm ? [A: 80 J .]

5.6 An object at rest with mass 2 kg starts moving and reaches a speed of 8 m/s over a distance of 5 m . What is the constant net force acting on this object and how much work has it performed? [A: 12.8 N and 64 J .]

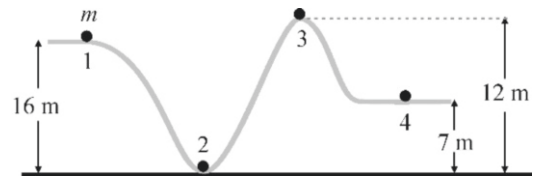
5.7 A cannon has a barrel-length of 6 m . When a shell of mass 20 kg is fired, it leaves the barrel at a speed of 200 m/s . (a) What is the kinetic energy of the shell? (b) What is the the work done on the shell by the average force inside of the barrel? (c) What is the average force exerted on the shell inside of the barrel? [A: (a) 400 kJ , (b) 400 kJ , (c) 67 kN .]

5.8 An athlete with a mass of 60 kg climbs up a hanging rope of length 10 m in 20 s . Calculate the power of the athlete in units of watts and horsepower. [A: 300 W , 0.4 HP .]

5.9 The engine of a car is able to drive it at a constant speed of 72 km/hour while delivering a power of 100 HP . As the traction force of the car is 1400 N , what percent of the power of the engine is lost to friction and air resistance? [A: 62% .]

5.4 Potential Energy

5.10 The normal length of a spring attached to the ceiling is 1 m . The length of this spring becomes 105 cm when a mass of 200 g is hung to its free end. (a) Calculate the spring constant. (b) How much will the spring extend when its potential energy is 5 J ? [A: (a) 40 N/m , (b) 50 cm .]

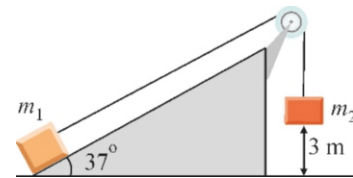


Problem 5.11

5.11 Calculate the gravitational potential energy of the mass $m = 3\text{ kg}$ at the points 1, 2, 3 and 4 shown in the figure. (Indicate where you choose your zero level.) [A: zero level on the ground: $480, 0, 360, 210\text{ J}$.]

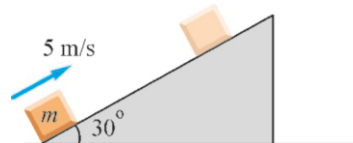
5.12 A telecommunication satellite with a mass of 3 tons was placed into orbit at a distance of $36\,000\text{ km}$ from the surface of the Earth. How much energy is required to lift this satellite to such a height? (Use the values $GM_E=4 \times 10^{14}\text{ m}^3/\text{kg}\cdot\text{s}^2$ and $R_E=6400\text{ km}$.) [A: $1.6 \times 10^{11}\text{ J}$.]

5.5 Law of Conservation of Energy



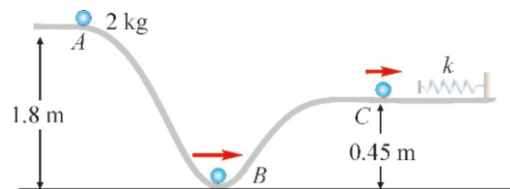
Problem 5.13

5.13 A mass of $m_1=1\text{ kg}$ is at rest on the bottom of a frictionless plane inclined at 37° . It is tied to another mass of $m_2=2\text{ kg}$ with a rope that passes through a frictionless pulley. At the start, the mass m_2 is released from a height of 3 m from the ground. What is the speed of mass m_2 when it reaches the ground? [A: 5.3 m/s .]



Problem 5.14

5.14 A block with mass m is thrown at a speed of 5 m/s up from the lower end of a plane inclined at an angle of 30° and with a coefficient of friction $\mu=0.5$. How much distance does it travel along the inclined plane until it stops? (Note: The mass m will be canceled at the end of the calculation.) [A: 1.34 m .]

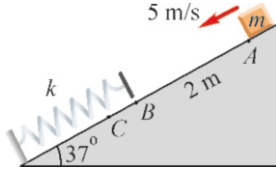


Problem 5.15

5.15 A spring with a constant of $k=200\text{ N/m}$ is fixed after point C along the frictionless path ABC in the figure. A ball

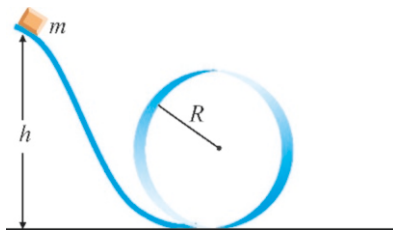
with mass 2 kg is released from rest at the point A. (a) What will be the speed of the ball at points B and C? (b) How much will the ball compress the spring?

[A: (a) $v_B = 6$, $v_C = 5$ m/s, (b) $x = 0.5$ m.]



Problem 5.16

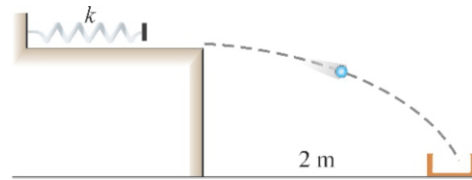
5.16 A spring with a constant of $k=200$ N/m is fixed to the lower end of a plane inclined at an angle of 37° and a coefficient of friction of $\mu=0.4$. A block with mass $m = 1$ kg is thrown from point A downwards at a speed of 5 m/s. The free end B of the spring is at a distance of $AB=2$ m. (a) At what speed will the block hit the free end of the spring? (b) What is the maximum compression of the spring, in other words, what is $BC = x$? [A: (a) 6 m/s, (b) 0.44 m.]



Problem 5.17

5.17 A portion of a frictionless rail track is bent as a circular loop with radius R . A block with mass m is released from rest at a height h from one end of the rail. From what height h should the block be released such that it can complete the loop without leaving the rails? (Hint: As long as the block touches the rails, there is some normal force acting on it. What should be its minimum value at the peak point?)

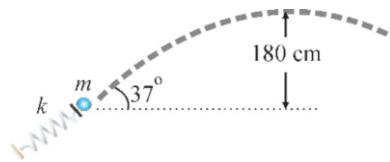
[A: $h = 5R/2$.]



Problem 5.18

5.18 A ball is compressed against a spring fixed on a frictionless table and released. We would like the ball to fly off of the edge of the table and fall inside a bowl on the floor at a distance of 2 m. In a first trial in which the spring is compressed by 20 cm, the ball falls 40 cm short of the bowl. How much should the spring be compressed so that the ball falls into the bowl in the second trial? (Hint: There is no need to completely solve the horizontal projectile problem. The ball's time of flight will be the same in both trials.)

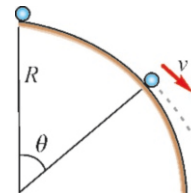
[A: 25 cm.]



Problem 5.19

5.19 A spring gun with a spring constant $k=500$ N/m is able to shoot bullets of 50 g. It is observed that a bullet fired by this gun at an angle of 37° to the horizontal reaches a height of 180 cm. (a) Find the initial speed of the bullet, (b) Find the amount of compression of the spring.

[A: (a) 10 m/s, (b) 0.1 m]



Problem 5.20

5.20 A ball released at the peak point of a frictionless spherical surface with a radius of R starts to slide down the surface. At what angle θ will it leave the surface? (Hint: The normal reaction force is zero where it leaves the surface. The centripetal force needed during the circular motion is simply the radial component of the weight.)

[A: $\cos \theta = 2/3$, $\theta = 48^\circ$.]

6

IMPULSE AND MOMENTUM



The forces generated during the collision of billiard balls have a very complex structure and are difficult to examine.

However, there is one quantity that always remains the same before and after the collision. We will find out what this is in this chapter. (Photo: Dave Jackson)

Collision problems are one of the cases in which direct application of Newton's laws is difficult. When two billiard balls collide, action-reaction forces arise between them during that very short contact. The details of these forces are complicated and very difficult to examine. However, these two balls have some well-defined velocities before and after the collision. Is it possible to find a relation between the initial and final states without looking into the internal mechanisms of the collision?

In this chapter, we will define the concepts of impulse and momentum, which are easy to interpret for the interaction of two or more particles. Furthermore, the momentum concept leads to another law of conservation that will make it easier to solve many problems.

6.1 IMPULSE AND MOMENTUM

Let us consider Newton's second law. We know that acceleration in this law is the derivative of velocity:

$$\vec{F} = m \vec{a} = m \frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt}$$

Mass m can be included in the derivative because it is constant. This expression shows that force itself is also the derivative of another quantity.

Definition: The **momentum vector** of an object with mass m and velocity vector \vec{v} is

$$\vec{p} = m \vec{v} \quad (\text{Momentum}) \quad (6.1)$$

The unit of momentum is kg.m/s and does not have a particular name. Newton's second law can thus also be written as follows (Newton had actually expressed the second law in this form):

$$\vec{F} = \frac{d\vec{p}}{dt} \quad (6.2)$$

Let the same object with mass m initially have a momentum $\vec{p} = m\vec{v}$ and then, let a constant force \vec{F} act on it during a short time interval Δt . After the interaction, the object may have a different momentum $\vec{p}' = m\vec{v}'$. If we rewrite Eq. (6.2) with Δt ,

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\vec{p}' - \vec{p}}{\Delta t}$$

or

$$\begin{aligned} \vec{p} + \vec{F}\Delta t &= \vec{p}' \\ m\vec{v} + \vec{F}\Delta t &= m\vec{v}' \end{aligned} \quad (6.3)$$

This expression resembles the work-energy theorem that we discussed in Chapter 5. There, the increase in kinetic energy was equal to the work performed by the force \vec{F} . And, in this formula, the increase in momentum is equal to the product of force with time. It is defined as a new quantity:

Definition: The integral of a variable force \vec{F} over a finite interval $[t_1, t_2]$,

$$\vec{J} = \int_{t_1}^{t_2} \vec{F} dt \quad (\text{Impulse}) \quad (6.4)$$

is called the **impulse** of the force \vec{F} . It has the same unit as momentum (kg.m/s), and is mostly used for forces that act for a short period of time.

Let us rewrite the above Eq. (6.3) in terms of impulse and momentum:

$$\vec{p} + \vec{J} = \vec{p}' \quad (\text{Impulse-momentum theorem}) \quad (6.5)$$

This result is called the **Impulse-momentum theorem**. We thus obtain a simple expression between the initial and final velocities of the object without having to know its acceleration. This is equivalent to Newton's second law.

Unlike work and energy, impulse and momentum are *vector* quantities. Their directions should be taken into consideration when used in formulas.

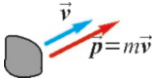
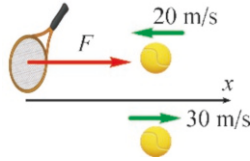


Figure 6.1: Momentum vector.

Example 6.1



A tennis ball incoming with a speed of 20 m/s is hit by a racket and is sent back at a speed of 30 m/s. The mass of the ball is 56 g and its contact with the racket lasts 0.05 seconds. Calculate the average force acting on the ball.

Answer

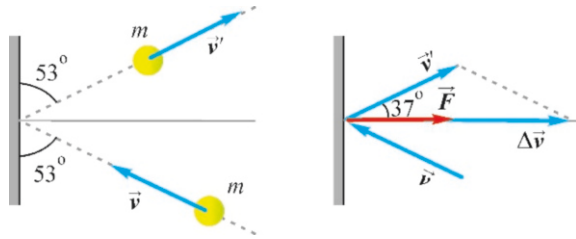
We should be careful to treat velocities as vectors when using the impulse-momentum theorem (Eq. 6.3). Choosing the direction of the outgoing ball as the positive x -direction, we write:

$$mv + F \Delta t = mv'$$

$$0.056 \times (-20) + F \times 0.05 = 0.056 \times (+30)$$

Solving for F , we find that
 $F = 56 \text{ N}$.

Example 6.2



A ball with a mass of 1 kg and a speed of $v=10 \text{ m/s}$ hits a wall at an angle of 53° and is reflected back by the same angle and at the same speed. If the ball is in contact with the wall for 0.02 s, find the direction and magnitude of the force applied by the wall on the ball.

dimensions:

$$\vec{F} \Delta t = m\vec{v}' - m\vec{v} \rightarrow \vec{F} = \frac{m}{\Delta t} \Delta\vec{v}$$

Hence, \vec{F} will have the same direction as the vector $\Delta\vec{v}$. As seen in the figure, when we calculate the vector $\Delta\vec{v} = \vec{v}' - \vec{v}$ by the triangle rule as the sum $[\vec{v}' + (-\vec{v})]$, it is found to be perpendicular to the wall.

Therefore, the vector \vec{F} that is proportional to $\Delta\vec{v}$ will also be perpendicular to the wall and in the $+x$ direction.

As the initial and final velocities have equal magnitude, we get $\Delta v = 2v \cos 37^\circ$. From this, we can calculate the magnitude of the force F :

$$F = \frac{m \Delta v}{\Delta t} = \frac{2mv \cos 37^\circ}{\Delta t} = \frac{2 \times 10 \times 0.8}{0.02} = 800 \text{ N}$$

Answer We write the impulse-momentum theorem in 2-

6.2 LAW OF CONSERVATION OF MOMENTUM

Rewriting Newton's second law in terms of momentum is not very interesting for a single object. Its real impact can be seen when expressed for two interacting objects.

Let us consider two masses with initial momentums $\vec{p}_1 = m_1 \vec{v}_1$ and $\vec{p}_2 = m_2 \vec{v}_2$. Let us further consider that no external force acts on this system and that the masses only interact with each other (Figure 6.2). Let the momentums become $\vec{p}_1' = m_1 \vec{v}_1'$ and $\vec{p}_2' = m_2 \vec{v}_2'$ after this interaction.

Let us write Newton's law for the mass m_1 in its Eq. (6.3) form:

$$\vec{p}_1 + \vec{F}_{21} \Delta t = \vec{p}_1'$$

In this equation, \vec{F}_{21} denotes the force applied by the second object on the first. Let us write the same law for the second object:

$$\vec{p}_2 + \vec{F}_{12} \Delta t = \vec{p}_2'$$

If we add these two equations, we get:

$$\vec{p}_1 + \vec{p}_2 + (\vec{F}_{21} + \vec{F}_{12}) \Delta t = \vec{p}_1' + \vec{p}_2'$$

The sum inside of the brackets is zero, because $\vec{F}_{21} = -\vec{F}_{12}$ according to the third law. From this, we get the **law of conservation of momentum**:

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' = \text{constant} \quad (\text{law of conservation of momentum}) \quad (6.6)$$

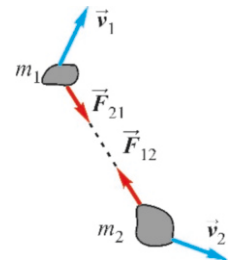


Figure 6.2: Two interacting masses.

Notice that there is no need to know the details of the interaction between the two objects.

This is a very general result, and is also valid for more than two objects. In such a case, **the total momentum of a system** is defined as follows:

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_N \quad (\text{total momentum}) \quad (6.7)$$

Therefore, the general law of conservation of momentum can be expressed as:

Law of Conservation of Momentum

If the external forces acting on a system are zero, the total momentum of the system will remain constant in time:

$$\sum_i F_i^{\text{ext}} = 0 \implies \vec{P} = \vec{P}' \quad (6.8)$$

$$\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_N = \vec{p}_1' + \vec{p}_2' + \cdots + \vec{p}_N'$$

This vector equation is true for each component. Let us write them for two objects:

$$\begin{aligned} m_1 v_{1x} + m_2 v_{2x} &= m_1 v'_{1x} + m_2 v'_{2x} \\ m_1 v_{1y} + m_2 v_{2y} &= m_1 v'_{1y} + m_2 v'_{2y} \end{aligned} \quad (6.9)$$

Important notice: The law of conservation of momentum is actually more general than the one that we stated above. Let us say briefly that the external forces need not be zero; it is sufficient that *their impulse be zero* for the the law of conservation of momentum to hold. For example, there may be external forces, but the time interval may be so short that their impulse can be neglected. We will not go into these fine points.

Example 6.3

A block with mass $m_1 = 1 \text{ kg}$ travels at a speed of 8 m/s on frictionless rails and collides into a block with a mass of $m_2 = 10 \text{ kg}$ traveling in the same direction at a speed of 5 m/s . After the collision, the block m_2 is observed to travel at a speed of 6 m/s in the same direction. Find the magnitude and direction of the final velocity of block m_1 .



Answer

Let us write the formulas (6.9) that we found for the conservation of momentum in one dimension (the subscripts are not necessary):

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$$

Choosing $+x$ in the direction of motion of the two blocks, we get

$$1 \times 8 + 10 \times 5 = 1 \times v'_1 + 10 \times 6$$

And, from here, we get $v'_1 = -2 \text{ m/s}$. The negative sign shows that the block m_1 is going backwards.

Example 6.4

A mass $M = 10 \text{ kg}$ traveling at a speed of 8 m/s on a frictionless horizontal road suddenly explodes and splits into two pieces. The piece with mass $m_2 = 7 \text{ kg}$ travels in the same direction with a speed of 14 m/s . Find the velocity of the piece with mass 3 kg .



Answer There is a single mass M at the start. If we indicate its velocity with V , conservation of momentum is expressed as follows:

$$\begin{aligned}
 MV &= m_1 v'_1 + m_2 v'_2 \\
 \text{Substitute } m_1=3, m_2=7 \text{ kg:} \\
 10 \times 8 &= 3v'_1 + 7 \times 14
 \end{aligned}$$

From here, we get $v'_1 = -6 \text{ m/s}$. The velocity of mass m_1 is in the opposite direction.

6.3 COLLISIONS IN ONE DIMENSION

In this section, we will examine collision problems in more detail. This is a very important class of problems that can arise, from billiard balls to shells hitting targets, from atomic particles smashing into each other to celestial bodies interacting over immense distances via the gravitational force.

We will first consider collisions in one dimension to minimize mathematical difficulties. Collisions are classified into two groups in this analysis.

Elastic Collisions

We have established that the total momentum of the system is conserved in all collisions in which the net external force is zero. Let us write this only for the x component:

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \quad (6.10)$$

(Here, we do not write the x -subscripts, but you must remember that these velocities are vectors, in other words, they can have positive or negative values.)

Collisions in which the total kinetic energy of the colliding objects is conserved are called elastic collisions. For example, in collisions between steel or glass balls, when the balls come into contact with each other, their surfaces stretch like a spring and store the energy. After the collision, as they return to their previous forms, this potential energy converts into kinetic energy. This property is not observed in collisions between objects made of clay or dough; they get deformed and lose the kinetic energy as heat.

According to this definition, kinetic energy is also conserved in elastic collisions:

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \quad (6.11)$$

Therefore, by solving the two Eqs.(6.10 and 6.11), we can find the velocities v'_1 and v'_2 after the collision. Explicit expressions of the general solution exist, but they are too complicated to reproduce here.

Let us seek the solution for a special case here. If mass m_2 is at rest before collision, we can set $v_2 = 0$ in the two equations above and simplify as follows:

$$\begin{aligned}
 m_1 v_1 &= m_1 v'_1 + m_2 v'_2 \\
 m_1 v_1^2 &= m_1 v_1'^2 + m_2 v_2'^2
 \end{aligned}$$

Let us isolate the unknown v'_2 in both equations:

$$\begin{aligned}
 m_1(v_1 - v'_1) &= m_2 v'_2 \\
 m_1(v_1^2 - v_1'^2) &= m_2 v_2'^2
 \end{aligned}$$

If we write the difference of two squares as $(v_1 - v'_1)(v_1 + v'_1)$ and divide both sides, we get

$$v_1 + v'_1 = v_2'$$

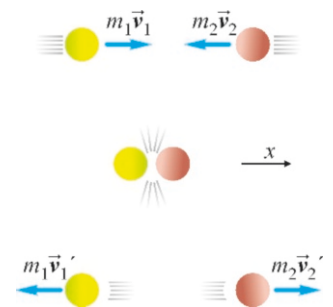


Figure 6.3: Collision in one dimension.

and we can use this equation instead of the energy equation with squares. Now, if we consider this last expression together with the equation for the conservation of momentum (6.10), we get the following solution:

$$\begin{aligned}v'_1 &= \frac{m_1 - m_2}{m_1 + m_2} v_1 \\v'_2 &= \frac{2m_1}{m_1 + m_2} v_1\end{aligned}\tag{6.12}$$

Now, let us draw some important conclusions from these results:

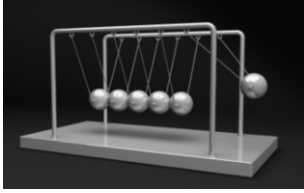


Figure 6.4: Newton's Cradle. Because the masses are equal, the momentum of the incoming ball is fully transferred to the last ball.

1. If the masses are equal ($m_1 = m_2$): We get $v'_1 = 0$ and $v'_2 = v_1$. In other words, they exchange their velocities; the incoming object stops and the other one starts with exactly the same velocity. We observe this in billiard balls in head-on collisions. The elegant toy called *Newton's cradle* uses this property (Figure 6.4): When the ball at one end is pulled and released, the balls in the middle remain motionless and the ball at the other end bounces off with the same velocity. Because the masses are equal, each ball in the middle transfers all of its energy and momentum to the next one and remains motionless.
2. If $m_1 > m_2$, in other words, if the incoming mass is heavier, Eqs. (6.12) tell us that the final velocities of both masses will be positive and both will continue in the same direction. A small mass at rest will never bounce a heavier mass back. When a truck crashes into a motionless automobile, it drags it along. (The truck-automobile collision is not elastic, but we may expect this behavior as an approximation.)
3. If $m_1 < m_2$, in other words, if the incoming mass is lighter, the final velocity of the incoming mass will be negative, which means that it will bounce back. When an automobile crashes into a truck, it may sometimes bounce back.
4. If the incoming mass is much much lighter ($m_1/m_2 \approx 0$), then the final velocities will be $v'_1 \approx -v_1$ and $v'_2 \approx 0$. In other words, the incoming mass will bounce back with the same velocity and the heavy mass will not move. It is as if it is bouncing back from a wall. Rutherford discovered the structure of the atom using this feature. When he bombarded heavy gold (Au) atoms with lighter alpha particles, he observed that some alpha particles bounced back with the same velocity. This was a clue to the existence of a heavy nucleus at the center of the atom.

Inelastic Collision

Collisions in which kinetic energy is not conserved are called **inelastic** collisions. In this case, the colliding objects can become deformed and kinetic energy can be transformed into heat. Thus, the total kinetic energy of the system will have decreased. The collisions of plastic balls used in sports or the collision of cars in traffic are inelastic collisions.

Conservation of momentum is still valid in inelastic collisions:

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2\tag{6.13}$$

However, we can no longer write conservation of kinetic energy. Therefore, it is difficult to find both of the final velocities. Additional information may be given in certain cases, so that final velocities can be calculated in such cases.

Totally Inelastic Collision

In this special case of inelastic collision, the two objects stick together after the collision. A meteor striking Earth or a bullet or an arrow hitting a target are examples of such types of collisions. In this case, the final velocities of the objects are equal and the conservation of momentum gives the solution:

$$v'_1 = v'_2 = v'$$

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v'$$

From here, we can find the common final speed:

$$v' = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \tag{6.14}$$



Figure 6.5: Cars are manufactured in such a way that their front and rear parts collapse readily in a collision. Thus, less energy and momentum will be transferred to the passengers.

Example 6.5

A ball with mass 3 kg travels at a speed of 10 m/s and collides into a ball with mass 5 kg at rest. As the collision is elastic, find the final velocities of the balls.

Answer

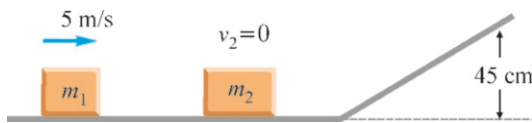
We can use Eqs. (6.12) derived for elastic collision, because one of the masses is at rest:

$$v'_1 = \frac{m_1 - m_2}{m_1 + m_2} v_1 = \frac{3 - 5}{3 + 5} \times 10 = -2.5 \text{ m/s}$$

The negative sign means that the ball m_1 bounces back. The velocity of the second ball is calculated using the same formula:

$$v'_2 = \frac{2m_1}{m_1 + m_2} v_1 = \frac{2 \times 3}{8} \times 10 = 7.5 \text{ m/s}$$

Example 6.6



A block with a mass of $m_1=1$ kg and a speed of 5 m/s hits another block with a mass of $m_2=2$ kg at rest. After the collision, the block m_2 climbs up to a height of 45 cm on a frictionless inclined plane. Calculate the velocities of these blocks right after the collision.

Answer

We first find the velocity of block m_2 on the ground level. According to conservation of energy, its potential energy at height h is supplied by the kinetic energy on the ground:

$$\frac{1}{2}m_2v_2'^2 = m_2gh \rightarrow v_2' = \sqrt{2gh} = \sqrt{2 \times 10 \times 0.45} = 3 \text{ m/s}$$

We then write conservation of momentum for the collision:

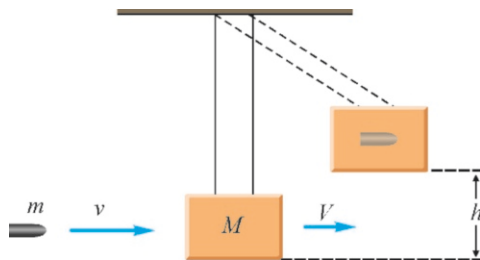
$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2$$

We calculate v'_1 by substituting the numerical values:

$$1 \times 5 + 0 = 1 \times v'_1 + 2 \times 3$$

$$v'_1 = -1 \text{ m/s}$$

Example 6.7



A **ballistic pendulum** is a device used to calculate the muzzle velocities of bullets. It is a pendulum constructed by hanging a wooden block with a very high mass at the end of a rope with length L . When a bullet with mass m traveling at a speed v embeds itself into the wooden block with mass M , the (bullet+block) system rises by h . Calculate the speed of the bullet in

terms of height h . Numerical example: $m = 50$ g, $M = 5$ kg, $L = 2$ m and $h = 80$ cm.

Answer

We first find the speed of the (bullet+block) system right after the collision, in other words, when the pendulum is in the vertical position. We write conservation of momentum for this inelastic collision:

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v'$$

Here, we have $m_1=m$ and $m_2=M$, and if we show the final velocity of the system with V , we get:

$$mv + M \cdot 0 = (m + M)V \rightarrow V = \frac{m}{m + M} v$$

The (bullet+block) system rises to a height h due to its kinetic energy. According to conservation of energy, all of the kinetic energy at the start will be converted into gravitational potential energy:

$$\frac{1}{2}(m + M) V^2 = (m + M) gh$$

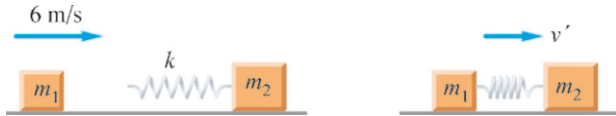
Substituting V , we solve for v and get

$$v = \frac{m + M}{m} \sqrt{2gh}$$

With the given numerical values, we get $v = 400 \text{ m/s}$.

Example 6.8

A spring with spring constant $k = 100 \text{ N/m}$ is mounted onto the front of a block with a mass of 2 kg at rest on a frictionless horizontal plane. Another block with a mass of 1 kg approaches at a speed of 6 m/s and collides with the spring. (a) How much will the spring get compressed and what will the velocities of the blocks be at that instant? (b) What will the velocities of the blocks be after they are separated?



Answer

This is also an elastic collision, because energy is stored by the spring during the collision and later given back. Accordingly, we can immediately solve item (b). However, let us first look into item (a). It seems that there are three variables (the final velocities of the blocks and the amount of compression of the spring) and we only have two equations. However, consider this point: at the instant of maximum compression, the velocities of the blocks will be equal for a short instant.

Conservation of energy and momentum is thus sufficient for the solution.

(a) If we use v' to show the common velocities of the blocks when the spring is compressed to a maximum, conservation of momentum is expressed as follows:

$$m_1 v_1 + 0 = (m_1 + m_2) v'$$

$$v' = \frac{m_1 v_1}{m_1 + m_2} = \frac{1 \times 6}{1 + 2} = 2 \text{ m/s}$$

Spring potential energy is also taken into consideration when writing conservation of energy:

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} (m_1 + m_2) v'^2 + \frac{1}{2} k x^2$$

From here, we can calculate the amount of compression of the spring:

$$x = \sqrt{\frac{m_1 v_1^2 - (m_1 + m_2) v'^2}{k}} = 0.49 \text{ m.}$$

(b) After the blocks are separated, the initial kinetic energy is regained, in other words, the collision is elastic. We use the formulas (6.12), because the block m_2 is at rest before collision:

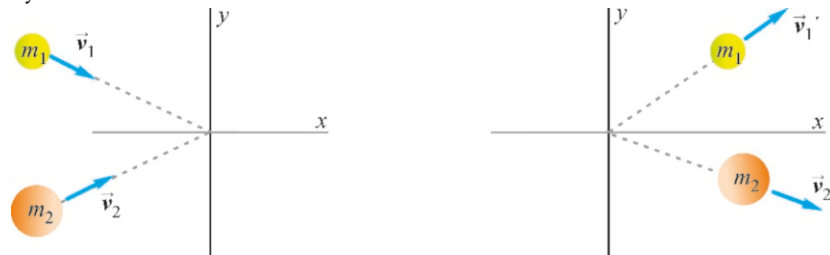
$$v'_1 = \frac{m_1 - m_2}{m_1 + m_2} v_1 = \frac{1 - 2}{1 + 2} \times 6 = -2 \text{ m/s}$$

$$v'_2 = \frac{2m_1}{m_1 + m_2} v_1 = \frac{2}{3} \times 6 = 4 \text{ m/s}$$

6.4 COLLISIONS IN TWO DIMENSIONS

Collisions in one dimension are rarely observed. Two colliding objects usually go off into different directions. In the most general case, collisions in three dimensional space can also be examined in two dimensions. This is because, the momentum vectors of the two objects intersect at the collision point and form a plane that is called the **collision plane**. Since total momentum is conserved, the final momentum vectors should also lie in this plane. Hence, it will be sufficient to study collisions in two dimensions.

Figure 6.6: Coordinates for a collision in two dimensions.



Eq. (6.6) which gives the law of conservation of momentum was found as a vector equation. Therefore, it should be valid separately for both components in a collision in two dimensions.

$$\vec{p}_1 + \vec{p}_2 = \vec{p}'_1 + \vec{p}'_2 \implies \begin{cases} m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x} \\ m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y} \end{cases} \quad (6.15)$$

There are four unknown variables in this two-equation system, because the final velocities have two components and the problem cannot be solved without additional information. Another equation is provided by the conservation of energy for elastic collisions:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \quad (\text{for elastic collision}) \quad (6.16)$$

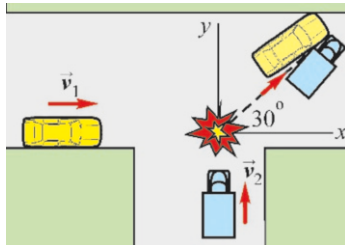
Hence, an additional equation is needed in order to solve the problem. In general, one of the variables is measured experimentally.

There is no such problem for totally inelastic collisions. If we write the final velocities as equal, we get

$$\begin{aligned} \vec{v}_1' &= \vec{v}_2' = \vec{v}' \\ m_1v_{1x} + m_2v_{2x} &= (m_1 + m_2)v'_x \\ m_1v_{1y} + m_2v_{2y} &= (m_1 + m_2)v'_y \end{aligned} \quad (6.17)$$

Two equations with two unknowns will give the components of the final velocity.

Example 6.9



A car with a mass of $m_1=800$ kg traveling East and a pickup truck with a mass of $m_2=1200$ kg traveling North collide head on in an intersection and stick together. Both drivers claim that the other entered the intersection with excessive speed. You arrive at the site as the expert. You measure the black tire tracks left on the tar surface and find that the tracks make an angle

of 30° with the eastern direction. Using only this information, calculate the ratio v_2/v_1 and determine which car was going faster.

Answer

Choosing the axes as shown in the figure, we use Eqs. (6.17) for a totally inelastic collision in two dimensions:

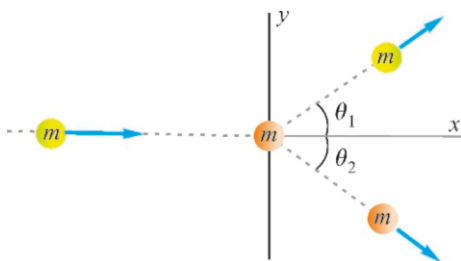
$$\begin{aligned} m_1v_{1x} + m_2v_{2x} &= (m_1 + m_2)v'_x \rightarrow m_1v_1 + 0 = (m_1 + m_2)v' \cos 30^\circ \\ m_1v_{1y} + m_2v_{2y} &= (m_1 + m_2)v'_y \rightarrow 0 + m_2v_2 = (m_1 + m_2)v' \sin 30^\circ \end{aligned}$$

Dividing both sides of the equation, we find the ratio v_2/v_1 :

$$\frac{v_2}{v_1} = \frac{m_1}{m_2} \tan 30^\circ = \frac{800}{1200} \times 0.58 = 0.38$$

According to this result, the car with mass m_1 was traveling approximately 3 times faster than the pickup truck. (Do not underestimate this example. Conservation of momentum and energy are the most important tools of traffic experts.)

Example 6.10



Proton-proton collisions are an important source of information in nuclear physics. A proton with a speed of 5×10^6 m/s has an elastic collision with another proton at rest. After the collision, the incoming proton is observed to scatter off at a speed of 3×10^6 m/s. Calculate the scattering angles of the protons and the speed of the second proton.

Answer

The velocity v_2' of the second proton can immediately be

found from conservation of energy ($m_1=m_2=m$):

$$\begin{aligned} \frac{1}{2}mv_1^2 + 0 &= \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 \\ v_2' &= \sqrt{v_1^2 - v_1'^2} = \sqrt{5^2 - 3^2} \times 10^6 = 4 \times 10^6 \text{ m/s} \end{aligned}$$

In order to find the angles, we must write conservation of momentum in two dimensions with axes as shown in the figure:

$$\begin{aligned} m_1v_{1x} + m_2v_{2x} &= m_1v'_{1x} + m_2v'_{2x} \rightarrow v_1 = v'_1 \cos \theta_1 + v'_2 \cos \theta_2 \\ m_1v_{1y} + m_2v_{2y} &= m_1v'_{1y} + m_2v'_{2y} \rightarrow 0 = v'_1 \sin \theta_1 - v'_2 \sin \theta_2 \end{aligned}$$

Substituting known velocities and simplifying, we get:

$$\begin{aligned} 5 &= 3 \cos \theta_1 + 4 \cos \theta_2 \rightarrow 4 \cos \theta_2 = 5 - 3 \cos \theta_1 \\ 0 &= 3 \sin \theta_1 - 4 \sin \theta_2 \rightarrow 4 \sin \theta_2 = 3 \sin \theta_1 \end{aligned}$$

We add the squares of both sides and use the identity ($\sin^2 \theta_2 + \cos^2 \theta_2 = 1$), finding that

$$16 = 9 \sin^2 \theta_1 + (9 \cos^2 \theta_1 - 30 \cos \theta_1 + 25)$$

We again use the identity ($\sin^2 \theta_1 + \cos^2 \theta_1 = 1$) and simplify:

$$\cos \theta_1 = 0.6 \quad \rightarrow \quad \theta_1 = 53^\circ$$

Substituting this θ_1 value in the equation for the y -component of momentum, we find θ_2 :

$$\sin \theta_2 = \frac{3}{4} \sin \theta_1 = \frac{3}{5} \rightarrow \theta_2 = 37^\circ$$

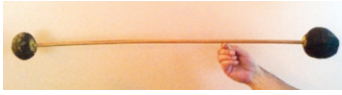


Figure 6.7: The point that balances a stick with a heavier end.

6.5 CENTER OF MASS

When we try to carry a piece of timber, we carry it by lifting it from its center. If one end of a stick is heavier than the other, we hold it somewhere near the heavier end (Figure 6.7). People on a boat will not gather at the same spot, but will sit at different places so that the boat will not capsize. These examples indicate that there is a mean point that represents the mass of extensive objects.

Definition: The “weighted” average of mass positions constituting a system is called the **center of mass** (CM).

By the weighted average, we understand that each position is included in the average as multiplied by the value of the mass at that position.

According to this definition, the expression for the center of mass of $m_1, m_2 \dots m_N$ located at $x_1, x_2 \dots x_N$ along the x -axis will be as follows:

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_N x_N}{m_1 + m_2 + \dots + m_N} = \frac{\sum_{i=1}^N m_i x_i}{\sum_{i=1}^N m_i} \quad (6.18)$$

The y - and z -coordinates of the center of mass can be similarly defined in three-dimensional space:

$$y_{\text{cm}} = \frac{\sum_i m_i y_i}{M}, \quad z_{\text{cm}} = \frac{\sum_i m_i z_i}{M} \quad (6.19)$$

Here, $M = m_1 + \dots + m_N$ is the total mass. If we write these three formulas as a single vector formula, we get:

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{M} \quad (\text{center of mass}) \quad (6.20)$$

In the simplest system, let us consider two point masses m_1 and m_2 located at positions x_1 and x_2 along the x -axis (Figure 6.8). Applying the formula for center of mass, we get

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad (6.21)$$

If the masses are equal ($m_1 = m_2$), the center of mass is the midpoint $x_{\text{cm}} = (x_1 + x_2)/2$. This expression conforms to our daily experience. We hold a straight stick from the center, but if a weight is put on one end, we hold it from a point nearer to that side.

Center of Mass of a Continuous Mass Distribution

For objects that have a continuous mass distribution, we can calculate their center of mass using either one of the following two methods:

1. **Symmetry.** As seen in Figure 6.9, objects may have certain geometrical shapes, such as a triangle, square, rectangle, disc or circle, that have a center of symmetry. In addition, if they are homogeneous, in other words, if their density is distributed evenly, then we can assume that the center of mass will be located at that center of symmetry. The summing expression for center of mass that we found for pointlike objects can thus also be applied to whole symmetrical parts.

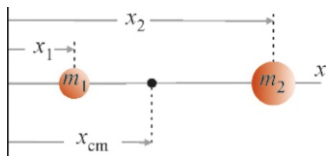


Figure 6.8: The center of mass of two objects.

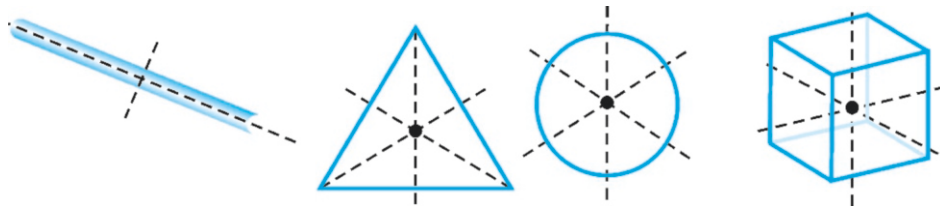


Figure 6.9: The centers of mass of symmetric objects whose mass is distributed homogeneously.

2. **Integration.** If the mass distribution in space is known as a function of position, integration technique can be used to find the center of mass. Let us see how the integral form is set up.

Let us divide the object with mass M into N number of small Δm_i masses located at the position (x_i, y_i, z_i) . Applying the formula for center of mass for these small parts, for example, for the x -component we get

$$x_{cm} \approx \frac{\sum_i x_i \Delta m_i}{\sum_i \Delta m_i} = \frac{\sum_i x_i \Delta m_i}{M}$$

The sum in the denominator is written as the total mass M . Similar formulas can be written for the y - and z -components.

Now, when we take the limit $\Delta m_i \rightarrow 0$, the sum in the numerator will become an integral. Therefore, the integral expressions of all three components can be written as follows:

$$x_{cm} = \frac{\int x dm}{M}, \quad y_{cm} = \frac{\int y dm}{M}, \quad z_{cm} = \frac{\int z dm}{M} \quad (6.22)$$

In these integrals, mass should be expressed as $dm = \rho(r) dr$ in terms of a density that depends on position. The details will be clear in the worked examples below.

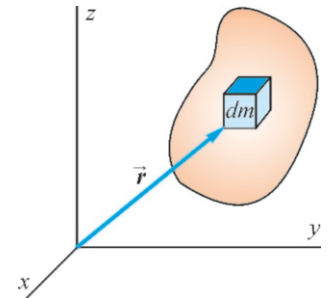
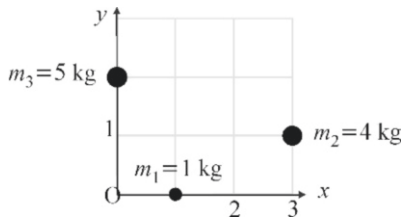


Figure 6.10: The dm mass element for continuously distributed mass.

Example 6.11



Determine the center of mass of the 3-mass system whose masses and positions on the xy plane are given in the figure.

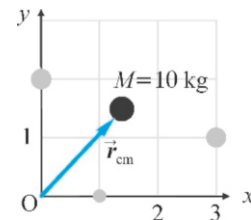
Answer

We write Eqs. (6.20), which we found for the coordinates of the center of mass, for the x - and y -components and calculate as follows:

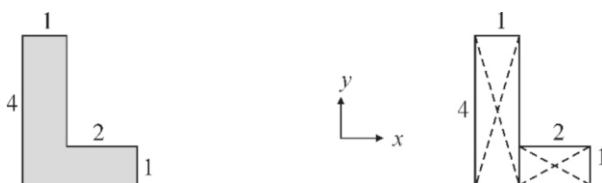
$$x_{cm} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{1 \times 1 + 4 \times 3 + 5 \times 0}{1 + 4 + 5} = 1.3 \text{ m}$$

$$y_{cm} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{1 \times 0 + 4 \times 1 + 5 \times 2}{10} = 1.4 \text{ m}$$

The position of the center of mass is indicated on the figure below.



Example 6.12



Find the center of mass of a plate shaped like the letter **L** with dimensions shown in the figure and with homogeneous density.

Answer Let us first separate the plate into two simpler pieces, as shown in the figure. The mass of each plate will be proportional to its surface area, because the same density coefficients will cancel each other out in the numerator and denominator.

The centers of mass of these pieces will be their symmetry center points, because their densities are homogeneous. Accordingly, the coordinates of the center of mass are calculated as follows:

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{4 \times 0.5 + 2 \times 2}{4 + 2} = 1 \text{ units}$$

$$y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{4 \times 2 + 2 \times 0.5}{6} = 1.5 \text{ units}$$

Example 6.13

Use the integration method to calculate the coordinates of the center of mass of a plate shaped like a right triangle, in which the perpendicular sides have the lengths a , b and the density is homogeneous.



Answer

Since the plate is homogeneous, its mass is proportional to its surface area. Let us choose the axes as shown in the figure

and consider a thin piece with thickness dx between x and $x + dx$ along the x -axis. The area of this piece is $y dx$ and its center of mass will be its own x coordinate.

The equation for a line passing through the origin with a slope of (b/a) is $y = (b/a)x$. Accordingly, Eq. (6.22) can be written as:

$$x_{\text{cm}} = \frac{1}{M} \int x dm = \frac{1}{ab/2} \int_0^a x (y dx)$$

$$= \frac{2}{ab} \int_0^a x (b/a)x dx = \frac{2}{a^2} \int_0^a x^2 dx = \frac{2}{a^2} \left[\frac{x^3}{3} \right]_0^a$$

$$x_{\text{cm}} = \frac{2}{3} a$$

Using a similar calculation, we find that $y_{\text{cm}} = \frac{1}{3} b$.

Dynamics of the Center of Mass

Consider a system consisting of two particles. In Chapter 4, we learned that forces acting on interacting masses could be separated into two groups: The external forces and the internal forces that they exert on each other. Accordingly, let us use \vec{F}_1^{ext} to show the external force exerted on m_1 and \vec{F}_{21} for the force exerted by the second mass. When writing Newton's law separately for each particle, let us write accelerations as the second derivative of position:

$$\text{for } m_1 : \quad \vec{F}_1^{\text{ext}} + \vec{F}_{21} = m_1 \vec{a}_1 = \frac{d^2}{dt^2} (m_1 \vec{r}_1)$$

$$\text{for } m_2 : \quad \vec{F}_2^{\text{ext}} + \vec{F}_{12} = m_2 \vec{a}_2 = \frac{d^2}{dt^2} (m_2 \vec{r}_2)$$

Adding these two equations, we get:

$$(\vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}}) + (\vec{F}_{21} + \vec{F}_{12}) = \frac{d^2}{dt^2} (m_1 \vec{r}_1 + m_2 \vec{r}_2)$$

According to the third law, the action and reaction forces between the masses will be equal and opposite, $\vec{F}_{21} = -\vec{F}_{12}$. Hence, the content of the second brackets on the left side is zero. Dividing and multiplying with the same total mass M , we obtain:

$$(\vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}}) = M \frac{d^2}{dt^2} \left(\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \right) = M \frac{d^2 \vec{r}_{\text{cm}}}{dt^2}$$

The second derivative on the right-hand side of the equation is just the acceleration \vec{a}_{cm} of the center of mass. Consequently, we obtain Newton's law, which determines the motion of the center of mass:

$$\sum_i \vec{F}_i^{\text{ext}} = M \vec{a}_{\text{cm}} \quad (\text{motion of center of mass}) \quad (6.23)$$

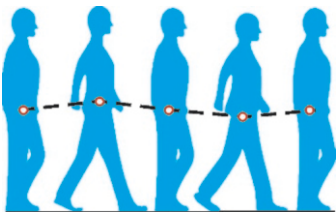


Figure 6.11: The center of mass of your body moves up and down as you walk, and you thus perform work against gravity. You will get less tired if you take smaller steps.

This result is important: *In many-particle systems, there is one point that moves with the resultant force of external forces, such that it obeys Newton's law as if the whole mass was concentrated at one point.* The expression (6.23) is actually the real definition of the center of mass.



Figure 6.12: The arms and legs of this boy jumping into the sea can follow many different paths, but he will always have a point that moves as if the whole mass was centered at that point. That point is the center of mass. Note that the center of mass of the boy, which is around his belly, draws a perfect parabola. (Photo: Erik Forsberg).

6.6 ROCKET MOTION

Airplanes fly by using the pressure differences of the air. For this reason, for many years, it was believed that it would be impossible to build crafts capable of flying in outer space, because there is no air to push. However, today, we have rockets that can fly in space using the principle of conservation of momentum and the reaction force of Newton's third law. A rocket moves with the forward momentum that it gains by ejecting its fuel backwards.

Let us examine the motion of the rocket with respect to an observer on the ground (Figure 6.13). In a gravity-free environment, let the total mass of the rocket and its fuel be m and velocity v in the positive direction at a given initial time t . Let a small mass Δm be ejected backwards as fuel during a time interval Δt with a speed v_{exh} with respect to the rocket.

Let the mass of the rocket be $m - \Delta m$ and its velocity $v + \Delta v$ at a later time $t + \Delta t$. We write the conservation of momentum for the (rocket+fuel) system:

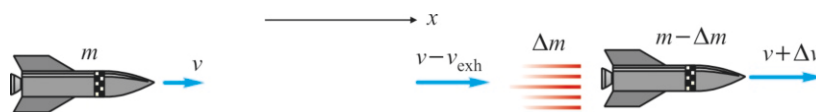


Figure 6.13: Initial and final velocities of the rocket-fuel system.

$$mv = (m - \Delta m)(v + \Delta v) + \Delta m(v - v_{\text{exh}})$$

Notice that we have written the velocity of the fuel with respect to the ground as $(v - v_{\text{exh}})$. Simplifying and neglecting the very small term $\Delta m \Delta v$, we find:

$$v_{\text{exh}} \Delta m = m \Delta v \quad (6.24)$$

We divide by Δt in order to form derivatives:

$$\underbrace{v_{\text{exh}} \frac{\Delta m}{\Delta t}}_{F_{\text{thr}}} = m \frac{\Delta v}{\Delta t}$$

The term on the left-hand side has the dimensions of a force and it is called the **thrust** of the fuel. And, in the limit $\Delta t \rightarrow 0$, the derivative on the right-hand side gives the acceleration of the rocket. In conclusion, we have Newton's law for rocket motion:

$$F_{\text{thr}} = v_{\text{exh}} \frac{\Delta m}{\Delta t} = ma \quad (\text{rocket motion}) \quad (6.25)$$

The ratio $\Delta m/\Delta t$ in the expression of the thrust on the right-hand side is the mass of fuel exhausted per unit time. We can see that, the higher the exhaust speed of the fuel, the higher the thrust will be.

During launch from Earth's surface, the weight $-mg$ should also be added to the left-hand side of this expression, in other words, the thrust should be greater than the weight: $F_{\text{thr}} - mg > 0$.

We now want to calculate the rocket speed as a function of its mass. Let us return to the expression (6.24):

$$v_{\text{exh}} \Delta m = m \Delta v$$

Here, Δm was the positive change in exhausted fuel mass. If we want to write the change in the rocket's mass, it will be just the opposite of this quantity. Hence, by taking $-\Delta m$ and dividing by m , we can write the equation governing the change in rocket mass:

$$v_{\text{exh}} \frac{(-\Delta m)}{m} = \Delta v$$

Let the rocket's initial mass be m_0 and its initial velocity v_0 , and let them reach the final values m and v . Taking the limits $(\Delta m, \Delta v) \rightarrow (dm, dv)$ and integrating both sides, we get:

$$\begin{aligned} -v_{\text{exh}} \int_{m_0}^m \frac{dm}{m} &= \int_{v_0}^v dv \\ -v_{\text{exh}} \ln \frac{m}{m_0} &= v - v_0 \end{aligned}$$

Hence, the final velocity of the rocket in terms of mass loss is:

$$v = v_0 + v_{\text{exh}} \ln \frac{m_0}{m} \quad (6.26)$$

where the property $\ln(a/b) = -\ln(b/a)$ of the logarithm was used to remove the negative sign and to obtain a positive logarithm.

Example 6.14

A rocket traveling in space at a constant speed of 80 m/s ejects an amount of fuel backwards with a ratio of 1/20 to its own mass at a speed of 200 m/s with respect to the rocket. What will the final velocity of the rocket be?

Answer

As the mass loss m_0/m is given, the velocity is calculated

using the formula (6.26):

$$v = v_0 + v_{\text{exh}} \ln \frac{m_0}{m}$$

Here, we have $m_0/m = 20/19$. Also, the numerical values of v_{exh} and v_0 are given in accordance with the definitions in the formula. From here, we can find the final velocity:

$$v = 80 + 200 \times \ln(20/19) = 80 + 200 \times 0.51 = 90 \text{ m/s}$$

Example 6.15

A rocket with a mass of 1000 kg is ejecting fuel at a rate of 10 kg per second and with a speed of 1200 m/s during its launch from the surface of the earth. (a) What is the thrust? (b) What is the initial acceleration?

Answer

(a) We use Eq. (6.25) for the thrust:

$$F_{\text{thr}} = v_{\text{exh}} \frac{\Delta m}{\Delta t}$$

We substitute the given ratio $\Delta m/\Delta t = 10$ and the exhaust

velocity, to find that

$$F_{\text{exh}} = 1200 \times 10 = 12000 \text{ N}$$

(b) In addition to thrust, there is now the weight of the rocket opposing motion. We write the second law for the net force acting on the rocket:

$$F_{\text{exh}} - mg = ma \rightarrow a = \frac{F_{\text{exh}} - mg}{m} = \frac{12000 - 10000}{1000} = 2 \text{ m/s}^2$$

This is only the initial acceleration; it will change later as the mass changes.

Multiple-choice Questions

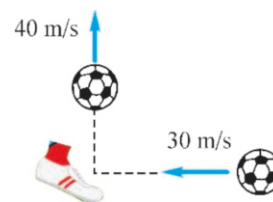
- Which quantity is preserved during all types of collisions?
 - Potential energy.
 - Kinetic energy.
 - Momentum magnitude.
 - Momentum vector.
- Which is incorrect?
 - Momentum is zero if the kinetic energy is zero.
 - Momentum is zero if the velocity is zero.
 - Momentum is zero if the potential energy is zero.
 - Momentum is constant if the net force is zero.
- Two objects with equal masses of m collide with equal and opposite velocities v . What is the total momentum of the system?
 - 0
 - mv
 - $2mv$
 - $-mv$
- A particle's speed is doubled and its mass is tripled. By what factor will its momentum increase?
 - 2
 - 6
 - 12
 - 18
- Which is correct for two objects?
 - If their momentum is equal, their kinetic energies are also equal.
 - If their momentums are equal, their velocities are also equal.
 - If their momentums are equal, they will move in the same directions.
 - If their momentums are equal, their masses are also equal.
- Which is correct for an elastic collision?
 - Only momentum is conserved.
 - Only kinetic energy is conserved.
 - Both momentum and kinetic energy are conserved.
 - None of the above.
- A block with a velocity of 3 m/s elastically collides with another block with equal mass at rest. What will be the final velocity of the incoming block?
 - 0
 - 3 m/s
 - +3 m/s
 - 6 m/s
- Which one of the following is the expression of kinetic energy in terms of momentum?
 - $K = p^2/2$
 - $K = p^2/2m$
 - $K = 2mp^2$
 - $K = 2p^2$
- A boy with mass $2m$ standing on a frictionless horizontal plane throws a package of mass m with a velocity v in the $+x$ direction. What will be the velocity of the boy?
 - 0
 - $-v$
 - $-2v$
 - $-v/2$
- Which of the following is correct if two masses collide and stick to each other?
 - Momentum is conserved.
 - Kinetic energy is conserved.
 - Both momentum and kinetic energy are conserved.
 - Potential energy is conserved.

11. An object at rest gains a velocity v under the action of a force F that lasts t seconds. How much velocity will the object gain by a force $3F$ acting for $3t$ seconds?
 (a) v (b) $3v$ (c) $9v$ (d) $v/3$
12. A spring is placed between two blocks with unequal masses and compressed. What will happen when the blocks are released?
 (a) They will move away with equal and opposite velocities.
 (b) The one with the larger mass will also have higher velocity.
 (c) The one with the lower mass will have lower velocity.
 (d) The one with the lower mass will have higher velocity.
13. A spring is placed between blocks with unequal masses and compressed. Which one will be true after the blocks are released?
 (a) The kinetic energies of the blocks will be equal.
 (b) Their momentums will be equal and opposite.
 (c) The one with the higher mass will have higher momentum.
 (d) The one with the lower mass will have higher momentum.
14. A car and a truck moving towards each other at the same speed collide and stick together. Which of the following is correct?
 (a) The force on the car is greater.
 (b) The force on the truck is greater.
 (c) The car will receive the kinetic energy lost by the truck.
 (d) The automobile will receive the momentum lost by the truck.
15. Which of the following is incorrect if the total momentum of a system is zero?
 (a) Its kinetic energy will also be zero.
 (b) Its center of mass will be at rest.
 (c) The net force acting on the system is zero.
 (d) All of the above.
16. Which of the following is correct if the net force acting on an object is zero?
 (a) The impulse is zero.
 (b) The momentum will remain constant.
 (c) Its center of mass will be in uniform linear motion.
 (d) All of the above.
17. If the position of mass m is $x_1 = 0$ and the position of mass $2m$ is $x_2 = 6$, then what will the position of the center of mass x_{cm} be?
 (a) 2 (b) 2.5 (c) 3 (d) 4
18. Two equal masses are at positions x_1 and x_2 . What will the coordinate of the center of mass be?
 (a) $x_2 - x_1$
 (b) $x_1 + x_2$
 (c) $(x_1 + x_2)/2$
 (d) $(x_2 - x_1)/2$
19. Which of the following is correct when a rifle is fired?
 (a) The momentum of the rifle is greater.
 (b) The momentums of the rifle and the bullet are equal and opposite.
 (c) The kinetic energies of the rifle and bullet are equal.
 (d) The kinetic energy of the rifle is greater than that of the bullet.
20. A spring is placed between two blocks at rest and compressed. What will happen when the blocks are released?
 (a) The center of mass will travel at constant velocity.
 (b) The center of mass will move towards the heavier block.
 (c) The center of mass will move towards the lighter block.
 (d) The center of mass will remain at rest.

Problems

6.1 Impulse and Momentum

6.1 A football player hits a ball of mass 400 g with an incoming speed of 30 m/s and sends it back in the same direction at a speed of 40 m/s. Since the contact of the ball with his foot took 0.04 s, what will the average force be on the football player's foot? [A: 700 N.]



Problem 6.2

6.2 A football player hits a ball of mass 400 g coming from

the right corner with a speed of 30 m/s and sends it towards the goal perpendicular to its incoming direction at a speed of 40 m/s. Since the contact of the ball with his foot took 0.05 s, find the average force acting on the ball and its direction.

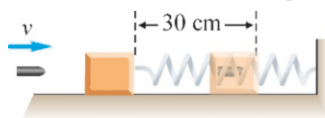
[A: 400 N and 53° with the corner.]

6.3 A ball with a mass of 0.5 kg is dropped from a height of $h_1=5$ m. The ball bounces back to a height of $h_2 = 3.2$ m after hitting the ground. What is the impulse given to the ball during the collision with the ground? [A: $J = 9$ kg·m/s.]

6.2-3 Conservation of Momentum-Collisions in One Dimension

6.4 A block with a mass of 1 kg traveling at a speed of 10 m/s on a frictionless horizontal road collides into another block with a mass of 4 kg that is at rest. If the collision is elastic, find the final velocities of the blocks. [A: -6, +4 m/s.]

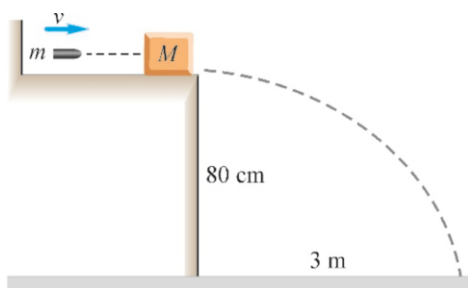
6.5 A block with a mass of 2 kg traveling at a speed of 8 m/s on a frictionless horizontal road collides into another block with a mass of 3 kg that is coming from the opposite direction with a speed of 6 m/s. After collision the blocks stick together. Find the velocity of the blocks after the collision. [A: -0.4 m/s.]



Problem 6.6

6.6 A wooden block with a mass of 950 g is attached to the free end of a spring with a spring constant of $k = 100$ N/m on a frictionless horizontal plane. The spring is initially at normal length. A bullet with a mass of 50 g and a speed v hits the block and is embedded in it. If the spring is compressed by 30 cm, what was the initial velocity of the bullet? [A: 60 m/s.]

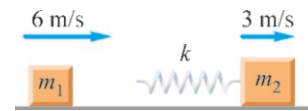
6.7 A block with a mass of 1 kg traveling at a speed of 30 m/s on a frictionless horizontal road collides with another block with a mass of 4 kg incoming from the opposite direction. Both blocks stay at rest after the collision. What was the initial velocity of the second block? [A: 7.5 m/s.]



Problem 6.8

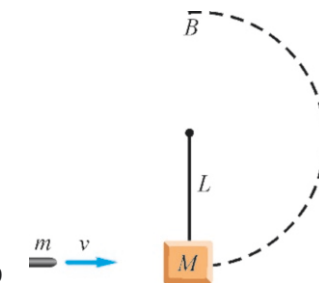
6.8 A bullet with a mass of $m = 50$ g is fired horizontally towards a wooden block with a mass of $M = 950$ g that is at rest near the edge of a tabletop 80 cm in height. The bullet is embedded in the block. The (block+bullet) system falls to the

ground at a horizontal distance of 3 m from the edge. Find the initial velocity of the bullet. [A: 150 m/s.]



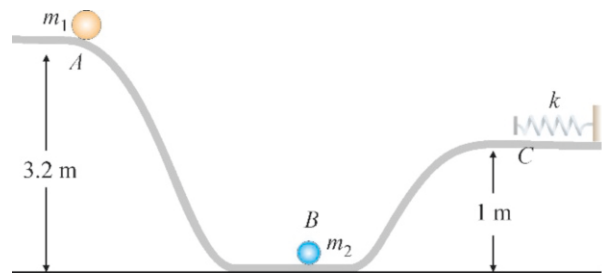
Problem 6.9

6.9 A spring with a constant of $k=100$ N/m is attached to the back of a block with a mass of 2 kg traveling at a speed of 3 m/s on a frictionless horizontal plane. Another block with a mass of 1 kg comes with a speed of 6 m/s and collides with the spring. What will the maximum compression of the spring be and what will the velocities of the blocks be at that instant? [A: 24 cm and 4 m/s.]



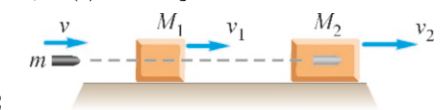
Problem 6.10

6.10 A bullet with a mass of $m=50$ g travels at a speed v and hits a block with a mass of $M=950$ g tied to the end of a rope of length $L = 2$ m. The bullet is embedded in the block. What should the initial velocity v of the bullet be such that the (bullet+block) system will make one complete loop without the rope becoming slack at point B? [A: 200 m/s.]



Problem 6.11

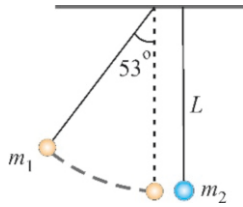
6.11 A ball with mass $m_1=3$ kg released from rest from point A at a height of 3.2 m travels along the frictionless path ABC, collides with and sticks to another ball with mass $m_2=1$ kg standing at rest at point B. After the collision, the $(m_1 + m_2)$ system climbs up to point C at a height of 1 m and compresses the spring with a constant of $k=400$ N/m, fixed to the plane. Calculate (a) the velocity right after collision, (b) the velocity at point C and (c) the amount of compression of the spring. [A: (a) 6 m/s, (b) 4 m/s, (c) 40 cm.]



Problem 6.12

6.12 Blocks with masses $M_1=5$ kg and $M_2=0.95$ kg are at rest on a frictionless horizontal surface. A bullet with mass $m=50$ g fired at a speed of $v=400$ m/s passes through the first block and embeds in the second block. After the collision, the

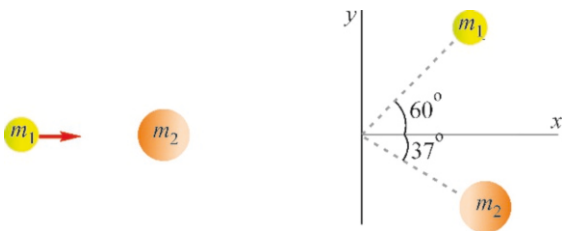
velocity of the first block is measured to be 3 m/s. Find the velocity of the second block. [A: 5 m/s.]



Problem 6.13

6.13 Two balls with masses $m_1=3$ kg and $m_2=2$ kg are hanging side by side on the ends of two strings of equal lengths L to form two pendulums side by side. The first ball is pulled aside and released at an angle of 53° from the vertical. If the balls collide elastically, what will be the maximum angle that the second ball makes with the vertical? [A: 65° .]

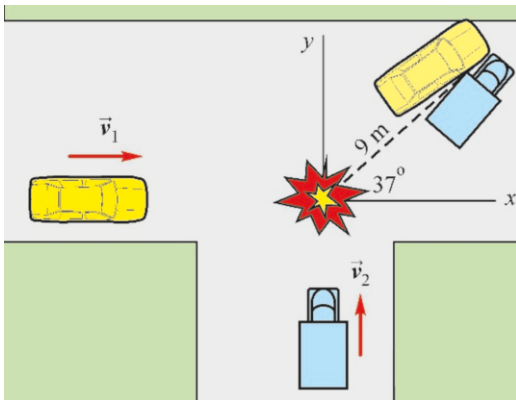
6.4 Collisions in Two Dimensions



Problem 6.14

6.14 A ball with $m_1=1$ kg travels on a frictionless horizontal plane at velocity $v_0=10$ m/s and collides with another ball with mass $m_2=2$ kg at rest. After the collision, the balls scatter, m_1 at an angle of 60° and m_2 at an angle of -37° . Calculate the final velocities of the balls.

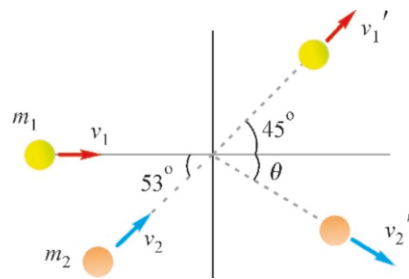
[A: $v_1 = 6.0$, $v_2 = 4.4$ m/s.]



Problem 6.15

6.15 A car with a mass of 800 kg traveling East and a pickup truck with a mass of 1200 kg traveling North collide at a junction and stick to each other. According to the tire tracks on the road, the vehicles had drifted off by 9 m in a direction that makes an angle of 37° . Given that the coefficient of friction of the tarmac road is $\mu=0.8$, determine the velocities of both vehicles before the collision.

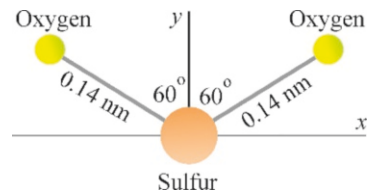
[A: Car: 24 m/s, pickup truck: 12 m/s.]



Problem 6.16

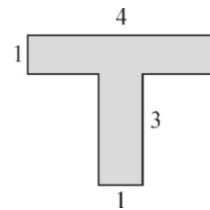
6.16 A ball with mass $m_1=1$ kg is traveling in the $+x$ direction at a velocity of $v_1=10$ m/s. It collides with another ball with a mass of $m_2=2$ kg incoming with a velocity of $v_2=5$ m/s that makes an angle of -53° with the x axis. After the collision, m_1 is scattered at a velocity of 15 m/s in a direction that makes an angle of $+45^\circ$ with the x -axis. Calculate the velocity components of m_2 . [A: 2.7, -1.3 m/s.]

6.5 Center of Mass



Problem 6.17

6.17 In a sulfur dioxide (SO_2) molecule, the S-O bonds have a length of 0.14 nm and the angle between them is 120° . Since the mass of sulfur is approximately twice that of oxygen, find the distance of the center of mass of this molecule to the S atom. [A: $y_{\text{cm}} = 0.035$ nm.]



Problem 6.18

6.18 Find the center of mass of a T shaped homogeneous plate with the dimensions given in the figure above.

[A: $y_{\text{cm}} = 2.6$]

6.6 Rocket Motion

6.19 A rocket in uniform linear motion in outer space with a constant speed of 120 m/s fires its engine and ejects an amount of fuel backwards at a ratio of 1/30 to its own mass at a speed of 300 m/s with respect to the rocket. What will the final velocity of the rocket be? [A: 130 m/s.]

6.20 A rocket with mass 800 kg is ejecting fuel at an amount of 500 kg per minute at a speed of 1500 m/s during its launch from the surface of the earth. (a) What is the thrust? (b) What is the initial acceleration? [A: (a) 12500 N, (b) 5.6 m/s².]

7

ROTATIONAL MOTION



Night sky in a time-lapse picture. The Earth rotates around its own axis, and we perceive it as the rotational motion of the stars.

Although each of these stars seems to have a different rotation velocity, there is one quantity that is common for all. In this chapter, we will learn what it is.

Up until now, we have considered an object as having a speed and acceleration as a whole. Now, consider the rotating wheel of a car. Each point of this wheel rotates at a different speed and acceleration. Therefore, we cannot examine the rotation of extended objects using the methods that we have learned so far.

In previous chapters, we have made calculations based on the assumption that all forces on an object acted on a single point. This is called a *pointlike object*. Although we drew objects such as cars, blocks and spheres, we always examined them as pointlike objects. However, there are extended objects in real life; they have volumes, and forces act on different parts of them. In this chapter, we will take this property of rigid bodies into consideration.

A body that takes up volume in space and does not change its shape is called a **rigid body**. (They are also called solid bodies.) Wheels, doors, automobiles, the Earth, the Moon, etc., are all examples of rigid bodies.

The motion of a rigid body is much richer than that of a pointlike object. Let

us consider the kinds of motions that rigid bodies can perform:

Figure 7.1: The most general motion of an object: Translation, rotation and vibration.

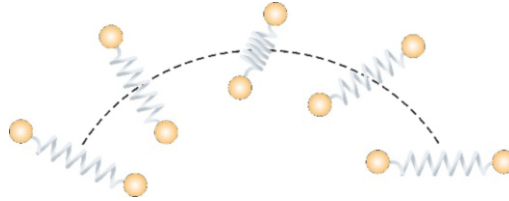


Figure 7.2: The net force acting on this ruler stick is zero. But it is not in equilibrium and will perform rotational motion.

1. The motion of the rigid body's center of mass is called the **translational motion**. Both ends of a baton thrown up in the air by a majorette can rise by rotating, but the center of mass of the stick travels on a parabolic trajectory, just like a pointlike stone.
2. Even if its center of mass remains at rest, a rigid body can perform **rotational motion** by rotating around its own center of mass. Let us apply a force \vec{F} perpendicular to one end of a ruler stick and a force $-\vec{F}$ to the other end (Figure 7.2). As the net force acting on the object is zero, according to the first law, its center of mass will either remain at rest or perform uniform linear motion. However, there is nothing preventing the stick from rotating around its own center of mass.
3. Bodies can also perform **vibrational motion**. If we push two masses connected by a string towards the center, the object will vibrate, even though the center of mass remains at rest. However, vibrational motion is not included in the definition of a rigid body, because the body changes shape and will be examined separately later in Chapter 9.

In Chapter 6, we saw that the center of mass moved as if all of the external forces were acting at the center of mass. Hence, the translational motion of the rigid body is the motion of its center of mass. It will thus be sufficient to examine the rotational motion of rigid bodies here.

7.1 ANGULAR KINEMATICS

The rotational motion of rigid bodies may seem complex at first. Each point of the body has a different position, velocity and acceleration, and you may guess the mathematical difficulty of examining all of these together.

However, we notice something here: Although each point of a wheel rotates at a different velocity, when the wheel makes one full revolution, each point will have made the same revolution. If one point rotates by a certain angle, all points will rotate by that angle. Therefore, rigid bodies should naturally be analyzed using angular coordinates. The topic that deals with concepts of angular velocity and angular acceleration is called **angular kinematics**.

Angular Position (θ)

Let us consider a point P rotating on a circular trajectory with radius r . Let us choose a **reference line** from which the angles are measured. If a point P travels some length of arc s from the reference line, the subtended angle,

$$\theta = \frac{s}{r} \quad (\text{angular position}) \quad (7.1)$$

is called the **angular position** of P (Figure 7.3). According to the most commonly accepted convention, *counterclockwise angles are taken as positive, and clockwise as negative*.

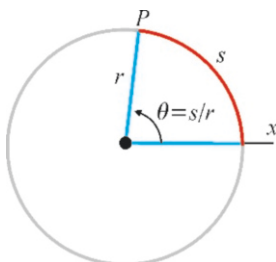


Figure 7.3: Angular position θ .

The unit of angle is the **radian** in angular kinematics. Using other units, like degrees ($^\circ$) or grades, will give incorrect results. The formula $\theta=s/r$ actually indicates that the radian is the natural unit of angle. Indeed, if we substitute the circumference of a circle as $s=2\pi r$, we get the angle 2π radians. Therefore, angles should be taken as radians in angular kinematics calculations. Remember the following rule for converting degrees into radians:

$$1 \text{ revolution} = 360^\circ = 2\pi \text{ radians}$$

Accordingly, $180^\circ=\pi$ radians, $60^\circ=\pi/3$ radians, $45^\circ=\pi/4$ radians, etc.

Angular Velocity (ω)

Let us again consider a point P rotating on a circle with radius r . If the angular position of this point at t_1 is θ_1 , and at a later time t_2 is θ_2 , the ratio

$$\omega_{\text{av}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \quad (7.2)$$

is the **average angular velocity** during this time interval.

The unit for angular velocity is radians/second (rad/s). Another unit that is used in industry is revolutions per minute (rpm):

$$1 \text{ rpm} = \frac{1 \text{ revolution}}{\text{minute}} = \frac{2\pi \text{ rad}}{60 \text{ s}} \approx 0.10 \text{ rad/s}$$

As in linear motion, **instantaneous angular velocity** is defined by taking the derivative at the limit $\Delta t \rightarrow 0$:

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad (\text{angular velocity}) \quad (7.3)$$

Angular Acceleration (α)

Angular acceleration is defined as the change of angular velocity in unit time and the ratio

$$\alpha_{\text{av}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \quad (7.4)$$

is the average angular acceleration during this time interval.

The $\Delta t \rightarrow 0$ limit of average acceleration is **instantaneous angular acceleration**:

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \quad (\text{angular acceleration}) \quad (7.5)$$

The unit of angular acceleration is rad/s^2 . Acceleration is positive if the object gets faster as it rotates in the counterclockwise direction and negative if it gets slower.

Let us emphasize once more: *Angular velocity and angular acceleration have the same value at all points of the rigid body.*

Motion With Constant Angular Acceleration

The angular acceleration α of a rigid body is constant if its angular velocity changes uniformly. In such a case, we get formulas for angular position and angular velocity, just as we did for linear motion in Chapter 2. There is no need to repeat this calculation. The results are displayed in the following table, together with the linear motion formulas:

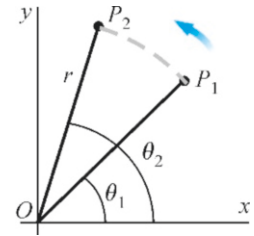


Figure 7.4:

Circular and linear motions with constant acceleration

Circular

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

Linear

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2} at^2$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

Example 7.1

A wheel at rest starts to rotate with constant angular acceleration and reaches an angular velocity of 90 rpm after 10 s.

- (a) What is the angular acceleration of the wheel?
 (b) How many revolutions has it performed during this period?

Answer

(a) We use the velocity formula for constant acceleration:

$$\omega = \omega_0 + \alpha t$$

The radian unit must be used in all calculations. The initial velocity is zero: $\omega_0 = 0$. Let us substitute the final velocity

with correct units rad/s:

$$90 \times \frac{2\pi \text{ rad}}{60 \text{ s}} = 0 + \alpha \times 10 \rightarrow \alpha = 0.3\pi \text{ rad/s}^2$$

(b) We use the angular position formula:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

Here, we take the point $\theta_0 = 0$ of the wheel:

$$\theta = 0 + 0 + \frac{1}{2} \times 0.3\pi \times 10^2 = 15\pi \text{ radians}$$

The angle is requested in terms of revolutions. One revolution is taken as 2π :

$$\theta = 15\pi \text{ radians} = \frac{15\pi}{2\pi} = 7.5 \text{ revolutions.}$$

Example 7.2

An electrical motor rotating with an angular velocity of 600 rpm starts to decelerate when the power is disconnected and its angular velocity decreases to 150 rpm in 3 s.

- (a) What is the angular acceleration?
 (b) How many revolutions will it perform during this time interval?
 (c) How many revolutions will it perform from the start until it stops?

Answer

(a) We substitute the given velocities in the angular velocity formula in terms of rad/s and find the angular acceleration:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{(150 - 600) \times \pi / 30}{3}$$

$$\alpha = -5\pi \text{ rad/s}^2$$

(b) We use the angular position formula:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 = 0 + \frac{600 \times \pi}{30} \times 3 + \frac{1}{2} \times (-5\pi) \times 3^2$$

$$\theta = 37.5\pi \text{ radians} = 18.8 \text{ revolutions.}$$

(c) The final velocity is zero when the motor stops. We use the velocity formula without time:

$$\omega^2 - \omega_0^2 = 2\alpha\theta \rightarrow \theta = \frac{\omega^2 - \omega_0^2}{2\alpha}$$

$$\theta = \frac{0 - (600\pi/30)^2}{2 \times (-5\pi)} = 40\pi \text{ radians} = 20 \text{ revolutions}$$

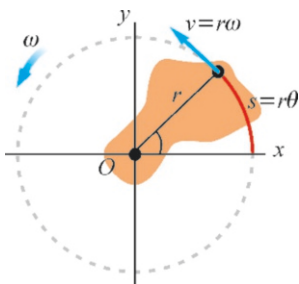


Figure 7.5: Angular and linear coordinates.

Relation Between Angular and Linear Kinematics

When a rigid body performs rotational motion, any point on it makes a circular motion with a certain radius. That point has some well-defined linear speed, tangential and centripetal accelerations. It is important to know how these quantities are related to angular velocity and acceleration.

Let us rewrite Eq.(7.1) which gives us the relation between angular and linear positions:

$$s = r\theta$$

As the angle θ changes over time, so will the arc s . Taking the derivative of both sides with respect to time t , we find that

$$\frac{ds}{dt} = \frac{d(r\theta)}{dt} = r \frac{d\theta}{dt}$$

where the constant radius r was moved outside of the derivative. The derivative on the right-hand side is the angular velocity ω and the derivative on the left-hand side is just the linear velocity v of point P :

$$v = r\omega \quad (7.6)$$

Acceleration has two components in circular motion (Figure 7.6). Let us remember the formula for the **centripetal acceleration**:

$$a_r = \frac{v^2}{r}$$

Substituting the expression $v = r\omega$ that we found above for linear velocity, we can write centripetal acceleration in terms of ω :

$$a_r = \frac{(r\omega)^2}{r} = r\omega^2 \quad (7.7)$$

For **tangential acceleration**, let us rewrite the expression (3.19) that we found in Chapter 3:

$$a_t = \frac{dv}{dt}$$

If we substitute the expression $v = r\omega$ that we found for linear velocity and take the derivative, as the radius r is constant, we get

$$a_t = r \frac{d\omega}{dt} = r\alpha \quad (7.8)$$

These formulas completely establish the relation between the rotational motion of the solid body and the linear motion of any point on it. Let us summarize these formulas in a table:

The relation between angular and linear kinematics		
	Angular	Linear
Position	θ	$s = r\theta$
Velocity	ω	$v = r\omega$
Acceleration	α	$\left\{ \begin{array}{l} a_r = r\omega^2 \text{ (centripetal acceleration)} \\ a_t = r\alpha \text{ (tangential acceleration)} \end{array} \right.$

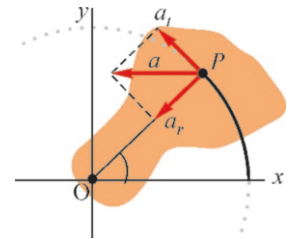


Figure 7.6: Tangential and centripetal accelerations.

Example 7.3

A wheel starts from rest and accelerates at a constant rate to reach a velocity of 60 rpm after 5 revolutions.

- What is the angular acceleration of the wheel?
- What are the linear velocity and tangential and centripetal accelerations of a point located at a distance of 2 m from the center of the wheel?

Answer (a) The angular velocity of the wheel is 60 rpm = $60 \times$

$\pi/30 = 2\pi$. We find angular acceleration using the velocity formula without time: $\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$

$$\alpha = \frac{\omega^2 - 0}{2(\theta - 0)} = \frac{(2\pi)^2}{2 \times 5\pi} = 0.4\pi \text{ rad/s}^2$$

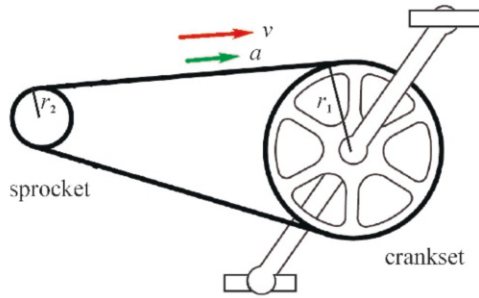
It is a good practice to leave π in angular calculations. They are either canceled out later or the value $\pi^2 \approx 10$ can be substituted.

(b) We use the formulas for conversion to linear kinematics for $r = 2$ m:

Linear velocity: $v = r\omega = 2 \times 2\pi = 4\pi \text{ m/s}$
 Tangential acceleration: $a_t = r\alpha = 2 \times 0.4\pi = 0.8\pi \text{ m/s}^2$

Centripetal acceleration: $a_r = r\omega^2 = 2 \times (2\pi)^2 = 8\pi^2 \approx 80 \text{ m/s}^2$.

Example 7.4



On a bicycle the crankset (toothed disk) fixed to the pedal has a radius of 10 cm and the sprocket (smaller toothed disk) at the center of the rear wheel has a radius of 5 cm. The two are connected by a chain.

- (a) If the angular velocity of the crankset is $\omega_1 = 40 \text{ rpm}$ and its angular acceleration is $\alpha_1 = 7 \text{ rad/s}^2$, what will be the angular velocity and angular acceleration of the rear wheel?
- (b) Calculate the centripetal accelerations of points on the rim of the crankset and of the sprocket.

Answer

(a) The chain is the connection between the two wheels. All of the points on this chain have equal linear speed and accel-

eration. In particular, the linear speeds of points on the rim of both disks must be equal:

$$v = v_1 = v_2$$

We use the relation $v = r\omega$ for each wheel:

$$r_1\omega_1 = r_2\omega_2 \rightarrow \omega_2 = \frac{r_1}{r_2}\omega_1$$

There is no need to change units here, because we are working with ratios:

$$\omega_2 = \frac{10 \text{ cm}}{5 \text{ cm}} \times 40 \text{ rpm} = 80 \text{ rpm}$$

Now, in addition to linear velocities, the linear acceleration of the chain is also the same everywhere. In particular, at the points on the rim, they become the tangential acceleration for each circular motion:

$$a = a_{1t} = a_{2t}$$

Using the relation $a_t = r\alpha$ that connects the tangential and angular acceleration for each wheel,

$$\alpha_2 = \frac{r_1}{r_2}\alpha_1 = \frac{10 \text{ cm}}{5 \text{ cm}} \times 7 \text{ rad/s}^2 = 14 \text{ rad/s}^2$$

(b) The formula $a_r = r\omega^2$ giving the centripetal acceleration is used for both disks:

$$a_{1r} = r_1\omega_1^2 = 0.10 \times (40 \times \pi/30)^2 = 1.8 \text{ m/s}^2$$

$$a_{2r} = r_2\omega_2^2 = 0.05 \times (80 \times \pi/30)^2 = 3.6 \text{ m/s}^2$$

7.2 TORQUE (MOMENT OF A FORCE)

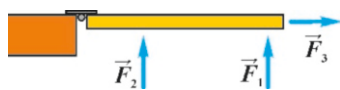


Figure 7.7: Which of these forces will rotate the door more easily?

How do we open a door? The door will open (Figure 7.7) if we push or pull with a perpendicular force (F_1) from the free side of the door. However, it will be difficult to open it if we apply the same force on a point near the hinges (F_2). Likewise, it will not move if we pull with a force along the door (F_3). The ability of the same force to rotate an object is different in all of these cases.

The ability of a force to rotate an object is called its **torque** (or, its moment). We shall introduce the torque in a simple way, then generalize.

Definition: Let \vec{F} be a force acting at a position \vec{r} (Figure 7.8). The magnitude of the **torque** about the origin O is,

$$\tau = F r \sin \theta \tag{7.9}$$

where θ is the angle between \vec{F} and \vec{r} .

The sign of the torque depends on the direction of rotation. According to the most commonly accepted convention, *the torque is positive if the force is trying to rotate the stick r in a counterclockwise direction*. The torque is negative if the force is trying to rotate in a clockwise direction.

The unit of torque is newton \times meters (N·m) and does not have a special name.

We can write the definition above in two different ways: In the first, the factor $F \sin \theta$ is the component F_\perp that is perpendicular to the direction r of the force

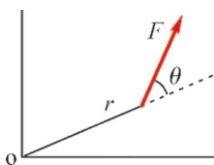


Figure 7.8: The torque of a force.

(Figure 7.9a):

$$\tau = \underbrace{F \sin \theta}_F r = F_{\perp} r \quad (7.10)$$

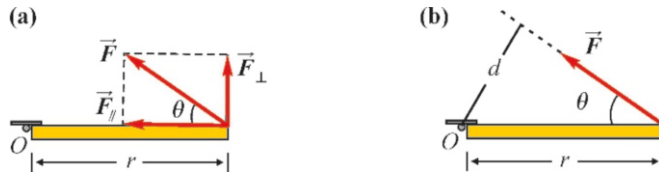


Figure 7.9: Two methods of calculating torque.

On the other hand, considering that the product $r \sin \theta$ is just the opposite perpendicular side of the triangle (Figure 7.9b), we can write

$$\tau = F \underbrace{r \sin \theta}_d = F d \quad (7.11)$$

The distance d here is called the **lever arm**. Whichever expression is suitable for the problem at hand should be used. Usually, the expression $F_{\perp} r$ is more practical.

Expression of Torque As a Vector Product

The factor $\sin \theta$ in the defining expression (7.9) of the torque suggests that it can be expressed as a vector product. Indeed, if the position vector \vec{r} is measured from the rotation axis to the application point of the force \vec{F} , the torque vector can be expressed as follows:

$$\vec{\tau} = \vec{r} \times \vec{F} \quad (7.12)$$

Figure 7.10 shows that this expression satisfies the definition given above: Let us assume that the vectors \vec{r} and \vec{F} are on the xy -plane. The direction of the vector product is found using the right-hand rule. According to this definition, the direction of the torque vector will be along z , which is the rotation axis. If the rotation of the force about the z -axis is counterclockwise, then the torque will be in the $+z$ direction, in other words, positive. If it rotates in the opposite direction, the torque will be negative.

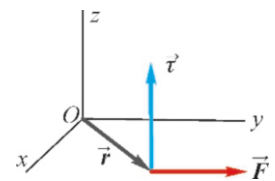
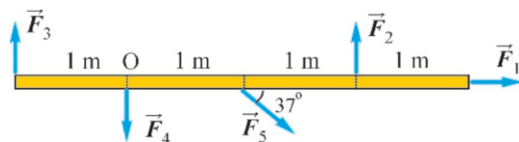


Figure 7.10: Expression of torque as vector product.

As the forces that we will take into consideration in this chapter are all on a plane, the torques will always be along the z -axis, with positive or negative components. We can thus work without referring to the vector nature of torque.

Example 7.5



All of the forces shown in the figure have the same magnitude of 10 N. Calculate the torques of these forces with respect to the point O.

Answer We use one of the expressions $\tau = r F_{\perp}$ or $\tau = F d$ for torque:

$\tau_1 = r F_{1\perp} = 0$, because the component of the force perpendicular to r is zero. Torque is always zero if the line of the force passes through the center of rotation.

$\tau_2 = r F_{2\perp} = 2 \times 10 = +20 \text{ N}\cdot\text{m}$. Torque is positive, because the direction of rotation of the force F_2 is counterclockwise.

$\tau_3 = -r F_{3\perp} = -1 \times 10 = -10 \text{ N}\cdot\text{m}$. Torque is negative, because the direction of rotation of the force F_3 is clockwise.

$\tau_4 = F_4 d = 0$. Torque is zero, because the lever arm $d=0$.
 $\tau_5 = -r F_5 \sin 37^\circ = -1 \times 10 \times 0.6 = -6 \text{ N}\cdot\text{m}$. Torque is negative, because the direction of rotation of the force F_5 is clockwise.

7.3 ROTATIONAL DYNAMICS

Rotational Dynamics of a Pointlike Object

Before expressing Newton's law for a rigid body, let us first demonstrate our method on a simpler motion.

Let a pointlike mass m rotate on a circular trajectory with radius r and with a net force \vec{F} acting on it (Figure 7.11). Let us separate this force into two components: The component F_r acts towards the center O , and thus causes the centripetal acceleration a_r . The other component F_t acts along the tangent and thus causes the tangential acceleration a_t .

Let us write the Newton's law for these two forces and convert linear accelerations into angular quantities:

$$F_r = ma_r = mr\omega^2$$

$$F_t = ma_t = mr\alpha$$

The first of these equations specifies the centripetal force required to rotate the object at the angular velocity ω . We set it aside to concentrate on the second equation which gives the tangential acceleration. Let us write it as multiplied by r :

$$F_t r = mr^2 \alpha$$

The left-hand side of the equation is the torque of the force F with respect to the center O :

$$\tau = (mr^2) \alpha \quad (7.13)$$

This expression resembles the Newton law $F=ma$. However, force F is replaced with the torque τ , acceleration a is replaced with angular acceleration α and the mass is replaced with product (mr^2) . This different expression of Newton's law is actually nothing new. But it gives us a clue as to how to write the rotational motion of the rigid body. The fact that all of the particles of the rigid body are rotating with the same angular acceleration α is explicit in the right-hand side of the equation.

Rotational Dynamics of a Rigid Body

Now, we can consider the dynamical law that determines the rotation of a solid object. Let us assume that a rigid body rotating about an axis (Figure 7.12) is made of N number of pointlike masses $\Delta m_1, \Delta m_2 \dots \Delta m_N$. Let the external forces acting on each of these masses be $\vec{F}_1, \vec{F}_2 \dots \vec{F}_N$. (There is no need to consider the internal forces because they will cancel each other out in the sum.)

For each small mass we write the equation (7.13) that we developed for a pointlike object, to get a system of equations:

$$\tau_1 = F_{1t} r_1 = (\Delta m_1 r_1^2) \alpha$$

$$\tau_2 = F_{2t} r_2 = (\Delta m_2 r_2^2) \alpha$$

$$\dots = \dots$$

$$\tau_N = F_{Nt} r_N = (\Delta m_N r_N^2) \alpha$$

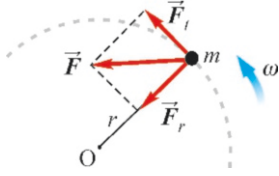


Figure 7.11: Tangential and radial components of forces acting on a point body in circular motion.

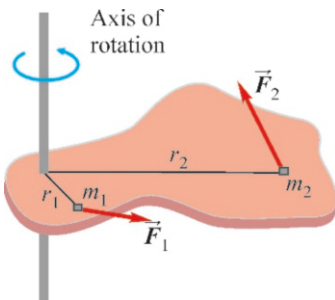


Figure 7.12: External forces acting on a rigid body.

Adding these equations on each side, the angular acceleration α on the right-hand side becomes the common factor:

$$\begin{aligned}\tau_1 + \tau_2 + \cdots + \tau_N &= (\Delta m_1 r_1^2 + \Delta m_2 r_2^2 + \cdots + \Delta m_N r_N^2) \alpha \\ \sum_i \tau_i &= \left(\sum_i \Delta m_i r_i^2 \right) \alpha\end{aligned}$$

The left-hand side of this equation is the total torque of the forces acting on the rigid body. The sum in the brackets on the right-hand side becomes a new term in which each point mass is multiplied by the square of its own radius r_i and added. This is called the **moment of inertia** of the rigid body and is shown with I :

$$I = \sum_i \Delta m_i r_i^2 \quad (\text{moment of inertia}) \quad (7.14)$$

Moment of inertia serves the function of mass in rotational motion; in other words, the larger I is, the more difficult it becomes to give angular acceleration to the object.

As the distances r_i in this definition vary depending on the chosen axis of rotation, the moment of inertia will depend on the chosen axis of rotation.

For a continuously distributed mass, the sum is replaced by an integral in the limit $\Delta m_i \rightarrow 0$:

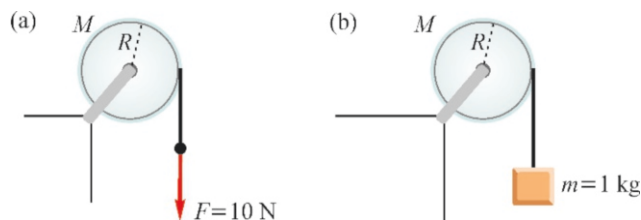
$$I = \int dm r^2 \quad (\text{moment of inertia}) \quad (7.15)$$

As a result, the equation for the rotational motion of a solid object is written as follows:

$$\tau_{\text{net}} = I \alpha \quad (\text{Rotational dynamics of a rigid body}) \quad (7.16)$$

We must emphasize one small point here: Previously, we had specified the positive direction of the torque as being counterclockwise. However, in calculations the positive direction of the torque can be chosen in the direction of motion, and not the clock direction. In other words, the torques in the direction that the rigid body is assumed to rotate are taken as positive and the torques in the opposite direction are taken as negative.

Example 7.6



A wheel with radius $R=2$ m and mass $M=5$ kg can rotate without friction about its center. Its moment of inertia is $I=\frac{1}{2}MR^2$.

(a) The wheel is pulled with a force $F=10$ N from the end of a rope tied around it. Find the angular acceleration of the

wheel.

(b) A block with mass $m=1$ kg is attached at the end of the rope and released. Calculate the accelerations and the tension in the rope.

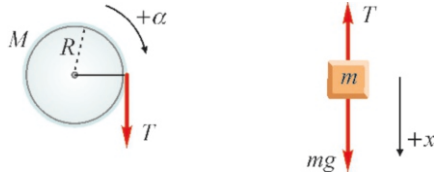
Answer

The reaction forces that may be generated on the axis of the wheel are not taken into account, because their torques will be zero as they pass through the axis of rotation. Accordingly, only the torque of force F will cause motion. We write the equation of rotational dynamics (7.16):

$$\tau_{\text{net}} = I \alpha \rightarrow F \cdot R = \left(\frac{1}{2} MR^2 \right) \alpha$$

$$\alpha = \frac{2F}{MR} = \frac{2 \times 10}{5 \times 2} = 2 \text{ rad/s}^2$$

(b) This time, there are two moving objects: The rotating wheel and the mass m in linear motion. The forces acting on each one are shown below as free-body diagrams:



Taking the direction of motion as positive for both objects, the equations of motion are as follows:

$$\text{For } M: \quad \tau_{\text{net}} = TR = I\alpha = \frac{1}{2}MR^2\alpha \quad (1)$$

$$\text{For } m: \quad F_{\text{net}} = mg - T = ma \quad (2)$$

These are two equations with three unknowns T , α , a .

Therefore, an extra equation is required. This is obtained from the relation between linear and angular kinematics. Since each point of the rope will have the same acceleration a , it will be the tangential acceleration of the wheel at the point of contact with the wheel. Therefore, the third equation is as follows:

$$a = a_t = R\alpha \quad (3)$$

Taking the value T from equation (1) and using it with the $a = R\alpha$ in equation (2), we get

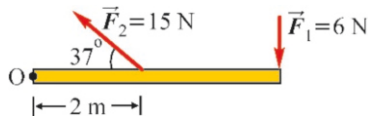
$$\alpha = \frac{mg}{(m + M/2)R} = \frac{10}{(1 + 5/2) \times 2} = 1.4 \text{ rad/s}^2$$

From here, we find the other unknowns:

$$a = R\alpha = 2.8 \text{ m/s}^2$$

$$T = m(g - a) = 7 \text{ N}.$$

Example 7.7



A rod of length $L = 5 \text{ m}$ can rotate without friction around its end O . The moment of inertia of the rod with respect to point O is $I_0 = 5 \text{ kg}\cdot\text{m}^2$. The two forces shown in the figure are acting on this rod. Find the direction of rotation and angular acceleration of the rod.

Answer

The torques of the two forces are in different directions. Let

us calculate their torques separately in order to determine which is greater:

$$\tau_1 = F_1 L = 6 \times 5 = 30 \text{ N}\cdot\text{m}$$

$$\tau_2 = r F_{2\perp} = r F_2 \sin 37^\circ = 2 \times 15 \times 0.6 = 18 \text{ N}\cdot\text{m}$$

The rod will rotate in the direction of F_1 , because the value τ_1 is greater.

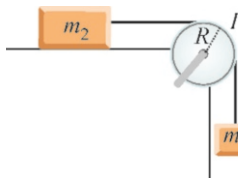
We choose the positive rotation direction for F_1 and write the rotational dynamics equation:

$$\tau_{\text{net}} = \tau_1 - \tau_2 = I_0 \alpha$$

From here, we find the angular acceleration:

$$\alpha = \frac{30 - 18}{5} = 2.4 \text{ rad/s}^2$$

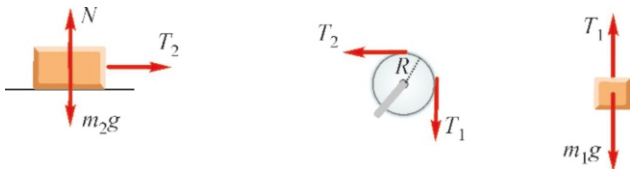
Example 7.8



A block of mass $m_1 = 1 \text{ kg}$ is attached to one end of a rope going around a pulley. Another block of mass $m_2 = 2 \text{ kg}$ at rest on a frictionless horizontal plane is tied to the other end of the rope. The pulley has a mass $m_2 = 2 \text{ kg}$, a radius $R = 20 \text{ cm}$ and a moment of inertia $I = 3 \text{ kg}\cdot\text{m}^2$. Calculate the linear and angular accelerations and the tensions in the ropes.

Answer

The important point to note here is that the tensions on two ends of the rope will be different. For, if the tensions were equal, there would be no net torque to rotate the pulley. The forces acting on each object can thus be shown as follows:



We write the equations of motion for each object (the weight of the mass on the plane and its reaction force N cancel each other out and therefore are not taken into consideration):

$$\text{For } m_1: \quad m_1 g - T_1 = m_1 a \quad (1)$$

$$\text{For } m_2: \quad T_2 = m_2 a \quad (2)$$

$$\text{For the pulley: } T_1 R - T_2 R = I \alpha \quad (3)$$

Additionally, we write the relation between the accelerations as follows:

$$a = R\alpha \quad (4)$$

We find the accelerations and tensions from these four equations. Taking the tensions from (1) and (2) and substituting in equation (3), we get

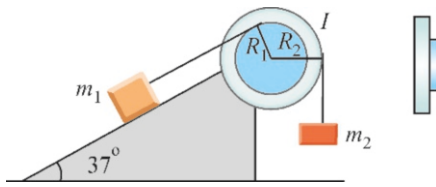
$$\alpha = \frac{m_1 g R}{I + (m_1 + m_2) R^2} = \frac{10 \times 0.2}{3 + 3 \times 0.2^2} = 0.64 \text{ rad/s}^2$$

Using these, we calculate the other unknowns:

$$a = R\alpha = 0.2 \times 0.64 = 0.13 \text{ m/s}^2$$

$$T_1 = m_1(g - a) = 9.9 \text{ N}$$

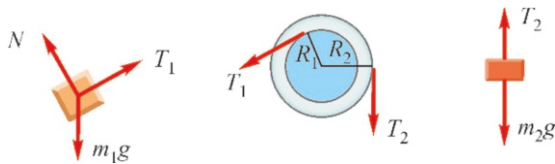
$$T_2 = m_2 a = 0.3 \text{ N}.$$

Example 7.9

A rope is wrapped around the smaller radius $R_1 = 10$ cm of a pulley with moment of inertia $I = 3 \text{ kg}\cdot\text{m}^2$ and the other end of the rope is tied to a block with mass $m_1 = 1$ kg on a frictionless inclined plane with 37° slope. A block with mass $m_2 = 2$ kg is tied to the end of a rope going around the outer radius $R_2 = 20$ cm of the pulley. Calculate the accelerations and the tensions in the ropes when this system is released.

Answer

The linear accelerations of the blocks are different in this problem because they are tied at different radii. Let us show the forces acting on each object:



Since $m_2R_2 > m_1R_1$, the direction of motion is obvious. We take the direction of motion of each object as positive and write the equations as follows:

$$\text{For } m_1: \quad T_1 - m_1g \sin 37^\circ = m_1a_1 \quad (1)$$

$$\text{For } m_2: \quad m_2g - T_2 = m_2a_2 \quad (2)$$

$$\text{For the pulley:} \quad T_2R_2 - T_1R_1 = I\alpha \quad (3)$$

As the linear accelerations will again be the tangential acceleration of the pulley, we write the relations between linear and angular accelerations:

$$a_1 = R_1\alpha \quad (4)$$

$$a_2 = R_2\alpha \quad (5)$$

We find the accelerations and tensions from these five equations. Taking T_1 and T_2 from (1) and (2) and substituting in equation (3), we find the angular acceleration:

$$\alpha = \frac{m_2gR_2 - m_1gR_1 \sin 37^\circ}{I + m_1R_1^2 + m_2R_2^2} = 1.1 \text{ rad/s}^2$$

We then calculate the accelerations and tensions using this value:

$$a_1 = R_1\alpha = 0.1 \times 1.1 = 0.11 \text{ m/s}^2$$

$$a_2 = R_2\alpha = 0.22 \text{ m/s}^2$$

$$T_1 = m_1g \sin 37^\circ + m_1a_1 = 6.1 \text{ N}$$

$$T_2 = m_2g - m_2a_2 = 19.6 \text{ N}.$$

7.4 MOMENT OF INERTIA CALCULATIONS

The moment of inertia, behaving like a mass in rotational motion, depends not only on the mass, but also on how the mass is distributed around the axis of rotation.

Moment of inertia can be calculated using two methods:

- If the rigid body consists of pointlike objects, the moment of inertia of each one is added using Eq. (7.14):

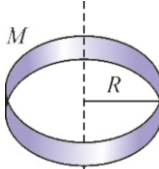
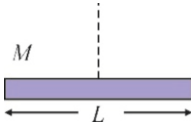
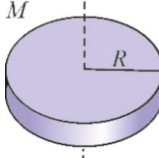
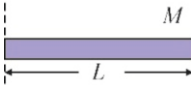
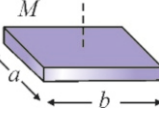
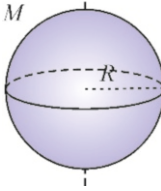
$$I = \sum_i m_i r_i^2$$

- If the distribution of mass is continuous, we take the integral Eq. (7.15):

$$I = \int dm r^2$$

Let us summarize the results of the calculation before delving into moment of inertia calculations:

Moments of inertia of various rigid objects

Ring : $I = MR^2$		Rod : $I = \frac{1}{12} ML^2$	
Disk or cylinder $I = \frac{1}{2} MR^2$		Rod : $I = \frac{1}{3} ML^2$	
Rectangular plate $I = \frac{1}{12} M(a^2 + b^2)$		Sphere : $I = \frac{2}{5} MR^2$	

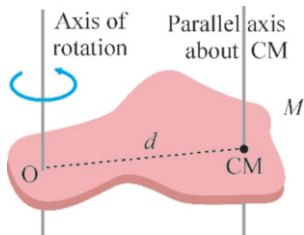


Figure 7.13: Parallel axis theorem.

Parallel Axis Theorem

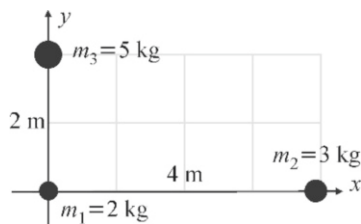
The value of moment of inertia depends on the chosen axis. The table above gives the moments of inertia I_{cm} of rigid bodies about an axis passing through their centers of mass. If the rigid body rotates about another axis, the moment of inertia can still be calculated using a result known as the **parallel axis theorem** (or, Steiner theorem):

If the moment of inertia of an object is I_{cm} with respect to an axis passing through its center of mass, then its moment of inertia with respect to another axis that is parallel at a distance d will be

$$I = I_{cm} + M d^2 \quad (\text{Parallel axis theorem}) \quad (7.17)$$

Here, M is the mass of the rigid body. We give this without a proof here. According to this theorem, considering that the term Md^2 is positive, the axis where the moment of inertia is the lowest, in other words, where it can rotate most easily, is the axis passing through its center of mass.

Example 7.10



Calculate the moment of inertia of a system consisting of the

three masses shown in the figure with respect to each of the x -, y - and z -axes.

Answer The moment of inertia will be $\sum_i m_i r_i^2$ for point masses. The mass m_1 will have no contribution, because it is on the axis, and therefore $r_1 = 0$.

We substitute the given values of m and r_i :

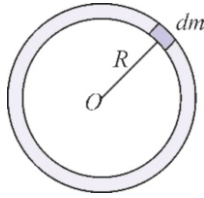
$$I_x = m_2 r_2^2 + m_3 r_3^2 = 0 + 5 \times 2^2 = 20 \text{ kg}\cdot\text{m}^2$$

$$I_y = m_2 r_2^2 + m_3 r_3^2 = 3 \times 4^2 + 0 = 48 \text{ kg}\cdot\text{m}^2$$

$$I_z = m_2 r_2^2 + m_3 r_3^2 = 48 + 20 = 68 \text{ kg}\cdot\text{m}^2$$

Example 7.11

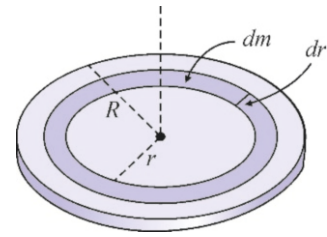
- (a) Find the moment of inertia of a ring with mass m and radius R about an axis passing through its center O and perpendicular to the plane of the ring.
- (b) Using the previous result, find the moment of inertia of a disk with mass M and radius R about a perpendicular axis passing through its center.

**Answer**

(a) We choose a small mass dm on the ring and use the integral formula (7.15):

$$I = \int dmR^2 = R^2 \int dm = MR^2$$

As each mass dm is located at a constant distance R , it is taken outside the integral. The sum of dm is equal to the total mass M .



(b) Let us consider a small ring between the radii r and $r + dr$ on the disk. Its surface area will be approximately $2\pi r dr$. If the total mass is M , we find the mass of this small disk using proportion:

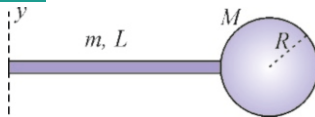
$$\frac{dm}{M} = \frac{2\pi r dr}{\pi R^2} \rightarrow dm = \frac{M}{R^2} 2r dr$$

We had found in item (a) that the contribution of this small disk is $dm r^2$. Therefore, their contributions are added, in other words, integrated from $r = 0$ to $r = R$:

$$I = \int dm r^2 = \int_0^R \left(\frac{M}{R^2} 2r dr \right) r^2 = \frac{2M}{R^2} \int_0^R r^3 dr$$

The value of the integral is $R^4/4$, and after simplification, we find the moment of inertia of the disk:

$$I = \frac{1}{2}MR^2$$

Example 7.12

A solid sphere with mass M and radius R is welded to the end of a rod with mass m and length L . Calculate the moment of inertia of this system with respect to the y -axis on the other end.

Answer

The moments of inertia of the rod and sphere with respect to their own centers of mass are given in the table on Page 124:

$$\text{Rod: } I_{\text{cm}} = \frac{1}{12} mL^2$$

$$\text{Solid sphere: } I_{\text{cm}} = \frac{2}{5} MR^2$$

The moment of inertia of each object with respect to the y -axis is found using the parallel axis theorem, Eq. (7.17). The distance of the parallel axis is $L/2$ for the rod and $(L + R)$ for the sphere:

$$\text{Rod: } I_y = I_{\text{cm}} + m \left(\frac{L}{2} \right)^2 = \frac{1}{12} mL^2 + \frac{1}{4} mL^2 = \frac{1}{3} mL^2$$

$$\text{Solid sphere: } I_y = I_{\text{cm}} + M(L + R)^2 = \frac{2}{5} MR^2 + M(L + R)^2$$

The total moment of inertia of the system is the sum of these two:

$$I_{y,\text{total}} = \frac{1}{3} mL^2 + \frac{2}{5} MR^2 + M(L + R)^2$$

Rotational Kinetic Energy

We cannot calculate the kinetic energy of a rotating rigid body using the formula $\frac{1}{2}mv^2$ because the linear velocity of each point is different. Instead, we have to write an expression in terms of angular velocity.

Let us assume that the rigid body consists of N number of small masses $m_1, m_2 \dots m_N$. Let the linear velocity of each of these masses be $v_1, v_2 \dots v_N$. The kinetic energy of the rigid body will be the sum of the kinetic energies of these pointlike masses:

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \dots + \frac{1}{2}m_Nv_N^2$$

Let each of these masses rotate at distances $r_1, r_2 \dots r_n$ from the axis. If we write the relation $v = r\omega$ between the linear and angular velocity for each one, we get

$$\begin{aligned} K &= \frac{1}{2}m_1(r_1\omega)^2 + \frac{1}{2}m_2(r_2\omega)^2 + \dots + \frac{1}{2}m_N(r_N\omega)^2 \\ &= \frac{1}{2} \left[\underbrace{m_1r_1^2 + m_2r_2^2 + \dots + m_Nr_N^2}_I \right] \omega^2 \end{aligned}$$

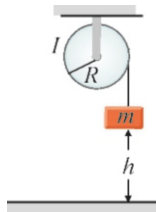
Therefore, rotational kinetic energy can be expressed as follows:

$$K = \frac{1}{2}I\omega^2 \quad (\text{rotational kinetic energy}) \quad (7.18)$$

Note the similarity of this expression with $\frac{1}{2}mv^2$ for translational kinetic energy. Therefore, the law of conservation of energy can still be valid using this expression for rigid bodies.

Example 7.13

A mass $m=1$ kg is attached to the end of a rope that is wrapped around a pulley with radius $R=20$ cm and with moment of inertia $I=0.5$ kg·m². The mass is released at a height of $h=2.7$ m from the ground. With what velocity will it hit the ground?



Answer We write conservation of energy both for the pul-

ley and the mass. The potential energy of the mass m will be completely converted into kinetic energy when the mass drops by h :

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

We write the relation between the linear velocity v of mass m and the angular velocity of the pulley:

$$v = R\omega$$

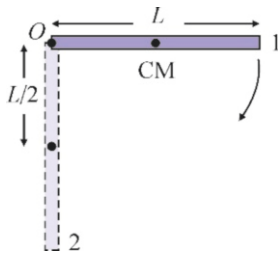
Substituting these values, we find the angular velocity:

$$\omega = \sqrt{\frac{2mgh}{I + mR^2}} = \sqrt{\frac{20 \times 2.7}{0.5 + 1 \times 0.2^2}} = 10 \text{ rad/s}$$

From this, we get the linear velocity:

$$v = R\omega = 2 \text{ m/s}$$

Example 7.14



A rod with mass m and length $L = 1.2$ m is hinged from one end to a wall. The rod is released in horizontal position. What will its angular velocity be once it reaches vertical position?

Answer

The potential energy of the rod will turn into kinetic energy once it reaches the vertical position and its center of mass descends by $L/2$:

$$mg(L/2) = \frac{1}{2}I_0\omega^2$$

The moment of inertia here is with respect to the axis of rotation O . We can calculate it from I_{cm} with the help of the parallel axis theorem:

$$I_0 = I_{\text{cm}} + m(L/2)^2 = \frac{1}{12}mL^2 + mL^2/4 = \frac{1}{3}mL^2$$

Substituting these values, we find the angular velocity:

$$\omega = \sqrt{\frac{3g}{L}} = \sqrt{\frac{30}{1.2}} = 5 \text{ rad/s.}$$

7.5 ROLLING MOTION

In the most general motion of rigid bodies, translation and rotation can occur simultaneously. This general motion may be very complex. But still, it can always be examined by separating it into two components:

1. **Translational motion of the center of mass:** This motion is determined with Newton's law:

$$\sum_i \vec{F}_i = m \vec{a}_{\text{cm}} \quad (\text{for translation}) \quad (7.19)$$

2. **Rotational motion about the center of mass:** This is determined with the rotational dynamics law:

$$\sum_i \tau_{i,\text{cm}} = I_{\text{cm}} \alpha \quad (\text{for rotation}) \quad (7.20)$$

The torques here should be taken with respect to the center of mass.

In the most general motion, there is no relation between linear acceleration a_{cm} and angular acceleration α , and each may have a different value. However, these two accelerations are related for an object that rolls on a surface without slipping.

Let us consider a cylinder that rolls on a horizontal plane without slipping. The velocity of each point of this cylinder is different, but there is one point that is at rest, in other words, has zero velocity for one instant. This is point P, where the object is in contact with the surface (Figure 7.14). Indeed, if the object is not slipping, the two common points where the surface and the object meet must have equal velocity. And this means that the velocity of the point on the object is zero.

This is true only for one instant. Immediately afterwards, this contacting point of the object will detach from the surface and gain velocity. However, another point will take its place and its velocity will also become zero at the moment that it contacts the surface. This point of contact is the **instantaneous axis of rotation**. Therefore, we can write the linear velocity and acceleration of the center of mass, which is at a distance R from this axis, in terms of angular quantities:

$$v_{\text{cm}} = R\omega \quad \text{and} \quad a_{\text{cm}} = R\alpha \quad (\text{Rolling conditions}) \quad (7.21)$$

Rolling motion can be easily calculated by adding these conditions to the aforementioned translation and rotation equations.

Important note: At the point of contact between the rolling object and the surface, there is also a friction force f . However, the formula $f = \mu N$ cannot be used here, because the object is not slipping and that friction has not reached its maximum value.

In rolling motion, as translation and rotation take place together, the kinetic energy is the sum of both types of energy:

$$K = \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2 \quad (\text{rolling kinetic energy}) \quad (7.22)$$

This kinetic energy expression is used in energy conservation problems that include rolling motion. (*Note:* It may seem surprising that it is possible to write the conservation of energy despite the fact that there is a friction force in rolling. However, the friction force performs no work, as the point of the object contacting the surface is motionless.)

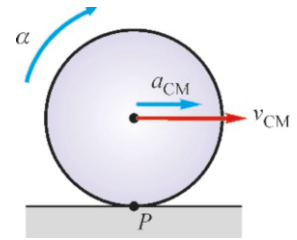
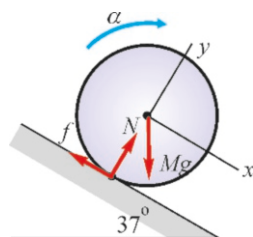


Figure 7.14: Rolling motion.

Example 7.15



Find the angular acceleration of a cylinder with radius $R = 50$ cm and mass $M = 4$ kg released on a plane inclined at an

angle of 37° .

Answer

The forces acting on the cylinder are shown in the figure. We can explain the direction of the friction force f as follows: If this was an inclined plane covered with ice, the cylinder would have slipped down without rolling. Therefore, friction would be upwards.

Let us write Eqs. (7.19 and 7.20) that we found for rolling motion, with the indicated axes (we denote $a_{\text{cm}} = a$). For translational motion, we have:

In the x -direction: $Mg \sin \theta - f = Ma$

In the y -direction: $N - Mg \cos \theta = 0$

(The y -equation will not be necessary, because the formula $f = \mu N$ does not apply.)

For rotational motion: The friction force f is what ensures rotation around CM, and the torques due to other forces are zero:

$$\tau = fR = I\alpha$$

Also, since the cylinder is not slipping, we have the relation

between the linear and angular accelerations:

$$a = R\alpha$$

f is eliminated between these equations:

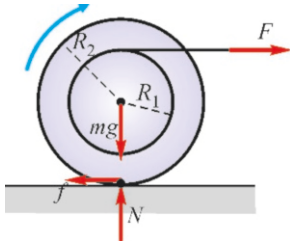
$$\alpha = \frac{MgR \sin 37^\circ}{I + MR^2}$$

Substituting the moment of inertia of the cylinder $I = \frac{1}{2}MR^2$ and the other numerical values,

$$\alpha = \frac{MgR \sin 37^\circ}{\frac{3}{2}MR^2} = \frac{2g \sin 37^\circ}{3R} = \frac{20 \times 0.6}{3 \times 0.5} = 8 \text{ rad/s}^2.$$

Example 7.16

Find the angular acceleration and the acceleration of the center of mass of a cylinder with radius $R_2 = 40$ cm and mass $M = 1$ kg pulled with force $F = 12$ N through a rope wrapped around the cylinder at $R_1 = 30$ cm from the center.



Answer

The forces acting on the cylinder are shown in the figure. We again apply the formulas (7.19 and 7.20):

Translation in the x -direction: $F - f = Ma$

Rotation around CM: $FR_1 + fR_2 = I\alpha$

The relation between the accelerations is: $a = R_2\alpha$

Eliminating the force f between these two equations, we find the acceleration α :

$$\alpha = \frac{F(R_1 + R_2)}{I + MR_2^2}$$

Substituting the moment of inertia of the cylinder $I = \frac{1}{2}MR_2^2$ and the other numerical values,

$$\alpha = \frac{F(R_1 + R_2)}{\frac{3}{2}MR_2^2} = \frac{12 \times (0.3 + 0.4)}{\frac{3}{2} \times 0.4^2} = 35 \text{ rad/s}^2$$

We find the linear acceleration of the center of mass, which is located at a distance of R_2 from the instantaneous center of rotation:

$$a = R_2\alpha = 0.4 \times 35 = 14 \text{ m/s}^2.$$

Example 7.17

A sphere with radius R and mass M released on an inclined plane at height h from the ground rolls down without slipping. What will the velocity of the center of mass be once it reaches the ground? (For sphere: $I_{\text{cm}} = \frac{2}{5}MR^2$)

Answer

According to conservation of energy, the initial potential energy of the sphere at height h is converted into translational and rotational kinetic energies at the bottom:

$$Mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Since there is no slipping, we can use the relation $v = R\omega$ between the linear and angular velocities. Substituting the expression $I = \frac{2}{5}MR^2$ in this equation as well, we get:

$$v = \sqrt{\frac{10gh}{7}} = \sqrt{1.4gh}$$

Considering that a pointlike object would have a velocity $v = \sqrt{2gh}$ after descending by h , we see that the sphere goes down slower, as the factor $\sqrt{1.4}$ is less than $\sqrt{2}$. The reason for this is that some of the energy is used for rotation.

7.6 ANGULAR MOMENTUM AND ITS CONSERVATION

In translational motion, momentum was defined as the product of mass and velocity: $\vec{p} = m\vec{v}$. Likewise, an angular momentum is also defined for rotational motion. By using moment of inertia I instead of mass and angular velocity ω instead of velocity, we get the expression

$$L = I\omega \quad (\text{angular momentum}) \quad (7.23)$$

defining the **angular momentum** of a rigid body. Its unit is $\text{kg}\cdot\text{m}^2/\text{s}$ and does not have a particular name.

Recall how we expressed Newton's law in terms of momentum in linear motion:

$$F = ma = m \frac{dv}{dt} = \frac{d(mv)}{dt} = \frac{dp}{dt}$$

The rotational motion equation can likewise be expressed as follows:

$$\begin{aligned}\tau &= I\alpha = I \frac{d\omega}{dt} = \frac{d(I\omega)}{dt} \\ \tau &= \frac{dL}{dt}\end{aligned}\quad (7.24)$$

In the simplest case, the angular momentum of a point object traveling at velocity v , with respect to an axis of rotation at a distance of r , is as follows:

$$\begin{aligned}L &= I\omega = (mr^2)\omega = mr(r\omega) \\ L &= mvr \quad (\text{angular momentum of a pointlike object})\end{aligned}\quad (7.25)$$

Angular momentum is actually a vector quantity. Its most general definition is given as the torque of the linear momentum vector \vec{p} (Figure 7.15):

$$\vec{L} = \vec{r} \times \vec{p}$$

Its direction is perpendicular to the object's plane of rotation. We will not use the vector properties of angular momentum here and provide it merely as information.

If the net torque of external forces acting on a rigid body is zero, then its angular momentum will remain constant, according to Eq. 7.24. This expression is the law of **conservation of angular momentum**:

$$\begin{aligned}\tau &= 0 \quad \Rightarrow \quad \frac{dL}{dt} = 0 \\ L_1 &= L_2 = \text{constant} \quad (\text{conservation of angular momentum})\end{aligned}\quad (7.26)$$

When an object is rotating, even if the torque of external forces is zero, its moment of inertia I may change with the impact of internal forces. For example, a man on a rotating platform may increase his moment of inertia by spreading his arms. As the final situation can again be considered as a rigid body, the angular velocity will decrease to conserve angular momentum.

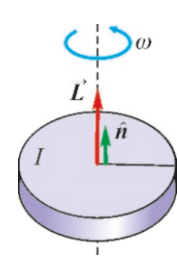
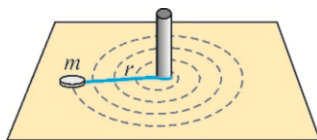


Figure 7.15: The angular momentum vector \vec{L} is perpendicular to the rotation plane.

Example 7.18



A mass m is rotating with angular velocity $\omega = 3 \text{ rad/s}$ at the end of a rope with length $r = 50 \text{ cm}$ tied to an axis on a frictionless table. As the rope starts to wrap around the axis, what will the angular velocity of the object be when the length of the rope shortens to 25 cm ?

Answer

We write the conservation of angular momentum for a pointlike object:

$$mvr = mv'r'$$

We write the linear velocities in terms of angular velocity using the formula $v = r\omega$:

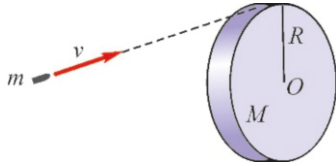
$$r^2\omega = r'^2\omega' \rightarrow \omega' = \frac{r^2}{r'^2}\omega$$

By substituting the value $r' = r/2$, we find the final angular velocity:

$$\omega' = 4\omega = 4 \times 3 = 12 \text{ rad/s.}$$

Example 7.19

A bullet with mass $m = 50$ g travels at a speed of $v = 200$ m/s and embeds itself into the rim of a disk at rest that can rotate freely, at a distance of $R = 60$ cm from the center. As the disk's mass is $M = 900$ g and the moment of inertia is $I = \frac{1}{2}MR^2$, find the angular momentum of the system $(M + m)$ after the collision.

**Answer**

The forces generated during the collision are the internal forces of the system (bullet+disk) and their net torque is zero.

We write the conservation of angular momentum:

$$L_{\text{bullet}} + L_{\text{disk}} = L'_{\text{bullet+disk}}$$

Initially, there is only the angular momentum mvR of the point mass m . After the collision, this mass will rotate with an angular velocity ω together with the disk at a distance R . Therefore,

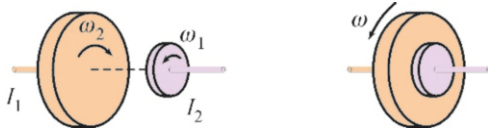
$$mvR + 0 = (I_{\text{disk}} + I_{\text{bullet}})\omega$$

Substituting the moments of inertia of the disk $I = \frac{1}{2}MR^2$ and of the bullet $I = mR^2$, we find that:

$$\omega = \frac{mvR}{(M/2 + m)R^2}$$

We find the angular velocity by substituting the numerical values:

$$\omega = \frac{mv}{(M/2 + m)R} = \frac{0.05 \times 200}{(0.9/2 + 0.05) \times 0.6} = 33 \text{ rad/s}$$

Example 7.20

In the transmission box of a car, two disks are rotating, one connected to the engine and the other connected to the transmission shaft. The moment of inertia of the disk connected to the engine is $I_1 = 2$ kg·m² and it rotates with angular velocity $\omega_1 = 2000$ rpm. The moment of inertia of the disk on the transmission shaft is $I_2 = 5$ kg·m² and it rotates in the opposite

direction with angular velocity $\omega_2 = 100$ rpm. What will be the final angular velocity when the disks are suddenly clamped together?

Answer We write the conservation of angular momentum:

$$I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$$

$$\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

If we take ω_1 in the positive rotation direction, then $\omega_2 = -100$ rpm. From here, we find the final angular velocity (there is no need to change units, as the ratios of I are used):

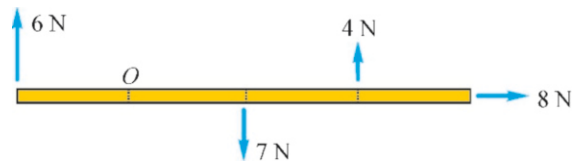
$$\omega = \frac{2 \times 2000 + 5 \times (-100)}{2 + 5} = 500 \text{ rpm}$$

Multiple-choice Questions

- Which of the following formulas is incorrect?
(a) $v = r\omega$ (b) $a_t = r\alpha$ (c) $a_r = r\omega^2$ (d) $L = I\omega^2$
- Which of the following is incorrect for the definition of the torque of a force?
(a) Force \times lever arm
(b) Distance \times perpendicular component of force
(c) Force \times distance
(d) Force \times distance²
- A disk's mass is doubled and its radius tripled. By what factor will its moment of inertia increase?
(a) 5 (b) 6 (c) 12 (d) 18
- Three forces act on an object perpendicular to the axis of rotation. The force F acts at a distance of $4R$, the force $2F$ acts at a distance of $3R$ and the force $3F$ acts at a distance of $2R$ from the axis. Which one applies the lowest torque?
(a) F (b) $2F$ (c) $3F$ (d) F & $3F$

- Which of the following is correct?

- An object will not rotate if the net force acting on it is zero.
- An object will not rotate if the net torque acting on it is zero.
- An object will rotate with a constant angular velocity if the net torque acting on it is zero.
- None of the above.

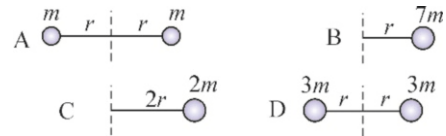


- The rod with four equal parts marked in the figure can rotate around the O-axis. Which of the forces shown will exert the largest torque?
(a) 4 N (b) 6 N (c) 7 N (d) 8 N

7. Which of the following is incorrect about the moment of inertia of an object?
- It increases with mass.
 - It is directly proportional to the square of the distance from the axis.
 - Its value depends on the chosen axis.
 - It increases with velocity.
8. Which of the following is incorrect if the angular acceleration of a rotating object is zero?
- Net torque is zero.
 - Angular velocity is zero.
 - Centripetal acceleration is zero.
 - Tangential acceleration is zero.
- (a) I (b) I & II (c) II & III (d) IV
9. A mass m is rotating around a circle with radius R . Another mass $2m$ is rotating around a circle with radius $R/2$. And a third mass $4m$ is rotating around a circle with radius $R/2$. Which one has highest moment of inertia?
- (a) m (b) $2m$ (c) $4m$ (d) m & $4m$
10. The same tangential force F is applied on a ring, on a disk, and on a sphere with the same masses and radii. Which one will accelerate faster?
- (a) Ring (b) Disk (c) Sphere (d) Equal
11. A rotating object's angular velocity is doubled and its moment of inertia is tripled. By what factor will its kinetic energy increase?
- (a) 5 (b) 6 (c) 12 (d) 18
12. A rotating object's angular velocity is tripled and its moment of inertia is doubled. By what factor will its angular momentum increase?
- (a) 5 (b) 6 (c) 12 (d) 18
13. What will happen to the Earth's rotation velocity if the icebergs melt as a result of "global warming"?
- It will increase.
 - It will decrease.
 - It will remain constant.
 - It is impossible to tell.
14. Which of the following is correct for a point A located at a distance of 1 cm and a point B at a distance of 2 cm from the center of a wheel rotating with constant angular velocity?

- Centripetal accelerations are equal.
- Linear velocities are equal.
- Tangential accelerations are equal.
- Angular velocities are equal.

15. The moment of inertia of a 3 kg object with respect to its center of mass is $I_{cm} = 5 \text{ kg}\cdot\text{m}^2$. What will its moment of inertia be in units of $\text{kg}\cdot\text{m}^2$ with respect to a parallel axis that is 2 m away?
- (a) 10 (b) 15 (c) 17 (d) 20



16. Which of the objects in the figure above has the highest moment of inertia?
- (a) A (b) B (c) C (d) D
17. Which of the following is incorrect for a rolling objects?
- The linear velocity is different at each point.
 - The velocity of the point in contact with the ground is zero.
 - The angular velocity is the same everywhere.
 - No friction force acts on the object.
18. The moment of inertia of a solid cylinder is $\frac{1}{2}MR^2$ and that of a hollow cylinder is MR^2 . Two cylinders, one solid and one hollow, with equal masses and radii are released at the same height on an inclined plane. Which one will roll down and reach the bottom first?
- The solid cylinder.
 - The hollow cylinder.
 - They will arrive at the same time.
 - It is impossible to tell.
19. Which of the following is incorrect if the net torque on an object is zero?
- The angular acceleration is zero.
 - The kinetic energy remains constant.
 - The angular momentum increases uniformly.
 - The angular momentum remains constant.
20. Which is correct for a rolling object?
- The axis of rotation is the center of mass.
 - The axis of rotation is the point in contact with the surface.
 - The translational kinetic energy is zero.
 - The rotational kinetic energy is zero.

Problems

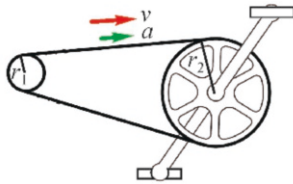
7.1 Angular Kinematics

7.1 A wheel at rest starts to rotate and reaches an angular velocity of 54 rpm in 3 s. Calculate the angular acceleration of the wheel and the number of revolutions it makes during this interval. [A: $0.6\pi \text{ rad/s}^2$ and 1.4 revolutions.]

7.2 A motor is rotating at angular velocity 90 rpm and, due to a power failure, stops after making 6 revolutions. Calculate its angular acceleration and the time that it takes to stop. [A: -0.38 rad/s^2 and 8 s.]

7.3 A wheel initially rotating at a certain angular velocity suddenly starts to accelerate and reaches an angular velocity of 120 rpm after making 3 revolutions in 2 seconds. Find the angular acceleration of the wheel and its initial velocity. [A: $\pi \text{ rad/s}^2$ and $2\pi \text{ rad/s}$.]

7.4 A wheel with a radius of 30 cm accelerates with angular acceleration 20 rad/s^2 to reach angular velocity 45 rpm. What will be the linear velocity and the tangential and centripetal accelerations of a point on the rim of the wheel? [A: $v = 0.45\pi \text{ m/s}$, $a_t = 6 \text{ m/s}^2$, $a_r = 6.8 \text{ m/s}^2$.]



Problem 7.5

7.5 On a bicycle, the sprocket (small disk connected to the rear wheel) has a radius of $r_1=5 \text{ cm}$ and the crankset (larger disk connected to the pedal) has a radius of $r_2=10 \text{ cm}$. The biker starts from rest and increases the angular velocity of the crankset to $\omega_2=30 \text{ rpm}$ in 2 seconds. (a) What is the angular acceleration of the crankset? (b) What will the angular velocity and acceleration of the sprocket be at the end of this time? (c) What are the final centripetal accelerations on the rims of both disks?

[A: (a) $\alpha_2 = 1.57 \text{ rad/s}^2$, (b) $\omega_1 = 6.28 \text{ rad/s}$, $\alpha_1 = 3.14 \text{ rad/s}^2$, (c) $a_{1r} = 2 \text{ m/s}^2$, $a_{2r} = 1 \text{ m/s}^2$.]

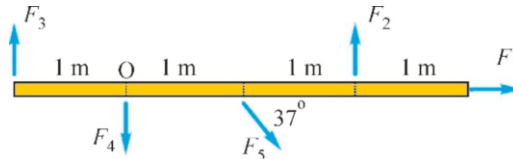
7.6 A rotating wheel with a radius of 80 cm starts to slow down when the linear velocity of a point on the rim is 24 m/s and stops in 5 seconds. (a) What are the angular acceleration and the number of revolutions that it will make until it stops? (b) What are the initial tangential and centripetal accelerations on the rim?

[A: (a) $\alpha = -6 \text{ rad/s}^2$, 12 revolutions, (b) $a_t = -4.8 \text{ m/s}^2$, $a_r = 720 \text{ m/s}^2$.]

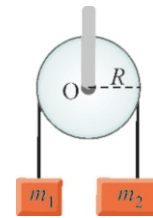
7.2-3 Torque - Rotational Dynamics

7.7 All of the forces shown in the figure below have the same magnitude of 10 N. As the moment of inertia of the

rod around the center of rotation O is $I=5 \text{ kg}\cdot\text{m}^2$, calculate the direction of rotation and angular acceleration of the rod. [A: 0.8 rad/s^2 , counterclockwise.]



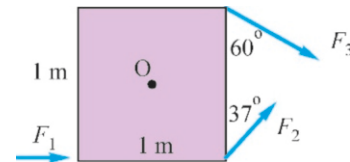
Problem 7.7



Problem 7.8

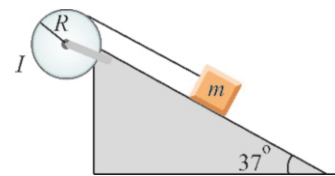
7.8 The pulley in the figure has a radius of $R=10 \text{ cm}$ and moment of inertia $I=0.1 \text{ kg}\cdot\text{m}^2$. The masses $m_1=1 \text{ kg}$ and $m_2=2 \text{ kg}$ are attached to the ends of a rope wrapped around the pulley. Calculate the accelerations and the tensions in the ropes when the system is released.

[A: $\alpha = 7.7 \text{ rad/s}^2$, $a = 0.77 \text{ m/s}^2$, $T_1 = 11 \text{ N}$, $T_2 = 19 \text{ N}$.]



Problem 7.9

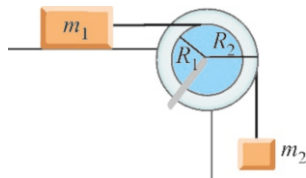
7.9 The square plate seen in the figure has a side length of 1 m and a moment of inertia $I_0=7 \text{ kg}\cdot\text{m}^2$ with respect to the center of rotation O. The forces $F_1=10 \text{ N}$, $F_2=20 \text{ N}$ and $F_3=30 \text{ N}$ are applied on the corners of this square in the directions shown in the figure. What is the angular acceleration of the plate? [A: 0.22 rad/s^2 .]



Problem 7.10

7.10 The pulley fixed to the upper end of the inclined plane shown in the figure has a radius of $R=20 \text{ cm}$ and moment of inertia $I=0.4 \text{ kg}\cdot\text{m}^2$. The slope angle of the plane is 37° and the coefficient of friction is $\mu=0.2$. A block with mass $m=1 \text{ kg}$ is attached to the end of the rope and released. Find the accelerations and the tension in the rope.

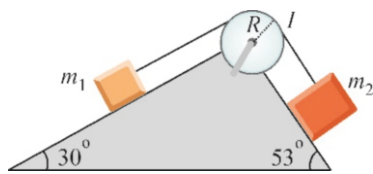
[A: 2 rad/s^2 , 0.4 m/s^2 , 4 N.]



Problem 7.11

7.11 The moment of inertia of the pulley shown in the figure is $I=0.45 \text{ kg}\cdot\text{m}^2$. A rope attached to a block with mass $m_1=1 \text{ kg}$ at rest on a horizontal plane is wrapped around the pulley at an inner radius of $R_1=10 \text{ cm}$. Another mass $m_2=2 \text{ kg}$ hanging freely is attached to another rope that is wrapped around the pulley at a radius of $R_2=20$. The coefficient of friction on the plane is $\mu=0.4$. Calculate the accelerations and the tensions in the ropes when the system is released.

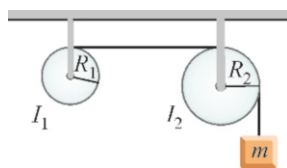
[A: $\alpha=7.2 \text{ rad/s}^2$, $a_1=0.7$, $a_2=1.4 \text{ m/s}^2$, $T_1=4.7$, $T_2=17 \text{ N}$.]



Problem 7.12

7.12 Two frictionless planes are inclined at angles of 30° and 53° . A block with mass $m_1=1 \text{ kg}$ is placed on the first and another block with mass $m_2=2 \text{ kg}$ on the second. The two blocks are tied to the two ends of a rope passing through a pulley with a radius of $R=50 \text{ cm}$ and moment of inertia $I=2 \text{ kg}\cdot\text{m}^2$. Find the accelerations and the tensions in the rope.

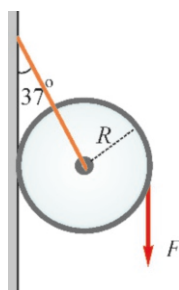
[A: 2 rad/s^2 , 1 m/s^2 , $T_1 = 6$, $T_2 = 14 \text{ N}$.]



Problem 7.13

7.13 A rope is wrapped around two pulleys, as shown in the figure, and its free end is attached to a block of mass $m=5 \text{ kg}$ hanging freely. The radii of the pulleys are $R_1=10 \text{ cm}$ and $R_2=20 \text{ cm}$, and their moments of inertia $I_1=0.1 \text{ kg}\cdot\text{m}^2$ and $I_2=0.2 \text{ kg}\cdot\text{m}^2$, respectively. Calculate the accelerations and the tensions in the ropes when the system is released.

[A: $a=2.5 \text{ m/s}^2$, $\alpha_1=25$, $\alpha_2=12.5 \text{ rad/s}^2$, $T_1=25$, $T_2=37.5 \text{ N}$.]

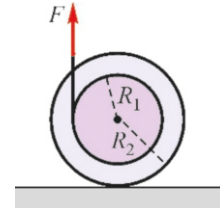


Problem 7.14

7.14 A cylindrical roll of paper with a radius of $R=10 \text{ cm}$ and moment of inertia $0.2 \text{ kg}\cdot\text{m}^2$ rests against a vertical wall by

a rope attached to its center. The rope makes an angle of 37° with the wall. The coefficient of friction of the wall is 0.3 . A tangential downward force $F=10 \text{ N}$ is applied on the roll. Find the angular acceleration of the roll. (Hint: Use the equations (7.19–7.20) of general motion.) [A: $\alpha = 4.1 \text{ rad/s}^2$.]

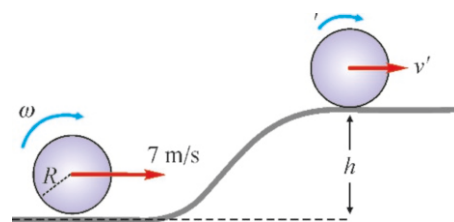
7.5 Rolling Motion



Problem 7.15

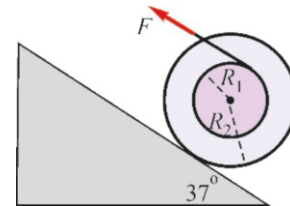
7.15 A wheel at rest on a horizontal plane has a mass 1 kg , a moment of inertia $I=0.2 \text{ kg}\cdot\text{m}^2$, an inner radius of $R_1=10 \text{ cm}$ and an outer radius of $R_2=20 \text{ cm}$. The wheel is slowly pulled vertically upwards with a force $F=8 \text{ N}$ by means of a rope wrapped around at radius R_1 . If the wheel is rolling without slipping, calculate the angular acceleration of the rolling motion and the linear acceleration of the center of mass.

[A: $\alpha = 3.3 \text{ rad/s}^2$, $a_{\text{cm}} = 0.67 \text{ m/s}^2$.]



Problem 7.16

7.16 A sphere with mass M and radius R is rolling on a horizontal plane with velocity $v=7 \text{ m/s}$ and then climbs to a plane at height $h=2.8 \text{ m}$. The moment of inertia of the sphere is $I=\frac{2}{5}MR^2$. Find the final linear velocity of the sphere. (Note: The mass and radius will cancel out in the final steps of calculation.) [A: 3 m/s .]



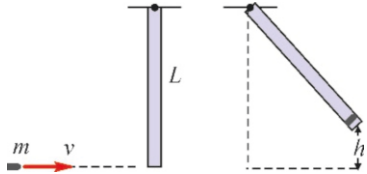
Problem 7.17

7.17 A wheel at rest on a plane inclined at 37° has a mass 1 kg , a moment of inertia $I=0.4 \text{ kg}\cdot\text{m}^2$, an inner radius of $R_1=10 \text{ cm}$ and an outer radius of $R_2=20 \text{ cm}$. The wheel is slowly pulled upwards with a force $F=15 \text{ N}$ parallel to the plane, by means of a rope wrapped around it at radius R_1 . If the wheel is rolling without slipping, calculate the angular acceleration of the rolling motion and the linear acceleration of the center of mass. [A: $\alpha = 7.5 \text{ rad/s}^2$, $a_{\text{cm}} = 1.5 \text{ m/s}^2$.]

7.6 Angular Momentum and Its Conservation

7.18 A disk with a moment of inertia $I_1=2 \text{ kg}\cdot\text{m}^2$ and rotating with angular velocity $\omega_1=8 \text{ rad/s}$ clamps together with

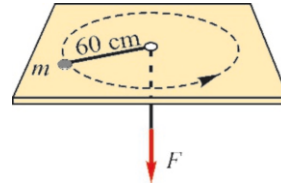
another disk with a moment of inertia $I_2=3\text{ kg}\cdot\text{m}^2$ and rotating in the opposite direction on the same axis with angular velocity $\omega_2=6\text{ rad/s}$. What will be the common angular velocity of the two disks? [A: -0.4 rad/s .]



Problem 7.19

7.19 A rod with mass $M=10\text{ kg}$ and length $L=1\text{ m}$ can rotate freely about its end hinged to the ceiling. A bullet with mass $m=100\text{ g}$ and a horizontal velocity $v=100\text{ m/s}$ hits and sticks to the free end of the rod. The moment of inertia of the rod with respect to its center of mass is $I_{\text{cm}} = ML^2/12$. (a) What will be the angular velocity of the (rod+bullet) system right after the collision? (b) How high will the bullet rise? (Note:

When the bullet rises by h , the center of mass of the rod will rise by $h/2$.) [A: (a) $\omega = 2.9\text{ rad/s}$, (b) $h = 0.29\text{ m}$.]



Problem 7.20

7.20 A rope tied to a mass $m=1\text{ kg}$ on a frictionless table is passed through a hole in the center of the table and is allowed to rotate in a circle when a force F is applied to the other end. (a) What should the force F be so that the mass m can rotate with angular velocity $\omega = 5\text{ rad/s}$ on a circular trajectory with radius 60 cm ? (b) What will the angular velocity be if the rope is pulled down and the rotation radius is brought to 30 cm ? [A: (a) $F = 15\text{ N}$, (b) $\omega' = 4\omega = 20\text{ rad/s}$.]

8

STATIC EQUILIBRIUM



Mostar Bridge in Bosnia and Herzegovina was commissioned by the famous Ottoman Grand Vizier Sokollu Mehmet Pasha near his birthplace in 1566. Its beauty and historical value has been recognized by UNESCO in its selection as a World Heritage Site.

Is it sufficient to have the resultant force be equal to zero to keep this bridge in equilibrium? Also, don't the forces need to have a certain distribution?

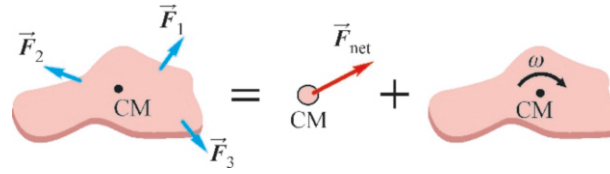
Stationary rigid structures are a part of our daily lives. The houses that we live in, the roads and bridges that we pass by in traffic, dams, etc. All of these structures are able to maintain their statical status because they can actually keep the forces acting on them in equilibrium, in other words, they remain in **static equilibrium**. The science that is concerned with objects and structures in equilibrium is called **Statics**; it is an important area of study in civil and mechanical engineering, architecture and many other branches of science and technology.

In this chapter, we will discuss the conditions of static equilibrium of rigid bodies.

8.1 TWO CONDITIONS OF STATIC EQUILIBRIUM

In examining rolling motion in Chapter 7, we saw that the most general motion of rigid bodies can be separated into two components:

Figure 8.1: The most general motion of a rigid body is separated into two as the translational motion of the center of mass and the rotational motion around the center of mass.



The first one is the **translational motion of the center of mass** and was determined by Newton's law (Equation 7.19):

$$\sum_i \vec{F}_i = m \vec{a}_{\text{cm}} \quad (\text{for translation}) \quad (8.1)$$

The second type of motion, **rotational motion about the center of mass**, is determined by the law of rotation dynamics (Equation 7.20):

$$\sum_i \tau_{i,\text{cm}} = I_{\text{cm}} \alpha \quad (\text{for rotation}) \quad (8.2)$$

Therefore, these two types of motion should be prevented for a rigid body to remain in static equilibrium. The condition required to prevent translational motion can be written directly as follows:

$$\sum_i \vec{F}_i = 0 \quad (8.3)$$

Writing this vector equation for the components, we have:

First Condition of Equilibrium

The net external force acting on an object should be zero:

$$\sum_i F_{i,x} = 0 \quad \text{and} \quad \sum_i F_{i,y} = 0 \quad (8.4)$$

Having the center of mass at rest, let us now examine the rotational motion of the object. To prevent rotation, we must have

$$\sum_i \tau_{i,\text{cm}} = 0$$

For a rotating body, this torque was calculated with respect to the center of mass. However, in the static condition the axis of rotation is irrelevant. This is because, if the object is not rotating, the net torque is zero regardless of the chosen axis. (Otherwise, it would rotate about that axis.) Therefore, we write the second condition of equilibrium as follows:

Second Condition of Equilibrium

The net torque of forces acting on an object about an arbitrary axis should be zero:

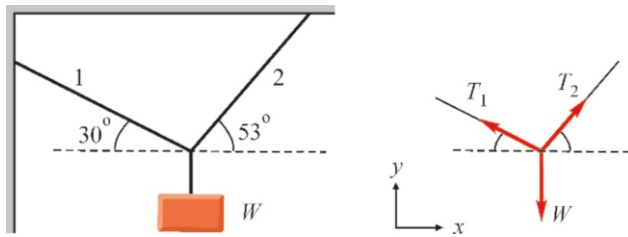
$$\sum_i \tau_i = 0 \quad (8.5)$$

These two conditions are, in principle, sufficient for static equilibrium. However, in real life, there are other parameters to consider, and our conditions may not be enough to calculate the static equilibrium of complex objects that consist of many parts (bridges, buildings, etc.) Other parameters are added to achieve the full solution of the problem. We will discuss these through worked examples.

8.2 APPLICATIONS

Now let us see how static problems can be solved using conditions of equilibrium, and other information particular to the problem at hand.

Example 8.1



A block with weight $W=10\text{ N}$ is in equilibrium hanging from two ropes as in the figure. Calculate the tensions in the ropes.

Answer

There will be no rotational motion, as the tensions in the ropes and the weight W meet at the same point and their torques are zero, hence the first condition will be sufficient.

We write it with the chosen axes:

$$\sum_i F_{i,x} = 0 \rightarrow -T_1 \cos 30^\circ + T_2 \cos 53^\circ = 0$$

$$\sum_i F_{i,y} = 0 \rightarrow +T_1 \sin 30^\circ + T_2 \sin 53^\circ - W = 0$$

We substitute the numerical values:

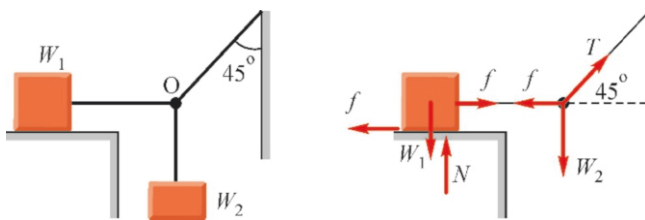
$$-0.87T_1 + 0.6T_2 = 0$$

$$0.5T_1 + 0.8T_2 = 10$$

From here, we find that $T_1 = 6\text{ N}$ and $T_2 = 8.7\text{ N}$.

Example 8.2

The weight $W_1=20\text{ N}$ is at rest on a horizontal plane with a coefficient of friction $\mu=0.4$. It is tied to a horizontal rope that connects to another weight W hanging freely. Both ropes are connected to a third rope attached to the wall at an angle of 45° . What is the maximum weight W_2 that can be attached without having the weight W_1 slide?



Answer

The free-body diagrams are shown above. The three forces W_2 , the tension T and the friction force f meet at point O, hence only the first condition is needed. The friction force can have any value, but it will be proportional to the normal force N_1 when the weight W_1 starts to slide:

$$f = \mu N_1 = \mu W_1$$

Therefore, we write the first condition of equilibrium at point O for these three forces:

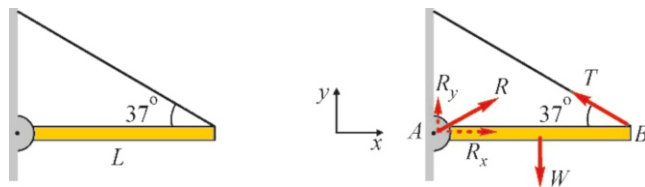
$$\sum_i F_{i,x} = -\mu W_1 + T \cos 45^\circ = 0$$

$$\sum_i F_{i,y} = +T \sin 45^\circ - W_2 = 0$$

By taking the unknown T from the second equation and substituting it in the first one, we get W_2 :

$$W_2 = \mu W_1 \tan 45^\circ = 0.4 \times 20 \times 1 = 8\text{ N}$$

Example 8.3



A beam with weight $W = 10\text{ N}$ is hinged to a wall from one end and tied to the wall from the other end with a cable, making a 37° angle with the horizontal. Calculate the tension in the

cable and the reaction force on the hinge.

Answer

There are three forces acting on the beam: The weight W , the tension T along the cable and the reaction force R on the hinge that can be in any direction.

We first write the first condition of equilibrium according to the axes shown in the figure:

$$\sum_i F_{i,x} = -T \cos 37^\circ + R_x = 0$$

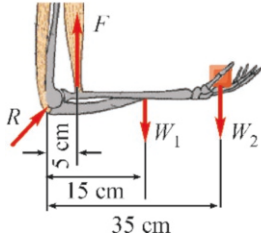
$$\sum_i F_{i,y} = +T \sin 37^\circ + R_y - W = 0$$

For the second condition, we are free to calculate the torque

with respect to any arbitrary axis. Hence, it is convenient to choose an axis that gives a simpler equation. Here, it will be smart to choose the axis at point A, because two unknowns (R_x, R_y) will both have zero torque and not be included in the equation:

$$\sum_i \tau_{i,A} = 0 \rightarrow -W \frac{L}{2} + T \sin 37^\circ L = 0$$

Example 8.4



The dimensions of an average human arm are shown in the figure. The biceps muscle acts at a distance of 5 cm from the elbow. The lower half of the arm has a weight of $W_1 = 20\text{ N}$ and its center of mass is located at 15 cm distance from the elbow. The distance of the hand to the elbow is 35 cm. How

We can immediately find T from this last equation:

$$T = \frac{W}{2 \sin 37^\circ} = \frac{10}{2 \times 0.6} = 8.3\text{ N}$$

Substituting this value in the other two equations, we find the components of the reaction force R_x, R_y :

$$R_x = 6.7\text{ N} \quad \text{and} \quad R_y = 5\text{ N}$$

much force F should the biceps muscle apply to lift a weight of $W_2 = 10\text{ N}$?

Answer

The forces on the arm are shown in the figure. We can take the torque with respect to the elbow, as we do not need the reaction force R :

$$\sum_i \tau_i = 0 \rightarrow F \times 0.05 - W_1 \times 0.15 - W_2 \times 0.35 = 0$$

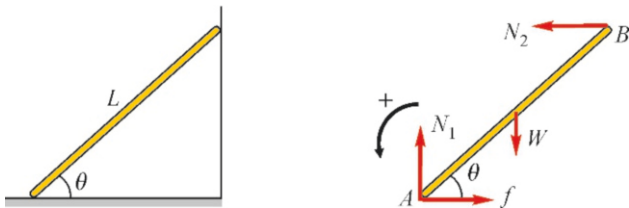
From here, we calculate the force F :

$$F = \frac{0.15W_1 + 0.35W_2}{0.05} = 150\text{ N}.$$

It may seem surprising that the force on the biceps is 15 times greater, but it is true. It is difficult for the arm to carry a load in the horizontal position, and it therefore gets tired very quickly.

Example 8.5

A ladder with weight W and length L is leaned against a wall with angle θ . The wall is frictionless, while the coefficient of friction is $\mu=0.5$ on the horizontal plane. What is the minimum angle θ at which the ladder can stand without sliding?



Answer

The forces acting on the ladder are shown in the figure. As the wall is frictionless, it will only have the perpendicular reaction force N_2 . The ground has both the normal reaction force N_1 and the friction force f .

The friction force f can have any value when the ladder is in equilibrium, but when the ladder starts to slide, it will reach its maximum value $f=\mu N_1$. Therefore, we will make the calculations for this limit value.

We write the first condition for equilibrium:

$$\sum_i F_{i,x} = \mu N_1 - N_2 = 0$$

$$\sum_i F_{i,y} = +N_1 - W = 0$$

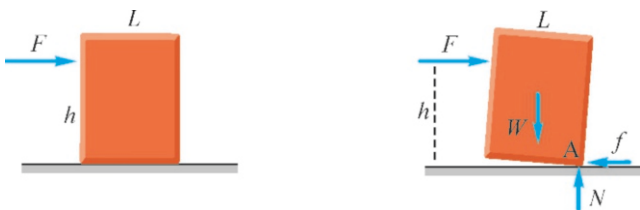
For the second condition, it is smart to choose point A to calculate the torque, then two unknowns (f and N_1) will not be included in the equation:

$$\sum_i \tau_{i,A} = 0 \rightarrow -W \frac{L}{2} \cos \theta + N_2 L \sin \theta = 0$$

By eliminating the unknowns N_1, N_2 from these equations, we find θ :

$$\tan \theta = \frac{2}{\mu} = \frac{2}{0.5} = 4 \rightarrow \theta = 76^\circ$$

Example 8.6



We want to push a chest without toppling it. The chest's weight is $W=500\text{ N}$ and its length is $L=2\text{ m}$ and it is at rest on a horizontal plane with a coefficient of friction $\mu=0.8$. (a) What is the minimum horizontal force F that can move the chest? (b)

At what maximum height h can we apply this minimum force F without toppling the chest?

Answer

(a) It is sufficient to apply a force greater than the friction force in order to move the chest. As the friction force will reach the value $f = \mu N$ once the chest starts to slide, we get

$$F = f = \mu N = \mu W$$

. We substitute the numerical values to find F :

$$F = 0.8 \times 500 = 400\text{ N}$$

(b) The chest, when it is about to roll over, will only be in

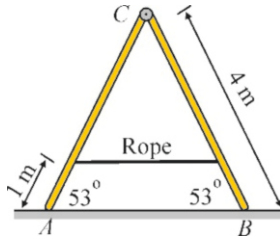
contact at point A. The chest will topple when the torque of the force F about point A is greater than or equal to the torque of the weight W . Calculating this limit condition, we get

$$\sum_i \tau_{i,A} = 0 \rightarrow -Fh + W \frac{L}{2} = 0$$

Substituting the value of F that we found in item (a),

$$h = \frac{L}{2\mu} = \frac{2}{2 \times 0.8} = 1.25 \text{ m}$$

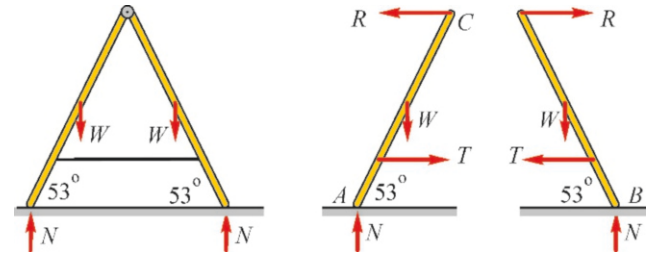
Example 8.7



The two arms of a ladder shaped like an ‘A’, standing motionless on a frictionless horizontal plane, are hinged from the top and tied to each other with a horizontal rope near the bottom. Each of the ladder arms has weight $W=30 \text{ N}$ and length $L=4 \text{ m}$. The points at which the rope is tied are 1 m from the lower end of the ladder. Find the tension in the rope and the reaction force on the hinge.

Answer

This problem is a good example of the isolation technique in the equilibrium problem of two objects. First, let us consider the ladder as a whole (the figure below). The forces on the hinge and the rope need not be shown, because they are internal forces of this system. We can only write $2N = 2W$ for the reaction forces on the ground. Hence, we will never be able to find T and the reaction force at the hinge by considering the whole ladder.



Therefore, one of the arms of the ladder should be isolated from the other (the figure on the right). According to the law of action-reaction, these forces will be equal and opposite on each arm. (According to symmetry, the force R must also be horizontal, otherwise it would be downwards in one and upwards in the other.)

We can now write the conditions of equilibrium for one arm. The first condition of equilibrium for the arm AC is:

$$\sum_i F_{i,x} = +T - R = 0$$

$$\sum_i F_{i,y} = +N - W = 0$$

From here, we get the results $T = R$ and $N = W$.

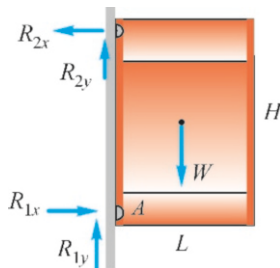
We write the second condition with respect to point A:

$$\sum_i \tau_{i,A} = R(4 \sin 53^\circ) - T(1 \sin 53^\circ) - W(2 \cos 53^\circ) = 0$$

Using the fact that $R = T$ in this equation, we find the result:

$$T = R = \frac{2W}{3 \tan 53^\circ} = \frac{2 \times 30}{3 \times 1.33} = 15 \text{ N}$$

Example 8.8



A barn door with height $H=2 \text{ m}$, width $L=1 \text{ m}$, and weight 50 N is hinged to the wall at its top and bottom edge corners. Calculate the horizontal and vertical components of the reaction forces on the top and bottom hinges.

Answer

This problem is interesting in that it shows how the number of equations may sometimes be insufficient for the full solution of static equilibrium problems. The forces acting on the door are shown in the figure. The

directions of the forces are chosen such that the top hinge will pull and the bottom hinge will push the door.

We write the first condition of equilibrium:

$$\sum_i F_{i,x} = R_{1x} - R_{2x} = 0 \rightarrow R_{1x} = R_{2x}$$

$$\sum_i F_{i,y} = R_{1y} + R_{2y} - W = 0 \rightarrow R_{1y} + R_{2y} = W$$

Let us take the torque with respect to point A:

$$\sum_i \tau_{i,A} = R_{2x}H - W \frac{L}{2} = 0$$

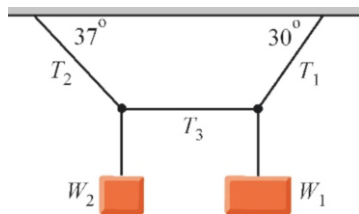
From here, we find the horizontal components of the reaction forces:

$$R_{1x} = R_{2x} = \frac{L}{2H} W = \frac{1}{4} 50 = 12.5 \text{ N}$$

We have no information for the vertical components of the reaction force other than the equation $R_{1y} + R_{2y} = W$ found above. Even if we took torque with respect to another point, only the *sum* of these two vertical components would appear in the equation. Therefore, it is not possible to calculate the vertical components separately; we can only know their sum:

$$R_{1y} + R_{2y} = W = 50 \text{ N}.$$

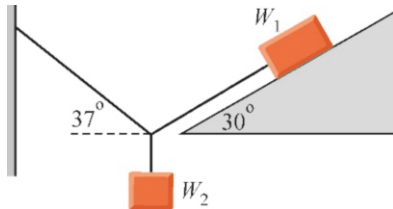
Problems



Problem 8.1

8.1 Two weights hanging from ropes as shown in the figure above, are in equilibrium. If $W_1=10\text{ N}$, find the weight W_2 and the tensions in the ropes.

[A: $W_2 = 13$, $T_1 = 20$, $T_2 = 22$, $T_3 = 17\text{ N}$.]



Problem 8.2

8.2 A block with weight $W_1=10\text{ N}$ is placed on a plane inclined at an angle of 30° and with a coefficient of friction $\mu=0.9$. Another weight W_2 is tied to the weight W_1 with a rope parallel to the inclined plane and with another rope at an angle of 37° to the wall. What is the largest weight W_2 that can be hung without having the weight W_1 slide?

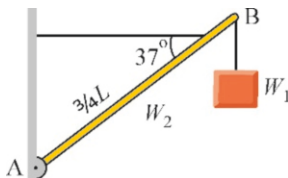
[A: 1.8 N .]

8.3 A weight $W_1=5\text{ N}$ is hung at the 10 cm mark of a uniform meter stick. The stick is balanced horizontally if a knife edge is placed under the 40 cm mark. What is the weight of the meter stick?

[A: 15 N .]

8.4 A horizontal beam of weight $W_1=10\text{ N}$ and length $L=1\text{ m}$ is supported by two vertical ropes attached to each end. A weight $W_2=30\text{ N}$ is hung at a distance $d = 30\text{ cm}$ from the right end. Find the tensions in the ropes.

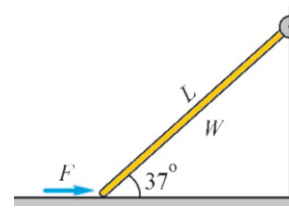
[A: 14 N and 26 N .]



Problem 8.5

8.5 In the figure, one end of a beam with weight $W_2=20\text{ N}$ and length L is hinged to the wall and a weight $W_1=10\text{ N}$ is hung on the other end. The beam is kept at an angle of 37° with the horizontal by means of a horizontal rope tied at a distance $3L/4$ from the hinged end A. Find the tension in the rope and the horizontal and vertical components of the reaction force at the hinge.

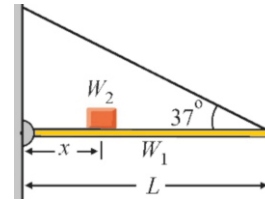
[A: $T = R_x=36$, $R_y=30\text{ N}$.]



Problem 8.6

8.6 A stick with negligible mass and a length of 5 m is hinged to a wall from one end, with its other end freely resting on a frictionless horizontal plane. A horizontal force $F=10\text{ N}$ is applied to the bottom end of the stick. Calculate the horizontal and vertical components of the reaction force on the hinge.

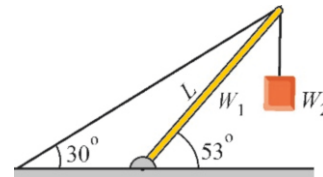
[A: $R_x = 10$, $R_y = 7.5\text{ N}$.]



Problem 8.7

8.7 A horizontal beam with length $L=1\text{ m}$ and weight $W_1=10\text{ N}$ is hinged to a wall from one end and tied to the wall from the other end with a rope making a 37° angle with the horizontal. A block with weight $W_2=40\text{ N}$ is placed on this beam at distance x . The rope is able to endure a maximum tension of 50 N . Find the maximum distance x that the block can go and the components of the reaction forces on the hinge.

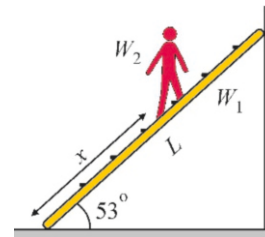
[A: $x = 0.63\text{ m}$, $R_x = 40$, $R_y = 20\text{ N}$.]



Problem 8.8

8.8 The beam in the figure above with length L and weight $W_1=10\text{ N}$ is hinged to the ground from one end at 53° angle with the horizontal and tied to the ground with a rope making a 30° angle with the horizontal. A weight $W_2=20\text{ N}$ is hung on the free end of the stick. Calculate the tension in the rope and the components of the reaction force on the hinge.

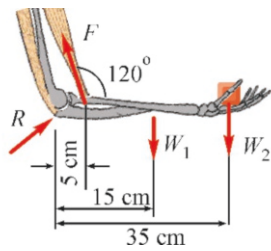
[A: $T = 38$, $R_x = 33$, $R_y = 49\text{ N}$.]



Problem 8.9

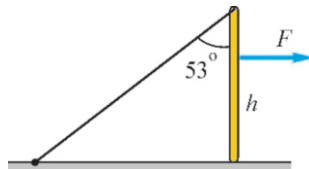
8.9 A ladder with weight $W_1=50\text{ N}$ and length $L=3\text{ m}$ is leaned against a frictionless wall at an angle of 53° with the horizontal. The coefficient of friction on the ground is 0.4 . A man with weight $W_2=700\text{ N}$ is climbing on the ladder. How

far from the lower end can the man climb before the ladder slides? [A: $x=1.61$ m.]



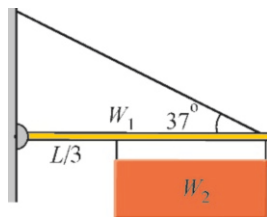
Problem 8.10

8.10 The dimensions of an average human arm are shown in the figure. The biceps muscle acts at a distance of 5 cm from the elbow. The lower half of the arm has a weight of $W_1=20$ N and its center of mass is located at 15 cm distance from the elbow. The distance of the hand to the elbow is 35 cm. How much force F should the biceps muscle apply to lift a weight of $W_2 = 50$ N? [A: 470 N.]



Problem 8.11

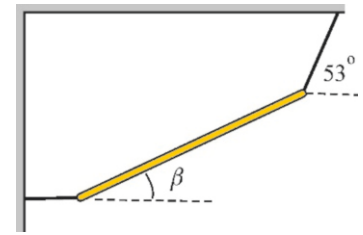
8.11 A rod with weight $W=10$ N and length $L=1$ m is freely placed vertically on a horizontal plane with a coefficient of friction $\mu=0.5$. The top end of the rod is tied to the ground using a rope making a 53° angle with the vertical. The rod is being pulled by a horizontal force $F=20$ N acting at height h from the ground. At what height h will the lower end of the rod start to slide? [A: $h = 0.55$ m.]



Problem 8.12

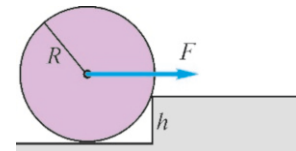
8.12 A beam with length L and weight $W_1 = 10$ N is hinged to a wall from one end, and tied to the same wall from the other end with a rope making a 37° angle with the horizontal. A signboard with mass $W_2 = 20$ N is hung on this beam. One end of the signboard is tied to the outer end of the beam and the other end at a distance of $L/3$ from the hinge. Calculate the tension in the rope and the components of the reaction force on the hinge. (*Hint:* The signboard applies equal forces

of $W_2/2$ to each of the hanging points.) [A: $T = 31$, $R_x = 24$, $R_y = 12$ N.]



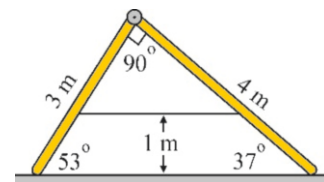
Problem 8.13

8.13 A uniform rod with two ropes tied at each end is in equilibrium, as shown in the figure. The rope attached to the wall is horizontal. The rope attached to the ceiling makes an angle of $\alpha=53^\circ$ with the horizontal. What is the angle β of the rod? (*Note:* The weight and length of the rod will not appear in the final expression.) [A: $\beta=34^\circ$.]



Problem 8.14

8.14 A cylinder with radius $R=50$ cm and a mass of 1 kg is pulled at its center by a force F along a horizontal plane. However, a step with height $h=10$ cm on the plane is preventing motion. What should the minimum value of the force F be for the cylinder to overcome the step? (*Hint:* What will the normal force be when the cylinder is just leaving the ground?) [A: $F = 7.5$ N.]



Problem 8.15

8.15 The two arms of a ladder shaped like a 'slanted A', standing on a frictionless horizontal plane, are hinged from the top and tied to each other with a horizontal rope near the bottom. The ladder arms have weights 30 N and 40 N, and lengths 3 m and $L=4$ m, respectively. The rope is at a height of 1 m from the ground. Find the tension in the rope, the normal forces on the ground, and the components of the reaction force on the hinge.

[A: $T = R_x = 40$, $R_y = 7$, $N_1 = 37$, $N_2 = 33$ N.]

9

HARMONIC MOTION



The *colibri* (hummingbird), with a size of 2 cm, is the smallest bird that is not extinct. It can flap its wings as fast as 80 beats per second.

Which concepts should be used to characterize the periodic motion? How can we associate these concepts with the laws of physics?

A motion that repeats itself regularly is called a **periodic motion**. It is the most frequently encountered type of motion in nature, in our daily lives and in technology. The oscillation of a pendulum, the vibration of a guitar string, our heartbeats or the vibration of our vocal cords, the ebb and tide of the sea, etc. Even an object at rest is made of vibrating atoms.

Oscillatory motion occurs when an object is disturbed from its equilibrium position and a *restoring force* exists that is trying to bring it back. In real life, there are also other *dissipative forces* that quickly dampen the oscillations.

We will examine the basic concepts and properties of this motion that repeats itself within a certain time interval.

9.1 SIMPLE HARMONIC MOTION

The oscillations of a mass connected to a spring represent the simplest motion in which we can observe the basic features of vibrational motion. Let us fix one end of a spring with spring constant k to a wall and attach a mass m to its other

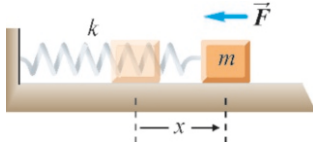


Figure 9.1: Spring force is always in opposite direction to extension.

end on a frictionless horizontal plane (Figure 9.1).

The mass starts to vibrate when we pull the spring by x from its normal length and release. The weight and the normal reaction force perpendicular to motion cancel each other out, and are thus not taken into consideration. Since the spring force causes the motion of the mass, let us write Newton's law:

$$\begin{aligned} F &= ma \\ -kx &= m \frac{d^2x}{dt^2} = m x'' \end{aligned}$$

Here, we used x'' to indicate the second derivative of position x . We have thus obtained an equation that determines the relation between the position and its second derivative:

$$x'' + \frac{k}{m} x = 0 \quad (9.1)$$

This expression is called the **differential equation of harmonic motion**. We will be able to determine the motion if we can find its solution in the form $x=x(t)$, in other words, the position as a function of time.

Differential equations are part of advanced mathematics, and their solution methods are known. However, we can find the solution here without using advanced techniques. Let us write the equation as follows:

$$x'' = -\frac{k}{m} x$$

What is the function $x(t)$ such that its second derivative will be proportional to its negative? We know these functions: They are sine and cosine functions. Therefore, we can predict the form of the solution:

$$x = A \cos \omega t \quad (9.2)$$

Here, A and ω are two constants to be determined later. We could have chosen the solution as a sine function as well (We will discuss when to choose sine or cosine below.)

Let us first make sure that this function satisfies the differential equation. Let us take the derivative twice:

$$\begin{aligned} x' &= -\omega A \sin \omega t \\ x'' &= -\omega^2 A \cos \omega t \end{aligned}$$

We substitute the expressions for x and x'' in Eq. (9.1) and simplify to get

$$\begin{aligned} -\omega^2 A \cos \omega t + \frac{k}{m} A \cos \omega t &= 0 \\ \left[-\omega^2 + \frac{k}{m} \right] A \cos \omega t &= 0 \end{aligned}$$

This equation must be true at every time t . The cosine is not always zero, therefore the expression in the brackets should be zero:

$$\left[-\omega^2 + \frac{k}{m} \right] = 0$$

From here, we find the constant ω , called the **angular frequency**, in terms of the mass and the spring constant:

$$\omega = \sqrt{\frac{k}{m}} \quad (\text{angular frequency}) \quad (9.3)$$

We will discuss the physical meaning of the angular frequency ω below. At this stage, the equation of motion of the mass m can be written as follows:

$$x = A \cos \omega t \quad (\text{simple harmonic motion}) \quad (9.4)$$

This motion is called **simple harmonic motion** or **sinusoidal motion**. Figure 9.2 shows how position x varies with respect to time t .

Selection of the cosine or sine function: We chose the cosine function for vibration in Eq. (9.4) above. We could have also chosen this as the sine function, because the sine also satisfies the same equation of motion. The initial position of the object determines which one is to be chosen (Figure 9.2).

We take $x = A \cos \omega t$ if the object is pulled to a certain distance and released from that distance at time $t = 0$. Indeed, we get $\cos 0^\circ = 1$ and $x = A$ at $t = 0$, as it should be. On the other hand, if the object is thrown from its equilibrium position at the start, then we should have ($x = 0$) at $t = 0$. Indeed, $x = A \sin \omega t$ fulfills this condition, because the sine function gives $x = 0$ at time $t = 0$.

(Actually, the cosine and sine are just the same curve. The name varies only with respect to the place where you draw the vertical axis. You get the cosine if you have the vertical axis pass through the maximum of the curve and the sine if you pass it through the zero value.)

In conclusion, our function selection becomes as follows:

$$x = \begin{cases} A \cos \omega t & (\text{if the object starts at maximum distance}) \\ A \sin \omega t & (\text{if the object starts at } x = 0) \end{cases} \quad (9.5)$$

Now, let us learn the basic concepts of the harmonic motion.

Amplitude (A)

When the cosine function takes values within the range $[-1, +1]$, the position $x = A \cos \omega t$ will also vary between the values $[-A, +A]$ (Figure 9.3). When we pull the mass attached to the spring by a distance A and release it, it does indeed start to go back and forth between $[-A, +A]$, and never venturing outside of this range. This quantity A , which is the absolute value of the maximum extension, is called the **amplitude**.

Period (T)

Harmonic motion is periodic, in other words, it repeats itself within a certain time interval. This complete motion in which the object passes through the same point in the same direction is called a **cycle**, and the time to complete one cycle is called the **period** (Figure 9.3). Let us find the period formula from the equation of motion.

Let us rewrite the position of the object at any time t :

$$x = A \cos \omega t$$

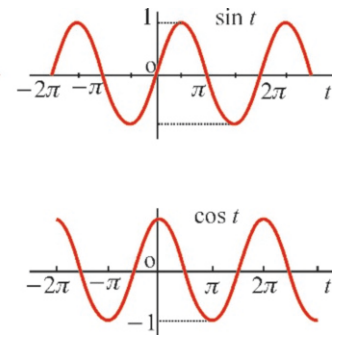


Figure 9.2: The sine and cosine functions.

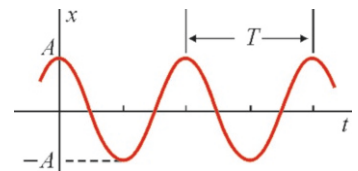


Figure 9.3: Amplitude and period.

There should pass a time interval T such that the object passes through the same x again:

$$\begin{aligned}x(t + T) &= x(t) \\ A \cos \omega(t + T) &= A \cos \omega t \\ \cos \omega(t + T) &= \cos \omega t\end{aligned}$$

The cosine function is periodic in the interval $[0, 2\pi]$, in other words, it repeats itself after 2π . Therefore, the difference between the arguments of the cosines on both sides of the equation must be 2π :

$$\begin{aligned}\omega(t + T) - \omega t &= 2\pi \\ \omega T &= 2\pi\end{aligned}$$

From here, we find the period of the harmonic motion:

$$\boxed{T = \frac{2\pi}{\omega} \quad (\text{period})} \quad (9.6)$$

This expression is true for all kinds of harmonic motions. In particular, if we use Eq. (9.3), which we found for the angular frequency of the mass-spring system, the period formula becomes:

$$\boxed{T = 2\pi \sqrt{\frac{m}{k}} \quad (\text{Period of spring-mass system})} \quad (9.7)$$

It can be checked that this expression gives the period unit in terms of seconds (s). According to this formula, when we attach an object with a known mass m to a spring with a known constant k and vibrate it, its period T is determined.

Frequency (f)

The number of cycles that the objects performs per unit time is called the **frequency** and is denoted by f . Accordingly, frequency is the reciprocal of period. For example, if the period is 3 seconds, then the number of oscillations in 1 second will be $1/3$. Therefore, the expression for frequency is:

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi} \quad (\text{frequency})} \quad (9.8)$$

This expression is true for all kinds of periodic motions. In particular, the frequency of the spring-mass system is

$$f = 2\pi \sqrt{\frac{k}{m}} \quad (9.9)$$

The unit of frequency is $1/\text{second} = \text{s}^{-1}$ and is also called the Hertz (Hz) in technology:

$$1 \text{ s}^{-1} = 1 \text{ Hz} \quad (9.10)$$

If we write the ω in the formula (9.8) in terms of frequency, we get

$$\omega = 2\pi f \quad (9.11)$$

This expression explains why ω is called the angular frequency. When the frequency is 1, in other words, when a cycle is completed in 1 second, ω takes the value of $360^\circ = 2\pi$ as if it were rotating in one complete circle.

As a conclusion, the equations of simple harmonic motion can be written in one of the three following forms:

$$x = \begin{cases} A \cos \omega t \\ A \cos \frac{2\pi}{T} t \\ A \cos 2\pi f t \end{cases} \quad (9.12)$$

Example 9.1

The position of an object with mass $m=3$ kg attached to a spring varies with time as

$$x = 6 \cos 0.8\pi t \text{ (meters)}$$

- What are the amplitude, angular frequency and period of the motion?
- What is the spring constant?
- What is the position of the object at time $t=5/12$ s?
- How many seconds does it take for the object to reach the positions $x=0$ m and $x=2$ m?

Answer

(a) Amplitude means maximum extension. As the cosine function varies within the range $[-1, +1]$, the object will go back and forth within the range $[-6, +6]$. Therefore, the amplitude will be $A=6$ m.

Angular frequency is the coefficient in front of the variable t . Therefore, we get $\omega=0.8\pi \text{ s}^{-1}=0.8\pi \text{ Hz}$. (It is convenient to leave the constant π in these calculations. It is sometimes canceled out and sometimes taken as $\pi^2 \approx 10$.)

We use Eq. (9.6), which relates period to angular frequency:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.8\pi} = 2.5 \text{ s}$$

(b) We use Eq. (9.3), which gives the relation between spring constant and angular frequency:

$$\omega = \sqrt{k/m} \rightarrow k = m\omega^2$$

$$k = 3 \times (0.8\pi)^2 = 1.92\pi^2 \approx 19 \text{ N/m.}$$

(c) We use $t = 5/12$ s in the equation of motion:

$$x = 6 \cos 0.8\pi \times \frac{5}{12} = 6 \cos \frac{\pi}{3}$$

Substituting $\cos \pi/3 = \cos 60^\circ = 0.5$, we find $x=3$ m.

(d) The distance from initial position $x=A$ to $x=0$ is one fourth of a cycle. Therefore, it will reach $x=0$ in $T/4=2.5/4=0.63$ s. In order to find the time to reach the other position, we substitute the value $x=2$ m in the equation of motion and calculate the t in the cosine:

$$x = 6 \cos 0.8\pi t \rightarrow 2 = 6 \cos 0.8\pi t$$

$$\cos 0.8\pi t = 1/3 = 0.33$$

We use the table in Appendix C to find the angle whose cosine is 0.33 as $71^\circ = 1.23$ radian. It is necessary to use radian units in these calculations. From here, we find time t :

$$0.8\pi t = 1.23 \rightarrow t = 0.49 \text{ s}$$

Example 9.2

A mass m attached to a spring with a constant $k=125$ N/m completes 5 cycles in 4 seconds.

- What are the period, angular frequency and m ?
- The mass is stretched by 60 cm from its equilibrium position and released. The time $t = 0$ is started on the stopwatch as it passes through the equilibrium position. Write the equation of the simple harmonic motion.

Answer

(a) Period is the time for one cycle:

$$T = 4/5 = 0.8 \text{ s.}$$

From here, we can calculate the angular frequency and the mass of the object:

$$\omega = \frac{2\pi}{0.8} = 2.5\pi \text{ Hz}$$

$$m = \frac{k}{\omega^2} = \frac{125}{(2.5\pi)^2} = 2 \text{ kg.}$$

(b) The object's position at $t = 0$ is $x = 0$. Therefore, we must use the sine function, which is zero at $t = 0$. Amplitude is given as $A = 0.6$ m. The equation becomes:

$$x = 0.6 \sin 2.5\pi t.$$

Velocity and Acceleration in Simple Harmonic Motion

We have learned how the position x of a mass attached to a spring varies with time t . What are the velocity and acceleration of this object at time t ? In order to determine these, let us start off by using the derivative definitions of

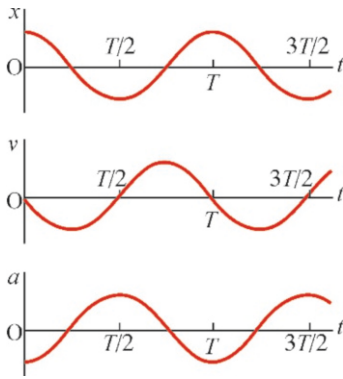


Figure 9.4: Position, velocity and acceleration.

velocity and acceleration:

$$x = A \cos \omega t$$

$$v = \frac{dx}{dt} = -\omega A \sin \omega t \quad (9.13)$$

$$a = \frac{dv}{dt} = -\omega^2 A \cos \omega t \quad (9.14)$$

These variations are shown in Figure (9.4). Let us emphasize the important points about the position, velocity and acceleration of harmonic motion.

- The velocity and acceleration are also harmonic; they oscillate as sine or cosine with the same frequency.
- The variations of position and velocity are opposite to each other: When position reaches its maximum value ($x=A$), velocity is zero. In contrast, as the object passes through the equilibrium point ($x=0$), its velocity is maximum.
- Velocity and acceleration also act in opposition to each other: The acceleration of the object is zero when its velocity is maximum (passing through the equilibrium point). In contrast, the object has maximum acceleration when it reaches maximum distance, where it has zero velocity.
- There is a relation between position and velocity that is always true regardless of what time t is equal to. We use the identity $\sin^2 a + \cos^2 a = 1$ to eliminate t between x and v :

$$\cos^2 \omega t = \frac{x^2}{A^2} \quad ; \quad \sin^2 \omega t = \frac{v^2}{\omega^2 A^2}$$

$$\sin^2 \omega t + \cos^2 \omega t = \frac{x^2}{A^2} + \frac{v^2}{\omega^2 A^2} = 1$$

Simplifying this expression, we get

$$v = \omega \sqrt{A^2 - x^2} \quad (9.15)$$

Hence, if either position or velocity is known, the other can be calculated without any need to know time.

- There is also a relation between position and acceleration:

$$a = -\omega^2 A \cos \omega t$$

$$a = -\omega^2 x \quad (9.16)$$

Therefore, if either position or acceleration of the object is known, the other can be calculated without reference to time t .

Example 9.3

A mass $m=2$ kg attached to a spring with spring constant $k=72$ N/m is stretched by 50 cm from its equilibrium position and released. (a) What will be its maximum velocity and acceleration? (b) What will its velocity and acceleration be as it passes through position $x = 30$ cm?

Answer (a) We first calculate the angular frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{72}{2}} = 6 \text{ Hz}$$

In Eqs. (9.13–9.14), which we found for velocity and acceleration, we set sine and cosine to their maximum values of 1 to calculate the maximum values of velocity and acceleration:

$$v_{\max} = \omega A = 6 \times 0.5 = 3 \text{ m/s}$$

$$a_{\max} = \omega^2 A = 6^2 \times 0.5 = 18 \text{ m/s}^2.$$

(b) We use Eq. (9.15), which gives the relation between position and velocity:

$$v = \omega \sqrt{A^2 - x^2} = 6 \sqrt{0.5^2 - 0.3^2} = 6 \times 0.4 = 2.4 \text{ m/s.}$$

Example 9.4

The angular frequency of an object performing simple harmonic motion is 5 Hz. This object passes through position $x = 3 \text{ m}$ with velocity $v = 20 \text{ m/s}$. (a) What is the amplitude of the motion? (b) What is its maximum velocity?

Answer

(a) We use Eq. (9.15), which gives the relation between position and velocity:

We use Eq. (9.16), which gives the relation between position and acceleration:

$$a = -\omega^2 x = -6^2 \times 0.3 = -11 \text{ m/s}^2.$$

$$v^2 = \omega^2 (A^2 - x^2)$$

We take the amplitude A from this formula:

$$A = \sqrt{x^2 + (v/\omega)^2}$$

$$A = \sqrt{3^2 + (20/5)^2} = 5 \text{ m.}$$

(b) Velocity will be maximum when $x = 0$ in the aforementioned formula, in other words, when the object passes through the equilibrium point. Accordingly,

$$v_{\text{max}} = \omega A = 5 \times 5 = 25 \text{ m/s.}$$

Energy of Harmonic Motion

Harmonic motion will have a kinetic energy dependent on the velocity of the mass and a potential energy dependent on the extension of the spring. Both types of energy will change with respect to time.

Using the expression for velocity from Eq. (9.13), in the definition of kinetic energy, we get

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t \tag{9.17}$$

Using Eq. (9.4) for position x in the elastic potential energy of the spring, we get

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2 \omega t \tag{9.18}$$

The variations of the kinetic and potential energies are shown in Figure 9.5a.

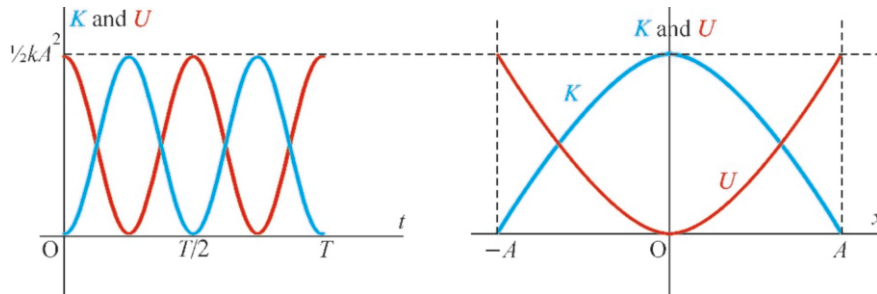


Figure 9.5: (a) The variations of kinetic and potential energies: (a) against time t , (b) against position x .

Let us calculate the total mechanical energy:

$$E = K + U = \frac{1}{2} \underbrace{m\omega^2}_k A^2 \sin^2 \omega t + \frac{1}{2}kA^2 \cos^2 \omega t = \frac{1}{2}kA^2 (\underbrace{\sin^2 \omega t + \cos^2 \omega t}_1)$$

$$E = \frac{1}{2}kA^2 = \text{constant}$$

Therefore, the total energy is constant in simple harmonic motion and has the value of the potential energy at maximum extension. This is easy to understand, because its velocity will be zero when it reaches maximum extension, in other words, all of it will have turned into potential energy:

$$E = K + U = \frac{1}{2}kA^2 \quad (\text{total energy of harmonic motion}) \tag{9.19}$$

Figure 9.5b also shows how kinetic and potential energies vary with position x and how their sum E remains constant.

Note that the energy is proportional to the square of the amplitude A . The energy is multiplied by a factor of 4 when the amplitude is doubled. This is why the destructive impact of earthquakes increases with amplitude.

Example 9.5

A mass $m=1$ kg attached to a spring with spring constant $k = 400$ N/m has a total energy of 800 J.

- (a) What is the amplitude of the oscillations?
 (b) What will the velocity of m be when the potential energy of the system is 200 J?

Answer

(a) We use Eq. (9.19), which we found for total energy:

$$E = K + U = \frac{1}{2} kA^2$$

$$A = \sqrt{\frac{2E}{k}} = \sqrt{\frac{1600}{400}} = 2 \text{ m}$$

(b) If the potential energy is 200 J, the kinetic energy will be $800 - 200 = 600$ J:

$$\frac{1}{2}mv^2 = 600 \rightarrow v = \sqrt{\frac{2 \times 600}{1}} = 35 \text{ m/s}$$

Example 9.6

In a system performing simple harmonic motion,

- (a) At which x/A ratio will the kinetic and potential energies be equal?
 (b) What will the ratio of the kinetic and potential energies be when $x = A/2$?

Answer

(a) We write the total energy expression:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

If the kinetic and potential energies are equal, this means

that they are both half of the total energy. This value is taken for potential energy:

$$\frac{1}{2}kx^2 = \frac{1}{2} \left(\frac{1}{2}kA^2 \right) \rightarrow \frac{x}{A} = \frac{1}{\sqrt{2}}$$

(b) Let us write the formula that gives velocity in terms of position:

$$v^2 = \omega^2 (A^2 - x^2)$$

We write the ratio of energies for the value $x = A/2$ in this expression:

$$\frac{K}{U} = \frac{mv^2}{kx^2} = \frac{\cancel{m}\omega^2(A^2 - A^2/4)}{\cancel{k}(A/2)^2} = 3$$

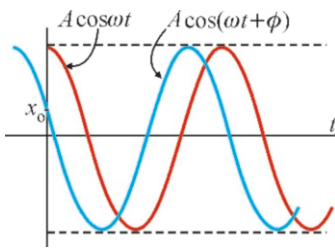


Figure 9.6: Phase angle ϕ .

Phase Angle

We had previously discussed when to use cosine or sine in the harmonic motion equation: We use sine if the object starts from the origin, and cosine if it is released from the maximum distance.

What if the motion starts at any place other than these two points? The answer to this question is seen in Figure 9.6. We can make the function start at any point by adding another term besides (ωt) as the argument of cosine. Having the unit of an angle, this constant argument is called the **phase angle** or phase difference and is indicated with ϕ :

$$x = A \cos(\omega t + \phi) \quad (9.20)$$

In this expression, we should find the sine function when we take $\phi = -90^\circ = -\pi/2$, in other words, when we start the cosine function 90° from behind. Indeed, the trigonometric identity $\cos(\alpha - 90^\circ) = \sin \alpha$ gives us the sine equation.

We can calculate the phase angle ϕ in terms of the initial position x_0 when we set $t = 0$ in this expression:

$$x_0 = A \cos \phi \quad \implies \quad \cos \phi = \frac{x_0}{A} \quad (9.21)$$

The phase difference will be important when we consider the superposition of two harmonic motions. We will return to this issue later, in the section on interference of waves.

Example 9.7

A mass-spring system can oscillate at an angular frequency $\omega=5$ Hz. The mass is stretched by 2 m from its equilibrium

position and released. The clock is started as it passes through position $x = 1$ m. Find the phase angle and write the equation of the harmonic motion.

Answer

We write the most general harmonic motion expression:

$$x = A \cos(\omega t + \phi)$$

We require that this equation satisfy the value $x = 1$ m at time $t = 0$:

$$1 = 2 \cos(0 + \phi) \rightarrow \cos \phi = 0.5$$

The angle whose cosine is 0.5 is $60^\circ = \pi/3$. Therefore, we write the equation of the motion as follows:

$$x = 2 \cos(5t + \pi/3)$$

Example 9.8

The equation of a harmonic motion is given as

$$x = 3 \cos(0.2\pi t - \pi/4)$$

- (a) What is its position at the start?
- (b) When will it reach the maximum position?

Answer

(a) We use the starting time, in other words, $t = 0$, in the

given equation, and find the position:

$$x = 3 \cos(0 - \pi/4) = 3 \cos(\pi/4) = 3 \times \sqrt{2}/2 = 2.1 \text{ m.}$$

(b) Maximum position is the amplitude $x = A$. We substitute the amplitude value for x :

$$3 = 3 \cos(0.2\pi t - \pi/4) \rightarrow \cos(0.2\pi t - \pi/4) = 1$$

The angles whose cosine is 1 are $0, 2\pi, 4\pi \dots$. We choose the smallest one 0 and find the time t :

$$0.2\pi t - \pi/4 = 0 \rightarrow t = 1.25 \text{ s.}$$

9.2 PENDULUM MOTION

Simple Pendulum

A mass tied to the end of a rope performs oscillatory motion. The period of a pendulum was used to measure time in old wall clocks. This motion can be analyzed as simple harmonic motion under certain conditions.

A mass m is tied to the end of a string with length L . Let us pull the string by angle θ from the vertical direction and release it. The forces acting on the rope at any instant are shown in Figure 9.7.

The tension force T has no effect on the motion, because it is along the string. However, the weight mg has a component in the tangential direction to the circle with radius L . The force that moves the mass along the path is the force $mg \sin \theta$.

When writing Newton's law along the tangent, we choose the direction towards which the angle θ increases as positive. If we also write acceleration as the derivative of velocity, we get

$$F_t = ma_t$$

$$-mg \sin \theta = m \frac{dv}{dt}$$

We write the velocity v in terms of angular velocity as the mass performs circular motion with radius L . The angular velocity is the derivative of the angle θ :

$$v = L\omega = L \frac{d\theta}{dt}$$

Substituting this expression and arranging the terms, we get

$$-mg \sin \theta = mL \frac{d^2\theta}{dt^2} \rightarrow \theta'' + \frac{g}{L} \sin \theta = 0$$

This equation looks similar to Eq. (9.1), which we found for harmonic motion of the mass-spring system, except that it is written for the angle θ . Also, we have $\sin \theta$ instead of θ , and therefore we cannot write the solution directly.

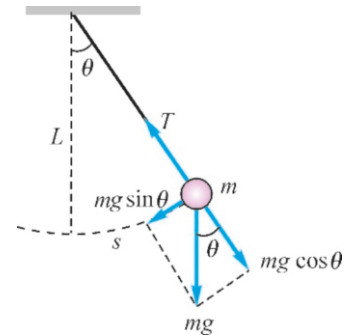


Figure 9.7: Simple pendulum.

However, if we consider oscillations with small amplitudes, we can approximately take $\sin \theta \approx \theta$. Then, we obtain the simple harmonic motion equation for the angle θ :

$$\theta'' + \underbrace{\frac{g}{L}}_{\omega^2} \theta = 0 \quad (9.22)$$

Comparing this expression with the equation (9.1), the coefficient of the term θ becomes the square of angular frequency ω^2 . From here, we find the period T of the pendulum using the relation $T = 2\pi/\omega$:

$$T = 2\pi \sqrt{\frac{L}{g}} \quad (\text{Period of simple pendulum}) \quad (9.23)$$

The maximum angle θ_{\max} is used instead of the amplitude A when writing the pendulum motion equation. Accordingly, the solution of pendulum motion is

$$\theta = \theta_{\max} \cos \omega t \quad (9.24)$$

and is valid for oscillations with small angles. Notice that Eq. (9.23) for the period of pendulum is independent of mass and amplitude. This property was used in the designing of clocks for centuries.

Physical Pendulum

The simple pendulum is an ideal case. Actually, every rigid body can perform pendulum motion. A rigid body that can rotate about an axis that is not through the center of mass is called a **physical pendulum**. The pendulums used in wall clocks and in the industry are physical pendulums.

Consider a rigid body with mass m and moment of inertia I_{cm} with a rotation axis a distance d from its center of mass (Figure 9.8). Only the $mg \sin \theta$ component of the weight causes the rotation motion.

Choosing the positive direction towards which the angle θ increases and writing the net torque with respect to the center of rotation O , we get,

$$\tau_o = F_t d = -mg \sin \theta d = I \alpha$$

If we write the angular acceleration α as the second derivative of the angular position θ and arrange the terms, we get

$$\theta'' + \frac{mgd}{I} \sin \theta = 0$$

We again use the approximate value $\sin \theta \approx \theta$ for small angles:

$$\theta'' + \underbrace{\frac{mgd}{I}}_{\omega^2} \theta = 0 \quad (9.25)$$

We again obtain the form of a simple harmonic motion equation. According to this equation, the coefficient of the term θ will be angular frequency ω^2 . We get the formula for period using $T = 2\pi/\omega$:

$$T = 2\pi \sqrt{\frac{I}{mgd}} \quad (\text{Period of physical pendulum}) \quad (9.26)$$

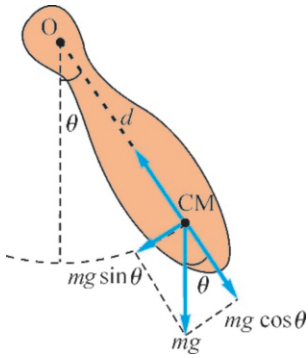


Figure 9.8: Physical pendulum.

Note that the moment of inertia I in this formula is with respect to the axis of rotation O . We use the parallel axis theorem to relate it to the I_{cm} values given in the tables (Chapter 7):

$$I = I_{\text{cm}} + m d^2$$

Example 9.9

We want to construct a simple pendulum with period 1 s.

- (a) What length of string should be used?
 (b) What length would be needed on planet Jupiter, where gravitational acceleration is 24 m/s^2 ?

Answer

We solve the formula (9.23), which we found for the period of a pendulum, for the length L :

$$T = 2\pi \sqrt{\frac{L}{g}} \rightarrow L = \frac{gT^2}{4\pi^2}$$

We use the values $T = 1 \text{ s}$ and $g = 10$:

$$L = \frac{10 \times 1}{4\pi^2} = 0.25 \text{ m}$$

(b) We can find the length of the pendulum on Jupiter by taking $g = 24$ in the same formula:

$$L = \frac{24 \times 1}{4\pi^2} = 0.60 \text{ m}.$$

Example 9.10

A physical pendulum is constructed by hanging a rod with length $L=60 \text{ cm}$ and mass m from one of its ends.

- (a) Calculate the period of the oscillations.
 (b) At what distance from its center of mass should it be hung for the period to be 2 s?

Answer

(a) The moment of inertia of the rod with respect to its center of mass is $I_{\text{cm}} = \frac{1}{12} mL^2$. We had previously found its moment of inertia with respect to one end to be $\frac{1}{3} mL^2$ using the parallel axis theorem. Therefore, if we calculate the period for $d = L/2$ using the formula (9.26), we get

$$T = 2\pi \sqrt{\frac{I}{mgd}} = 2\pi \sqrt{\frac{mL^2/3}{mgL/2}} = 2\pi \sqrt{\frac{2L}{3g}}$$

$$T = 2\pi \sqrt{\frac{2 \times 0.60}{3 \times 10}} = \frac{2\pi}{5} = 1.26 \text{ s}.$$

(b) If we write the value $I = I_{\text{cm}} + md^2$ for the axis at distance d and then solve for d , we get

$$T = 2\pi \sqrt{\frac{\frac{1}{12}mL^2 + md^2}{mgd}} \rightarrow d^2 - \frac{gL^2}{4\pi^2}d + \frac{L^2}{12} = 0$$

Substituting $T = 2 \text{ s}$ and the other numerical values, we get

$$d^2 - d + 0.03 = 0$$

The two solutions to this equation are $d = 0.03$ and 0.97 and the solution greater than $L = 0.6$ is not taken into consideration. Therefore, the solution is as follows:

$$d = 0.03 \text{ m}$$

9.3 DAMPED HARMONIC MOTION

An oscillating body left to itself will stop after some time. If a swing is not pushed, its amplitude gradually decreases and then stops. The energy of a mass attached to a spring decreases due to friction. However, we sometimes deliberately want the oscillation to stop. For example, when a car travels over a bump, the passengers are protected by oscillations resulting from the springs in the shock absorbers that last some reasonable time. Sometimes, we do not want oscillations to last long. The needle of a bathroom scale should stop without oscillating too much.

Therefore, the damping effect should also be taken into consideration in a real oscillatory motion. In the simplest case, let us reconsider a mass attached to a spring, but this time, let us place the mass inside of a box filled with oil. The viscosity or the friction of the oil during the motion of the mass will provide the necessary damping.

In liquids, the friction force is proportional to the velocity of the moving object. We can feel this when trying to walk in the sea. We can move easily with

slow steps, but the resistance of the water suddenly increases when we try to run. Using F_d to show this friction force, we get

$$F_d = -bv \quad (9.27)$$

and the constant b is called the **damping coefficient**. The negative sign in this expression shows that the force is always in the opposite direction to the velocity.

In this new setup, there are two forces acting on the mass m (Figure 9.9): The spring force is F and the friction force is F_d . Let us write Newton's law:

$$\begin{aligned} F + F_d &= ma \\ -kx - bv &= m \frac{d^2 x}{dt^2} \end{aligned}$$

Let us rearrange the terms and write the differential equation of motion:

$$\frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \quad (9.28)$$

We see that a new term with a first derivative is added.

We can likewise guess the solution to this equation without going into the topic of differential equations. In simple harmonic motion, we had found the solution by guessing that cosine and sine are the functions proportional to their own second derivative. Here, we have the additional friction term with the first derivative. Likewise, let us consider which function is proportional to its own derivative. This is the exponential function. Therefore, we look for a solution that includes both behaviors simultaneously, in other words, the product of the exponential and cosine functions. We can then determine the necessary coefficients by requiring that this solution satisfy the equation. Let us directly write the solution here without going into too much detail of this long operation:

$$x = A e^{-(b/2m)t} \cos \sqrt{\omega_0^2 - \frac{b^2}{4m^2}} t \quad \text{with} \quad (\omega_0^2 = k/m) \quad (9.29)$$

We can check that this solution satisfies Eq. (9.28). The parameter $\omega_0^2 = k/m$ is just the angular frequency of the simple oscillator with no damping.

Let us examine what types of motion this expression involves:

- As shown in Figure 9.10, this is a sinusoidal motion whose amplitude decreases exponentially over time.
- It is easy to see that this equation reduces to simple harmonic motion if friction is zero, in other words, if $b = 0$.
- **Critical damping:** When the argument of the cosine is zero, we get $\cos 0^\circ = 1$ and the amplitude decreases exponentially without the object making any oscillation. The following must be true for this to happen:

$$\omega_0^2 - \frac{b^2}{4m^2} = 0 \quad \implies \quad b = \sqrt{4mk}$$

No oscillation is observed for a coefficient of friction b above this limiting value and the object exponentially comes to rest. In this case, it is said to be **overdamped**.

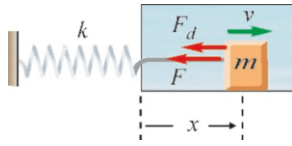


Figure 9.9: Damped oscillator.

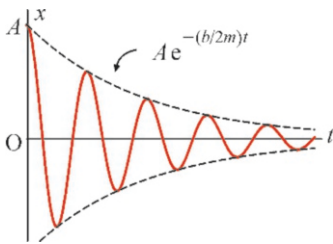


Figure 9.10: Damped harmonic motion. It performs sinusoidal oscillations as it decreases exponentially.

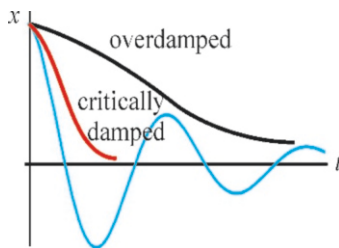


Figure 9.11: Critically damped and overdamped oscillations.

Regardless of the type of damping, the mechanical energy of the system vanishes at the end and all of the energy is converted into heat. An important application of damped motion in technology is the **shock absorber**. It provides for easy driving by preventing vibrations on cars and motorcycles (Figure 9.12). Large shock absorbers are used to dampen vibrations caused by earthquakes in modern buildings and bridges.

9.4 DRIVEN HARMONIC MOTION – RESONANCE

Did you ever think about how we push a child on a swing? We push the swing gently at the end of each oscillation. In other words, we apply a *periodic force*. And to do this, we wait for the instant when the swing is about to go back. In other words, we apply a force *in phase* with the swing.

When an external force is applied on a harmonic oscillator, the effect of friction is partly compensated and much richer behaviors are observed. Let us again use the setup of the damped motion. This time, we apply an external force $F_a \cos \omega t$ with angular frequency ω (Figure 9.13).

We write Newton's law for the sum of the three forces acting on the system:

$$\begin{aligned} F + F_d + F_a \cos \omega t &= ma \\ -kx - b \frac{dx}{dt} + F_a \cos \omega t &= m \frac{d^2x}{dt^2} \end{aligned}$$

Let us rearrange the equation using the expression $k/m = \omega_0^2$:

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega_0^2 x = F_a \cos \omega t \quad (9.30)$$

This time, we have a new term that is not dependent on the unknown x , but is rather a function of time.

Again, we shall only write the solution of this equation without going into its explicit solution:

$$x = A \cos(\omega t + \phi) \quad (9.31)$$

$$A = \frac{F_a/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + b^2\omega^2/m^2}} \quad (9.32)$$

Let us emphasize the important properties of the driven harmonic motion:

- The motion is again simple harmonic motion, due to the $\cos \omega t$ term. Its angular frequency is equal to the angular frequency of the driving force and follows it with a phase difference.
- The most important property is that the amplitude A is dependent on the frequency of the applied force. This amplitude is shown in Figure 9.14 as a function $A(\omega)$.
- **Resonance.** $\omega_0 = \sqrt{k/m}$ is determined by the properties of the spring and mass that constitute the oscillator and is called the **natural frequency**. The frequency ω of the external force can be arbitrarily varied.

As seen in the figure, the amplitude of the oscillator increases enormously as $\omega \rightarrow \omega_0$. This is called **resonance**. It is surprising that it has a large impact even if the amplitude F_a of the external force is small.



Figure 9.12: The shock absorber of a motorcycle.

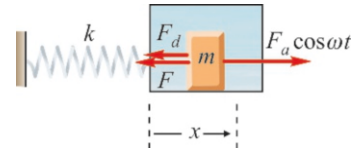


Figure 9.13: Oscillator driven by a periodic force.

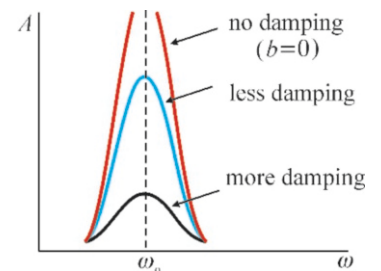


Figure 9.14: Variation of amplitude with respect to external frequency in driven harmonic motion.

Resonance is a very important phenomenon in everyday life and in technology. In some cases, it is completely intentional:

- In musical instruments, each string is tuned such as to resonate at a different frequency.
- When searching for a radio station, the receiver is made to resonate with the broadcast frequency of the transmitter that we want.
- A microwave oven emits electromagnetic waves that resonate the water molecules inside of food.
- **Magnetic Resonance.** The neutrons and protons in the nucleus of all atoms have a magnetic property. They receive and return energy only at certain frequencies to and from external magnetic fields. Certain atoms are made to resonate by applying a magnetic field close to their frequencies, and the location of the radiation that they emit can be measured to map the profile of organs.



Figure 9.15: Magnetic resonance imaging (MRI) device.

Sometimes, resonance is unwanted:

- Soldiers are not marched in unison when crossing a bridge. In this phenomenon, discovered during World War I, the periodic steps of soldiers may cause a bridge to resonate and collapse.
- Some accessories and equipment made of crystal glass may resonate and break due to a surrounding vibration.
- Attention is paid to ensure that the natural vibration frequencies of buildings are not close to earthquake or wind gust frequencies, because, even if the building is strong, it may resonate and collapse. One example of this was the collapse of the Tacoma River Bridge in the United States due to the periodic wind gusts. In order to prevent this hazard, the natural frequencies of buildings can be determined and their amplitude can be reduced using shock absorbers.

Multiple-choice Questions

- Which is incorrect for simple harmonic motion?
 - The number of oscillations per unit time is the frequency.
 - The time for one oscillation is the period.
 - The maximum oscillation distance is the amplitude.
 - Acceleration is constant.
- The angular frequency ω is equal to which of the following?
 - $2\pi/f$
 - $2\pi/T$
 - $2\pi/A$
 - $2\pi A$
- If a harmonic oscillator passes through the equilibrium position at time $t = 0$, which of the following describes its motion?
 - $A \cos \omega t$
 - $A \sin \omega t$
 - $A \tan \omega t$
 - $A \cos(\omega t + \pi/3)$
- If the amplitude of a harmonic oscillator with total energy E is doubled, what will be its total energy?
 - $E/2$
 - $2E$
 - $4E$
 - $E/4$
- Which is incorrect for simple harmonic motion?
 - Velocity is zero when the position is maximum.
 - Position is zero when the velocity is maximum.
 - Acceleration is zero when the velocity is maximum.
 - Acceleration is zero when the position is maximum.
- Which of the following are incorrect for the period of a mass+spring system?
 - It depends on the amplitude.
 - It depends on the mass.
 - It depends on the spring constant.
 - It depends on the phase angle.
 - I & III
 - I & IV
 - II & III
 - II & IV

7. By what factor will the period of a pendulum increase if its length is increased by a factor of 4?
(a) 2 (b) 4 (c) 6 (d) 8
8. Which is incorrect for the energy of a harmonic oscillator?
(a) Potential energy is maximum at the maximum position.
(b) Kinetic energy is maximum at the maximum velocity.
(c) Kinetic+potential energy is constant.
(d) Kinetic energy is zero at the equilibrium position.
9. In a mass+spring system, the spring constant is doubled and the mass is halved. By what factor will the period decrease?
(a) 2 (b) 4 (c) 6 (d) 8
10. The oscillation period of a simple pendulum depends on which of the following?
(a) Mass
(b) Amplitude
(c) Length
(d) Kinetic energy
11. A simple pendulum used as a clock is lagging. How can you fix it?
(a) Increase its length.
(b) Decrease its length.
(c) Increase its mass.
(d) Decrease its mass.
12. While a simple pendulum is oscillating, half of the mass on its end breaks off and falls. How will the motion change?
(a) It will get slower.
(b) It will get faster.
(c) It will not change.
(d) It will stop.
13. The period of a harmonic oscillator is doubled. Which of the following is correct?
(a) Its frequency will double.
(b) Its angular frequency will be halved.
(c) Its amplitude will double.
(d) Its amplitude will be halved.
14. Which is incorrect when the amplitude of a harmonic oscillator is doubled?
(a) Maximum velocity will double.
(b) Maximum acceleration will double.
(c) Energy will increase by a factor of 4.
(d) Period will increase by a factor of 4.
15. Which of the following will not help to increase the maximum velocity of a mass+spring system?
(a) Increasing its amplitude.
(b) Increasing its angular frequency.
(c) Increasing its energy.
(d) Increasing its phase angle.
16. How will a pendulum clock behave when taken to the surface of the Moon?
(a) It will lag.
(b) It will run too fast.
(c) It will not change.
(d) It will stop.
17. What is the source of damping in damped harmonic motion?
(a) Spring force.
(b) Friction force.
(c) Amplitude.
(d) Potential energy.
18. The period of a physical pendulum does not depend on which of the following?
(a) Moment of inertia.
(b) Mass.
(c) Distance between the axis and the center of mass.
(d) Amplitude.
19. Which will change when we choose another starting time $t = 0$ in a harmonic oscillator?
(a) Amplitude
(b) Frequency
(c) Energy
(d) Phase angle
20. An oscillator consisting of a mass+spring system is supplied with a periodic external force. Which of the following is correct?
(a) It will approach resonance as the amplitude of the external force increases.
(b) It will approach resonance as the amplitude of the external force decreases.
(c) It will approach resonance as the external frequency approaches the natural frequency.
(d) No resonance will be observed.

Problems

9.1 Simple Harmonic Motion

9.1 The position of an object with mass $m=2$ kg undergoing simple harmonic motion varies with time as,

$$x = 3 \cos 4\pi t \text{ (meters).}$$

(a) What are the amplitude, angular frequency and period of the motion? (b) What is the spring constant? (c) What is the position of the object at time $t=1/24$ s? (d) How many seconds does it take for the object to reach the positions $x = -3$ m and $x = 1.5$ m?

[A: (a) $A = 3$ m, $\omega = 4\pi$ Hz, $T = 0.5$ s, (b) $k = 320$ N/m, (c) 2.6 m, (d) $1/4$ and $1/12$ s.]

9.2 A mass $m=0.5$ kg attached to a spring completes 25 cycles in 10 s. (a) What are its period, angular frequency and spring constant k ? (b) This object is stretched by 40 cm from its equilibrium position and released at the time $t=0$. Write the equation for the simple harmonic motion equation.

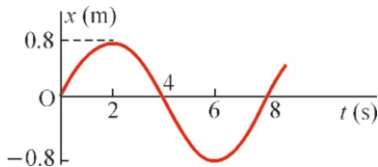
[A: (a) $T=0.4$ s, $\omega=5\pi$ Hz, $k=125$ N/m, (b) $x=0.4 \cos 5\pi t$.]

9.3 A mass $m=3$ kg attached to a spring with spring constant $k=75$ N/m is pulled by 50 cm from its equilibrium point and released. (a) What will be its maximum velocity and acceleration? (b) What will its velocity and acceleration be as it passes through position $x = 20$ cm?

[A: (a) 2.5 m/s, 12.5 m/s², (b) 2.3 m/s, -5 m/s².]

9.4 The angular frequency of an object performing simple harmonic motion is 2 Hz. This object passes through position $x = 1$ m with velocity $v = 4.8$ m/s. (a) What is the amplitude of the motion? (b) What is its maximum velocity?

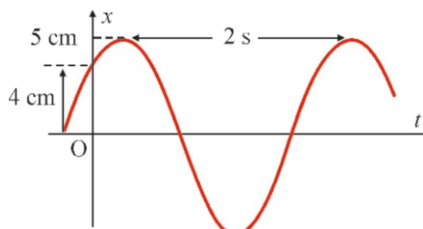
[A: (a) 2.6 m, (b) 5.2 m/s.]



Problem 9.5

9.5 The figure above shows the motion of a mass $m = 2$ kg attached to a spring. (a) Find the angular frequency and the spring constant. (b) Write the equation of this motion.

[A: (a) $\omega = \pi/4$, $k = 1.25$ N/m, (b) $x = 0.8 \sin \pi t/4$.]



Problem 9.6

9.6 Write the equation of the simple harmonic motion shown in the figure.

[A: $x = 0.05 \cos(\pi t - 37^\circ)$.]

9.7 A system with $m=2$ kg attached to a spring with spring constant $k=72$ N/m has a total energy of 900 J. (a) What is the amplitude of the oscillation motion? (b) At a certain moment, the kinetic energy is 500 J. What is the extension of the spring at that moment? [A: (a) 5 m, (b) 3.3 m.]

9.8 When the body of a car with mass 1200 kg is pressed from above and released, it vibrates at a frequency of 3 Hz. What will the vibration frequency of the car be if 5 people, each weighing 70 kg, get into the car? [A: 2.6 Hz.]

9.9 A system performs simple harmonic motion. (a) At which x/A ratio will the kinetic energy be twice the potential energy? (b) What will the ratios of the kinetic and potential energies be when $x = A/5$?

[A: (a) $x/A = 1/\sqrt{3}$, (b) $K/U = 24$.]

9.10 A harmonic oscillator with angular frequency 3 Hz is pulled by 5 m from its equilibrium position and released. The time $t=0$ is initiated as it passes through position $x=4$ m. Find the phase angle and write the equation of the harmonic motion.

[A: $x = 5 \cos(3t + 37^\circ)$.]

9.11 The equation of a harmonic motion is given as

$$x = 4 \cos(\pi t - \pi/3)$$

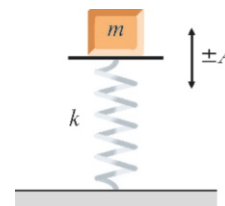
(a) What is its position at the start? (b) When will it reach the maximum position? [A: (a) 2 m, (b) 0.33 s.]



Problem 9.12

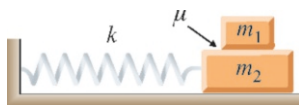
9.12 A bullet with mass $m=50$ g and velocity $v = 200$ m/s hits and embeds in a wooden block with mass $M=950$ g, which is attached to a spring with constant $k=64$ N/m. Calculate the angular frequency and amplitude of the resulting harmonic motion.

[A: $\omega = 8$ Hz, $A = 1.25$ m.]



Problem 9.13

9.13 A mass $m=1$ kg is placed on the plate of a spring with constant $k=50$ N/m that can vibrate in the vertical direction. How large can the amplitude of the harmonic motion be such that the mass does not leave the plate? The mass of the plate can be neglected. (Hint: Consider the maximum acceleration of the harmonic motion.) [A: $A \leq 0.2$ m.]



Problem 9.14

9.14 A mass $m_1=1$ kg is placed on another mass $m_2=2$ kg lying on a frictionless horizontal plane. The friction coefficient between the two blocks is $\mu=0.5$. The mass m_2 is attached to a spring with spring constant $k=60$ N/m and set into harmonic motion. What is the maximum amplitude such that both masses can oscillate together without the top mass sliding? (*Hint*: Compare the maximum acceleration that can be given to the mass m_1 solely by the friction force and the maximum acceleration of the harmonic motion.)

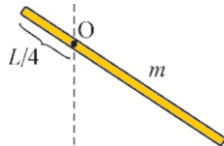
[A: $A \leq 0.25$ m.]

9.2 Pendulum Motion

9.15 The combined effect of the centripetal force at the equator and the oblate shape of the Earth result in a $g=9.78$ m/s² at the equator and a $g=9.83$ m/s² at the poles. By how much will a simple pendulum clock with a period of 1 s on the equator lag behind on the poles in 1 day? [A: 220 s.]

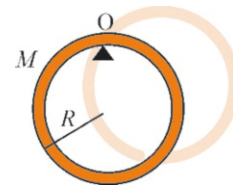
9.16 A simple pendulum with length 2.5 m is pulled by 5° from its equilibrium position and released. (a) In how many seconds will it reach its equilibrium position? (b) In how many seconds will it reach the angle $\theta = 2.5^\circ$?

[A: (a) $T/4 = \pi/4 = 0.79$ s, (b) $\pi/6$ s.]



Problem 9.17

9.17 A stick with mass m and length $L=1.20$ m can rotate about an axis O located at distance $L/4$ from one end. Calculate the period of the pendulum motion. [A: 1.67 s.]



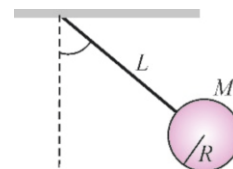
Problem 9.18

9.18 A ring with mass M and radius R is hung on a nail, as shown in the figure. Show that the period of the harmonic motion about the axis O is,

$$T = 2\pi \sqrt{\frac{2R}{g}}$$

9.19 When a rigid body with mass $m=1$ kg is oscillating about an axis located at a distance of 20 cm from its center of mass, its period is 2 s. Calculate the moment of inertia of the rigid body with respect to its center of mass.

[A: 0.16 kg·m².]



Problem 9.20

9.20 Calculate the period of oscillation of the solid sphere with radius $R=20$ cm attached to the end of a rope with length $L=1$ m, as shown in the figure. The moment of inertia of the solid sphere with respect to its center of mass is $I_{cm}=\frac{2}{5}MR^2$.

[A: 2.2 s.]

10

WAVES



Surfing is the art of riding a wave. The size of the *swell* (ocean surface wave) depends on the strength of the wind and the area of open water. What really moves forward in a wave motion? And what does a wave carry?

We all know how waves are formed by a pebble thrown into a still pond. The ripples move in ever growing circles. Likewise, waves propagate along a rope when we shake it from one end. The propagation of any change, vibration or perturbation in a medium is called a **wave**. There is a broad range of wave motions in our everyday life and in nature: Sound waves, water waves, electromagnetic waves that form light and radio-TV signals, etc.

The source of all of these waves is a vibration: we shake a rope, and a wave propagates along the rope; a tuning fork is set to vibrate, and sound waves appear, etc.

Notice that it is not the matter itself but the disturbance that propagates. If you examine a bottle floating on wavy water, you will notice that the bottle goes up and down as the wave passes, but does not move forward with the wave.

What is it that moves forward in wave? It is energy and momentum that propagate. Each water molecule transfers the energy and momentum that it receives to the next one as it goes up and down. Energy and momentum transferred in a certain harmony mediate the propagation of waves.



Figure 10.1: A bottle does not move forward as a wave does, it only moves up and down.

10.1 GENERAL PROPERTIES OF WAVES

Waves can be classified according to their various features. According to the type of perturbation, they can be separated into two distinct groups (Figure 10.2):

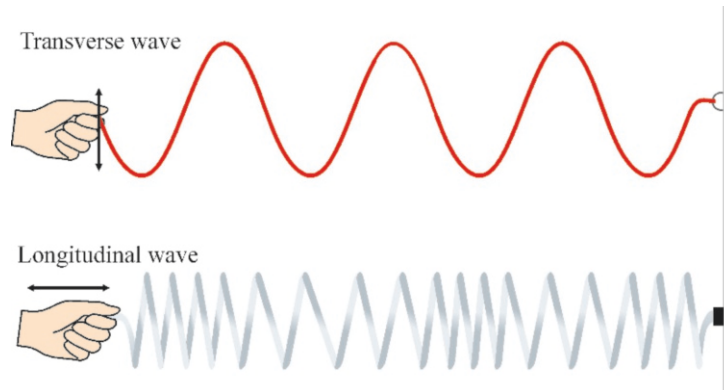


Figure 10.2: Transverse and longitudinal waves.

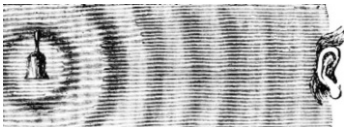


Figure 10.3: Sound wave is a longitudinal wave.

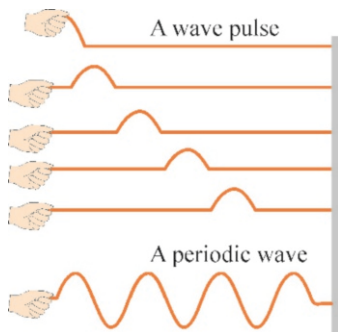


Figure 10.4: Impulse wave and periodic wave.

- **Transverse waves:** The perturbation is perpendicular to the wave's direction of propagation. The waves on a string are a good example of this type.
- **Longitudinal waves:** The perturbation is along the wave's direction of propagation. When we compress and release a spiral spring (Figure 10.2), a wave is observed to propagate along the spring. Sound waves are good examples of longitudinal waves. Sound waves consist of the back and forth motion of air molecules.

Another classification is according to the medium in which the wave propagates:

- **Mechanical waves:** A flexible or elastic medium is needed for the wave to propagate. The wave on a string propagates by moving the string. A sound wave is produced by the density fluctuations in the surrounding air. Mechanical waves must have a medium in which to travel.
- **Electromagnetic waves:** They can propagate in empty space and need no medium. In the modern view, an electromagnetic field fills all space, and perturbations in this electromagnetic field form the wave.

We will discuss only mechanical waves in this chapter. Electromagnetic waves will be mentioned in the discussion on light.

Another classification of waves is in terms of periodicity:

- **Periodic wave:** The perturbation propagates by repeating itself in time. For example, if one end of a string vibrates, the wave propagating along the string will be periodic.
- **Wave pulse:** This represents the propagation of a single disturbance. For example, if we shake the end of the string once, we will only observe the progress of this perturbation.

Regardless of type, every wave has a **propagation speed**, which is the rate at which the perturbation propagates. This speed, denoted by v , will be constant if the medium is homogeneous, and is independent of the cause of the waves.

Wave Speed

The wave speed in a medium depends on the various properties of said medium, such as density, heat, pressure, tension in the rope, etc. It can be calculated for a particular type of wave and medium. Without considering how these speeds are calculated, each of which requires a different method, let us only give the results:

- **Wave speed on string:** The linear mass density of a string is defined as $\mu = m/\ell$ (kg/m). Accordingly, if a tension F is applied between the two ends of a string, the wave speed is given with the formula,

$$v = \sqrt{\frac{F}{\mu}} \quad (\text{Wave speed on a string}) \quad (10.1)$$

(The tension is indicated here as F to prevent confusion with the period T .) The more tautly the string is stretched, the faster the wave speed will be. Wave speed is lower in strings made of denser materials.

- **Speed of sound in air:** If a gas with mass density ρ (kg/m³) has pressure P , the speed of sound waves in this gas is

$$v = \sqrt{\frac{\gamma P}{\rho}} \quad (\text{Speed of sound in air}) \quad (10.2)$$

Here, γ is a dimensionless constant that describes the thermodynamic properties of the gas and is approximately $\gamma = 1.4$ for air.

- **Liquid waves:** Wave speed depends on the wavelength λ and the depth h of the liquid:

$$v = \begin{cases} \sqrt{gh} & (\text{In shallow waters} \quad : \quad h \ll \lambda) \\ \sqrt{\frac{g\lambda}{2\pi}} & (\text{In deep waters} \quad : \quad h \gg \lambda) \end{cases} \quad (10.3)$$

- **Electromagnetic waves:** The speed of electromagnetic waves in vacuum is equal to the speed of light c :

$$v = c = 2.997 \times 10^8 \text{ m/s}$$

The speed of electromagnetic waves decreases when they enter a material medium. Electromagnetic wave speed in a medium with refractive index n is $v = c/n$.

Wave Function

What is the mathematical expression of a propagating wave? Figure 10.5 shows the images of a pulse wave propagating with speed v in the $+x$ direction at two different times. Let the profile of the string be the function $f(x)$ at time $t = 0$. What will be the mathematical expression of this wave at a later time t ?

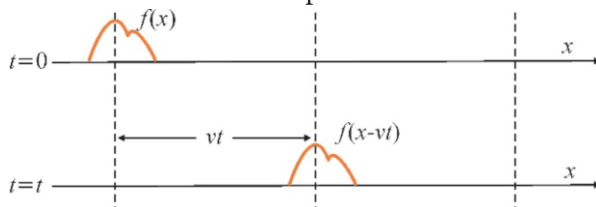


Figure 10.5: The image of a wave propagating at speed v at various times.

Note that the profile given with the function f is only displaced without changing its form. The wave at time $t = 0$ is displaced by a distance $+vt$ at time t . Therefore, it is sufficient to write the same function by displacing it by $+vt$. And this becomes the function $f(x - vt)$. (You may check this with a simple function: The function $y = x$ crosses the x -axis at 0, but the function $y = x - a$ crosses it at $+a$ and the function $y = x - 2a$ crosses it at $+2a$...)

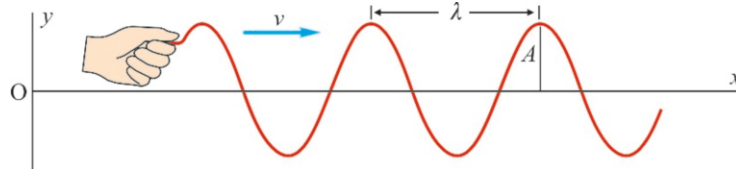
Therefore, every function in the form $f(x - vt)$ represents a wave propagating in the $+x$ direction with speed v .

The reverse is also true: Every function in the form $f(x + vt)$ represents a wave propagating in the $-x$ direction with speed v .

Sinusoidal Wave

When we vibrate the free end of a string in a periodic fashion, a sinusoidal wave is produced that propagates along the string. Figure 10.6 shows the displacement along the y -axis of a string extended along the x -axis. This profile can be shown as a sine or cosine wave. Now let us examine the main parameters of sinusoidal waves.

Figure 10.6: The distance between two consecutive crests is the wavelength λ in a sinusoidal wave.



The distance between two consecutive crests (maximum points) of a wave is called the **wavelength** and is indicated with λ (lambda).

If this wave is generated from a source that vibrates with a period T and the resulting wave propagates with speed v , the distance λ between the two crests can be expressed as follows:

$$\begin{aligned} \text{wavelength} &= \text{speed} \times (\text{duration of one vibration}) \\ \lambda &= vT \end{aligned} \quad (10.4)$$

The maximum displacement of a wave is called the **amplitude** A . Let us write the profile of the string at $t = 0$ as a sine function. This function with wavelength λ and amplitude A is

$$y = A \sin 2\pi \frac{x}{\lambda} \quad (10.5)$$

The factor 2π was added to ensure that the wave is periodic at multiples of λ .

Now, let us turn this string profile at $t = 0$ into a wave that propagates at speed v . We have seen how to do this above: Replacing the function $f(x)$ with $f(x - vt)$ produces a function that moves in the $+x$ direction at speed v . Therefore, a sinusoidal function propagating at speed v can be written as:

$$y = A \sin \frac{2\pi(x - vt)}{\lambda}$$

We can substitute v as $v/\lambda = 1/T$ to make this expression more symmetrical and find the sinusoidal wave expression as follows:

$$y(x, t) = A \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T} \right) \quad (\text{Sinusoidal wave function}) \quad (10.6)$$

The displacement $y(x, t)$ is a function with two variables, depending on both position x and time t . This expression clearly shows that the wave is periodic in both space and time: As $\sin(\alpha + 2\pi) = \sin \alpha$, the displacement y of the positions x and $x + \lambda$ will always be equal; likewise, the displacement at times t and $t + T$ will be equal.

For another form of the wave equation, a quantity called a **wavenumber** and indicated with k is defined as follows:

$$k = \frac{2\pi}{\lambda} \quad (\text{wavenumber}) \quad (10.7)$$

The wavenumber is a measure of the number of wavelengths per unit length, but gives it in terms of 2π . For example, if the wave length is 1 m, $\frac{1}{2}$ m, $\frac{1}{3}$ m, \dots , then the wavenumber will be $k = 2\pi, 4\pi, 6\pi, \dots$.

On the other hand, let us remember the angular frequency ω that we know from Chapter 9:

$$\omega = \frac{2\pi}{T} = 2\pi f \quad (10.8)$$

The sinusoidal wave is written as follows in terms of these two quantities:

$$y(x, t) = A \sin(kx - \omega t) \quad (\text{Sinusoidal wave functions}) \quad (10.9)$$

If the wave is propagating in the $-x$ -direction, we take $A \sin(kx + \omega t)$.

This expression is very convenient in calculations, because the coefficients of x and t can be immediately identified as k and ω . Then, other quantities can be calculated in terms of ω and k as:

$$v = \frac{\omega}{k}, \quad \lambda = \frac{2\pi}{k}, \quad T = \frac{2\pi}{\omega} \quad (10.10)$$

If these wave functions also have a phase angle ϕ , then the equation becomes $y = A \sin(kx - \omega t + \phi)$. We will add the phase angle where necessary.

Example 10.1

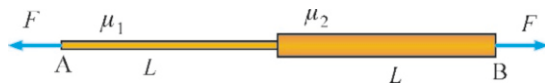
The speed of sound in air is 340 m/s. The note A (La) coming out of a musical instrument has a frequency of 440 Hz. Calculate the wavelength of the note A (La).

Answer

We write the relation between frequency and wavelength:

$$\lambda = vT = \frac{v}{f} = \frac{340}{440} = 0.77 \text{ m} = 77 \text{ cm}.$$

Example 10.2



Two strings, each with length $L=1$ m, are attached end to end. The linear density of the first is $\mu_1=0.1$ kg/m and that of the second is $\mu_2=0.2$ kg/m. The tension $F=40$ N is applied to the outer ends of the strings. How much time does it take for a wave to travel from end A to end B on this composite string?

Answer

We first calculate the wave speed for each string with Eq. (10.1):

$$v_1 = \sqrt{F/\mu_1} = \sqrt{40/0.1} = 20 \text{ m/s}$$

$$v_2 = \sqrt{F/\mu_2} = \sqrt{40/0.2} = 14 \text{ m/s}$$

A wave with speed v travels the distance L in time L/v .

Accordingly, we calculate the total time as follows:

$$t = t_1 + t_2 = \frac{L}{v_1} + \frac{L}{v_2} = \frac{1}{20} + \frac{1}{14} = 0.12 \text{ s}$$

Example 10.3

The sinusoidal wave on a string with linear density $\mu=0.1$ kg/m is given as

$$y(x, t) = 0.03 \sin(5\pi x - 40t) \quad (\text{meters})$$

- What are the wave amplitude, wavelength, period and wave speed?
- What is the tension in the string?

Answer

(a) We can directly identify the following quantities, as the wave is given in the form $A \sin(kx - \omega t)$:

$$A = 0.03 \text{ m}, \quad k = 5\pi \text{ m}^{-1}, \quad \omega = 40 \text{ Hz}$$

The others are found using Eqs. (10.10):

$$v = \frac{\omega}{k} = \frac{40}{5\pi} = 2.5 \text{ m/s}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{5\pi} = 0.4 \text{ m}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{40} = 0.05\pi \text{ s}$$

(b) We use Eq. (10.1) for the wave speed on the string:

$$v = \sqrt{F/\mu} \rightarrow F = \mu v^2$$

We substitute the numerical values:

$$F = 0.1 \times (2.5)^2 = 0.63 \text{ N.}$$

Example 10.4

Determine the direction of propagation and wave speed for the following wave functions.

- (a) $y = 2 \sin(3x - 18t)$
 (b) $y = 3 \cos(4t - 10x)$
 (c) $y = 4 \sin(3x + 21t)$

Answer

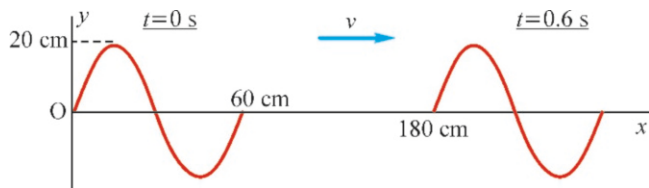
If a function has the form $f(x - vt)$, then it is a wave propagating in the $+x$ -direction with speed v . However, if its form

is $f(x + vt)$, then it propagates in the $-x$ -direction.

We then convert the given functions into this form (without considering the amplitudes or the signs):

- (a) $\sin(3x - 18t) = \sin[3(x - 6t)]$
 $\rightarrow v = 6 \text{ m/s, in the } +x \text{ direction}$
 (b) $\cos(4t - 10x) = \cos[-(4t - 10x)] = \cos[10(x - \frac{2}{5}t)]$
 $\rightarrow v = \frac{2}{5} \text{ m/s, in the } +x \text{ direction}$
 (c) $\sin(3x + 21t) = \sin[3(x + 7t)]$
 $\rightarrow v = 7 \text{ m/s, in the } -x \text{ direction}$

Example 10.5



The figure shows the profile of a sinusoidal wave on a string at times $t = 0$ and $t = 0.6 \text{ s}$. Find the wave function.

Answer

The amplitude and wavelength can be immediately identified by examining the figure:

$$A = 0.20 \text{ m, } \lambda = 0.60 \text{ m}$$

To find the wave speed, let us note that the point at the origin at time $t = 0$ was displaced by 1.80 m at time $t = 0.6 \text{ s}$. Accordingly, we calculate the wave speed as follows:

$$v = \frac{1.80}{0.6} = 3 \text{ m/s}$$

From here, we calculate the period:

$$T = \frac{\lambda}{v} = \frac{0.6}{3} = 0.2 \text{ s.}$$

We find the wave function based on this information:

$$y = A \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T} \right) = 0.2 \sin 2\pi \left(\frac{x}{0.6} - \frac{t}{0.2} \right)$$

Example 10.6

Two ends of a string with linear density $\mu = 0.25 \text{ kg/m}$ are stretched with a pair of forces $F = 100 \text{ N}$. The string is transversely pulled from one end by 3 cm and released. It vibrates with a frequency of 8 Hz.

- (a) Find the wave speed, period and wavelength.
 (b) Write the equation of the sinusoidal wave produced.

Answer

(a) We use Eq.(10.1) for speed:

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{100}{0.25}} = 20 \text{ m/s.}$$

We calculate the period and the wavelength:

$$T = \frac{1}{f} = \frac{1}{8} = 0.125 \text{ s}$$

$$\lambda = vT = 20 \times 0.125 = 2.5 \text{ m.}$$

(b) We use Eq. (10.6), which expresses the wave function in terms of T and λ :

$$y = A \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T} \right) = 0.03 \sin 2\pi \left(\frac{x}{2.5} - \frac{t}{0.125} \right)$$

10.2 INTERFERENCE, REFLECTION AND TRANSMISSION OF WAVES

Superposition Principle

What will be the combined effect of two or more waves in the same medium? For example, the sounds from many musical instruments in an orchestra arrive at our ears simultaneously. Likewise, electromagnetic waves from many TV stations affect the television antenna simultaneously.

The superposition principle determines the combined effect of waves:

Superposition Principle

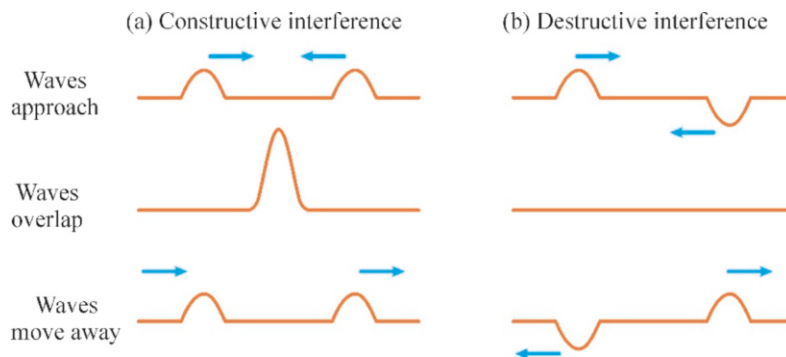
The total effect of two or more waves propagating in the same medium is equal to the algebraic sum of their wave functions:

$$y(x, t) = y_1(x, t) + y_2(x, t) \quad (10.11)$$

We can explain the source of this principle in mathematical terms as follows: Functions in the form $f(x-vt)$ satisfy a differential equation that has the following property: If y_1 and y_2 are two separate solutions, then the function $y_1 + y_2$ is also a new solution. Experiments show that the principle is true for waves with small amplitudes.

Let us emphasize the most important conclusions to be drawn from the superposition principle:

- If two waves reach a point with the same sign, the displacement of that point increases. This is called **constructive interference** (Figure 10.7a).



- If the algebraic sum of two waves at a point is zero, then that point will remain motionless. This is called **destructive interference** (Figure 10.7b).
- Waves continue along their paths with their former shapes after overlapping and separating.

Interference

Let us examine in detail the constructive and destructive interference that we mentioned above. Consider two sinusoidal waves propagating in the same medium and in the same direction (Figure 10.8). In the simplest case, let the amplitudes, wavelengths and frequencies of both waves be equal. Let the only difference between them be a phase difference ϕ on one of the waves:

$$y_1 = A \sin(kx - \omega t)$$

$$y_2 = A \sin(kx - \omega t + \phi)$$

According to the superposition principle, the total wave in the medium will be the algebraic sum of these two:

$$y = y_1 + y_2 = A \sin(kx - \omega t) + A \sin(kx - \omega t + \phi)$$

If we apply an identity from trigonometry on this sum, we get

$$\sin a + \sin b = 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$$

$$\sin(kx - \omega t) + \sin(kx - \omega t + \phi) = 2 \sin(kx - \omega t + \phi/2) \cos(-\phi/2)$$

Figure 10.7: (a) Constructive interference and (b) Destructive interference in the superposition of two waves.

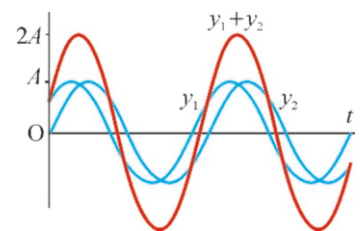


Figure 10.8: The interference of two identical waves with a small phase difference.

Since $\cos(-a) = \cos a$, the total wave function can ultimately be written as follows:

$$y = \underbrace{[2A \cos(\phi/2)]}_{A'} \sin(kx - \omega t + \phi/2) \quad (10.12)$$

The term $\sin(kx - \omega t + \phi/2)$ here shows that this expression is again a wave propagating in the $+x$ direction with the same wavelength and same frequency, but with a phase difference of $\phi/2$. The factor $2A \cos \phi/2$ at the front of this expression is independent of x and t and is considered the amplitude of the new wave:

$$y = A' \sin(kx - \omega t + \phi/2) \quad \text{with} \quad A' = 2A \cos(\phi/2) \quad (10.13)$$

Let us consider two special cases of the phase angle:

- If $\phi = 0^\circ$, in other words, if the two waves are in equal phase, we get $\cos 0^\circ = 1$. In this case, the wave function y becomes the same function with amplitude $2A$. This is called **constructive interference**.
- If $\phi = 180^\circ$, in other words, if the two waves are in opposite phase, we get $\cos 90^\circ = 0$. In this case, we get $y = 0$ for every value of x and t . And this is called **destructive interference**.

We have considered here the interference of two identical waves as the simplest case. In more general cases, interference between two waves with different wavelengths and frequencies exhibits more interesting features, as the following example shows.

Standing Waves

Another important case of interference is that of two waves propagating in opposite directions. Let us consider two waves with equal amplitude, equal wavelength and equal frequency, one traveling in the $+x$ -direction and the other in the $-x$ -direction:

$$y_1 = A \sin(kx - \omega t) \quad (\text{wave traveling in the } +x\text{-direction})$$

$$y_2 = A \sin(kx + \omega t) \quad (\text{wave traveling in the } -x\text{-direction})$$

According to the superposition principle, the total wave function will be

$$y = y_1 + y_2 = A \sin(kx - \omega t) + A \sin(kx + \omega t)$$

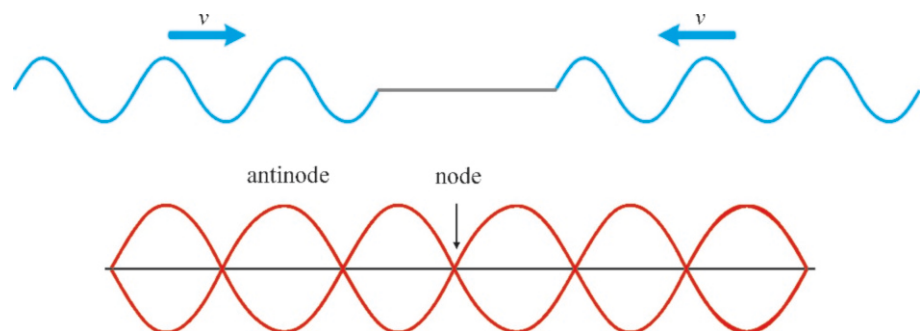


Figure 10.9: Standing wave produced by two waves propagating in opposite directions.

If we again use the trigonometric identity above for $\sin a + \sin b$,

$$y = 2A \sin kx \sin \omega t \quad (10.14)$$

This result shown in Figure 10.9 is called a **standing wave**. Let us emphasize the important features of standing waves:

- This wave is no longer a propagating wave, because it does not have the form $f(x \pm vt)$. If you examine the wave produced on a guitar string, you will notice that it does not propagate (Figure 10.10). Each point makes a vibration motion with a different amplitude. This is the vibration motion $\cos \omega t$ with amplitude $A' = 2A \sin kx$:

$$y = \underbrace{[2A \sin kx]}_{A'} \sin \omega t$$

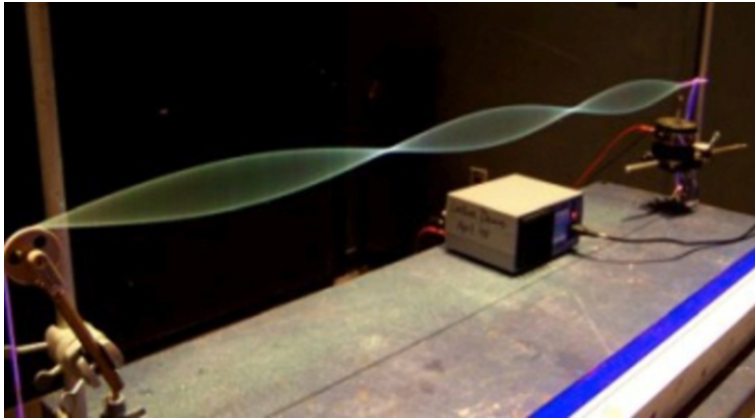


Figure 10.10: A standing wave produced in laboratory (Harvard Natural Sciences Lecture Demonstrations).

- The points at certain x values do not move. These points, called **nodes**, occur at the values $\sin kx = 0$:

$$\sin kx = 0 \quad \longrightarrow \quad kx = 0, \pi, 2\pi, 3\pi \dots$$

Using the definition $k = 2\pi/\lambda$ for the wavenumber, we get

$$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2} \dots = n \frac{\lambda}{2} \quad (n = 0, 1, 2, \dots) \quad (\text{Nodes}) \quad (10.15)$$

- The points at certain x values vibrate at maximum amplitude. These points, called **antinodes**, occur at the values $\sin kx = 1$:

$$\sin kx = 1 \quad \longrightarrow \quad kx = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \dots = (2n + 1) \frac{\lambda}{4} \quad (n = 0, 1, 2, \dots) \quad (\text{Antinodes}) \quad (10.16)$$

Harmonics of a Musical Instrument

Waves are generated on stringed instruments like the violin or the guitar, upon which a string has two fixed ends. According to our results above, the two fixed ends of such a string must be a node. Therefore, on a string with length L , the points $x = 0$ and $x = L$ should fulfill the following condition:

$$\sin k \cdot 0 = \sin kL = 0 \quad \Longrightarrow \quad kL = 0, \pi, 2\pi, 3\pi \dots$$

Writing this for wavelength $\lambda = 2\pi/k$ we have,

$$L = n \frac{\lambda}{2} \quad \Longrightarrow \quad \lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3 \dots)$$

The length L of the string should accommodate half-wavelengths. According to this result, only waves with wavelength λ_n can be generated in a string with two fixed ends (Figure 10.11).

We use the formula $f = v/\lambda$ to convert these into frequencies:

$$f_n = n \frac{v}{2L} \quad (n = 1, 2, 3 \dots) \quad (10.17)$$

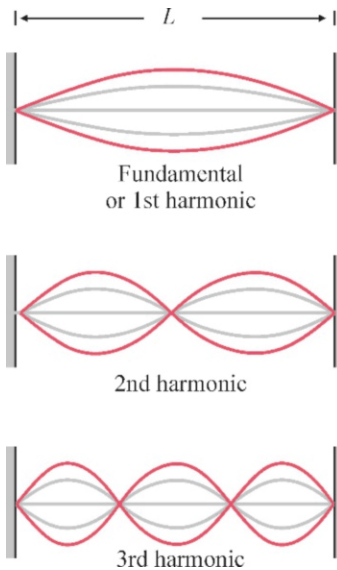


Figure 10.11: Natural frequencies or harmonics on a string of length L .

These frequencies that can be generated by a string with a certain length are called **natural frequencies** or resonance frequencies. In musical terms, they are called **harmonics**. The one with $n = 1$ is called the **fundamental frequency** or the **first harmonic**; and $f_n = 2f_1, 3f_1 \dots$ are called the n^{th} **harmonics**:

$$f_1 = \frac{v}{2L} \quad (\text{The fundamental frequency or the first harmonic}) \quad (10.18)$$

Here, let us also use Eq. (10.1) $v = \sqrt{F/\mu}$, which we previously gave for the wave speed on a string:

$$f_n = n f_1 = n \frac{v}{2L} = n \frac{1}{2L} \sqrt{\frac{F}{\mu}} \quad (n = 1, 2, 3 \dots) \quad (10.19)$$

When “tuning” a musical instrument, we vary the tension in the strings to ensure that it generates standing waves at the required frequency.

Timbre: The property that distinguishes the sound of a violin from that of a guitar is called *timbre*. Timbre is a result of mixing harmonics. In each instrument, together with the fundamental frequency, a few more of these harmonics are generated. The mixing rates of these harmonics depends on the type of musical instrument. A different harmonic mixture is generated in the violin than is generated in the guitar.

Reflection and Transmission of Waves

What happens when a wave encounters an obstacle or reaches the boundary between two mediums? If the second medium is also flexible, some of the wave will be transmitted and some will be reflected back. Let us review the general rules of reflection and transmission before going into their detailed calculation.

Figure 10.12 shows a wave on a string with one end fixed to the wall, with the wave moving towards the wall. When the wave reaches the wall, it will be fully reflected back, because it cannot pass to the other side. However, the point A of the string connected to the wall must always remain motionless, due to the superposition principle. The only way to ensure this is for the reflected wave to have opposite phase with the incoming wave. It means that the reflected wave is inverted.

Figure 10.12: A wave reflected from a fixed end will have a 180° opposite phase.



Therefore, *a wave reflected from a fixed end will be inverted*. Another way to see this is as follows: When the wave hits the wall, according to Newton’s law, the wall will push the string back with an equal and opposite force. In other words, the wave will be pushed downwards if it tries to get displaced upwards.



Figure 10.13: A wave reflected from a free end will have the same phase.

On the other hand, the end of the string could be free to move. For example, we can consider a ring that can slide freely up or down the wall (Figure 10.13). In this case, the reflection will be in the same phase, because this end of the string is displaced in the same direction for the waves in both directions.

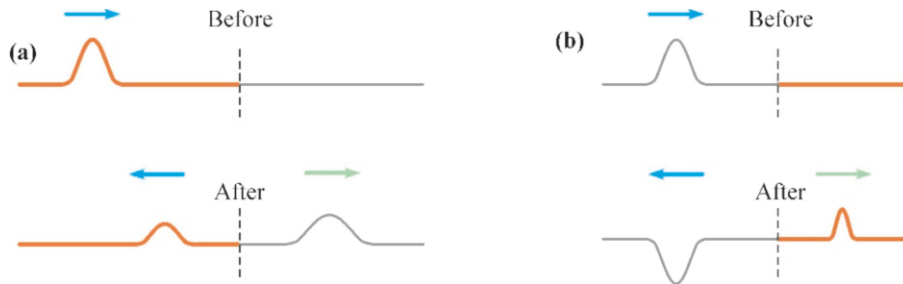


Figure 10.14: (a) A wave coming from a denser medium will be reflected in the same phase. (b) A wave trying to enter a denser medium will be reflected in the opposite phase.

Apart from these two extreme cases, a wave is usually partially transmitted into a second flexible medium and partially reflected. Whether or not there is a phase difference during the reflection depends on the densities of the two mediums. It gets reflected back without phase difference if it comes from a denser medium. In contrast, the reflected wave will be inverted if transmitting into a denser medium. Both cases are summarized in Figure 10.14.

Example 10.7

Two waves propagating on a string are given as

$$y_1 = 3 \sin(5\pi x - 4t)$$

$$y_2 = 3 \sin(5\pi x + 4t)$$

- (a) Determine the nodes and antinodes of the standing wave.
- (b) Calculate the amplitude at the point $x = 0.05$ m.

Answer

(a) The standing wave generated by the waves $A \sin(kx \pm \omega t)$ propagating in opposite directions was given by Eq. (10.14):
 $y = 2A \sin kx \cos \omega t$

Therefore, the values $k = 5\pi$ and $\omega = 4$ are read from this expression.

We first calculate the wavelength to find the nodes and antinodes:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{5\pi} = 0.4 \text{ m.}$$

The nodes will be located at the integer multiples of $\lambda/2$ and antinodes at the odd multiples of $\lambda/4$:

- Nodes: $x = 0, \lambda/2, \lambda, 3\lambda/2 \dots = 0, 0.2, 0.4 \dots$ m
- Antinodes: $x = \lambda/4, 3\lambda/4, 5\lambda/4 \dots = 0.1, 0.3, 0.5 \dots$ m

(b) The amplitude of the vibrational motion of any position x will be the coefficient of $\cos \omega t$:

$$A' = 2A \sin kx = 6 \sin 5\pi x$$

We find the amplitude by substituting the value $x=0.05$:

$$A' = 6 \sin 5\pi \times 0.05 = 6 \sin \pi/4 = 6/\sqrt{2} = 4.2 \text{ m.}$$

Example 10.8

A tension of $F = 100$ N is applied on both ends of a string with length 1 m and linear density 0.1 kg/m. What are the fundamental frequency (1st harmonic) and 2nd harmonic frequency generated on this string?

Answer

Let us first calculate the wave speed on the string:

$$v = \sqrt{F/\mu} = \sqrt{100/0.1} = 33 \text{ m/s}$$

We had previously found the formulas for the fundamental and harmonic frequencies. However, it is more convenient to keep in mind the wavelength condition, rather than memorize these formulas.

The fundamental frequency should be such that a half-

wavelength can fit in the string:

$$L = \frac{\lambda_1}{2} \rightarrow \lambda_1 = 2L$$

Frequency can be calculated if the wavelength and speed v are known:

$$f_1 = \frac{v}{\lambda_1} = \frac{33.3}{2} = 17 \text{ Hz}$$

Likewise, two half-wavelengths should fit in the length L to obtain the 2nd harmonic:

$$L = 2 \frac{\lambda_2}{2} \rightarrow \lambda_2 = L$$

From here, we calculate the frequency:

$$f_2 = \frac{v}{\lambda_2} = \frac{33.3}{1} = 33 \text{ Hz}$$

Example 10.9

The wave speed on a string with length 80 cm is 240 m/s.

- (a) What is the fundamental frequency?
 (b) Considering that the human ear can hear sounds with a maximum frequency of 20 000 Hz, what is the highest harmonic that can be heard on this string?

Answer

(a) We use the formula (10.17), which gives the harmonic frequencies:

$$f_n = n \frac{v}{2L} \quad (n = 1, 2, 3 \dots)$$

We take $n=1$ for the fundamental frequency:

$$f_1 = \frac{v}{2L} = \frac{240}{2 \times 0.80} = 150 \text{ Hz.}$$

(b) We write the harmonic frequencies in terms of the fundamental frequency:

$$f_n = n f_1$$

In order to be audible by the human ear, the frequency f_n should be as close as possible to the value 20 000 Hz:

$$n f_1 \leq 20\,000 \rightarrow n \leq \frac{20\,000}{150} = 133.3$$

The closest value to this is $n = 133$:

$$f_{133} = 133 \times 150 = 19\,950 \text{ Hz.}$$

Example 10.10

A guitar string with linear density $\mu = 0.001 \text{ kg/m}$ and length 60 cm was tuned so that its fundamental frequency produces the note C (Do, frequency 262 Hz).

- (a) What is the tension in the string?
 (b) Where should the string be pressed so that it can produce the note D (Re, frequency 294 Hz)?

Answer

We first calculate the wave speed:

$$v = \lambda f$$

The wavelength of the fundamental frequency is such that $\lambda = 2L$. We can calculate the wave speed from this:

$$v = 2L f = 2 \times 0.60 \times 262 = 314 \text{ m/s}$$

We calculate the tension F giving this speed:

$$v = \sqrt{F/\mu} \rightarrow F = \mu v^2$$

$$F = 0.001 \times 314^2 = 99 \text{ N}$$

(b) The wave speed v is the same, because the tension of the string has not changed. However, this time, the fundamental frequency is changed to f' , because the length of the string is changed to L' :

$$v = \lambda' f' = 2L' f' \rightarrow L' = \frac{v}{2f'}$$

We use the frequency of the note D and the wave speed found in item (a):

$$L' = \frac{314}{2 \times 294} = 0.53 \text{ m.}$$

Accordingly, one must press at $60 - 53 = 7 \text{ cm}$ from one end of the string.

10.3 THE DOPPLER EFFECT AND SHOCK WAVES

In this section, we will discuss two interesting effects caused by waves: The Doppler effect is observed when the wave source or the observer is moving. A shock wave is observed when the wave source moves faster than the wave.

Both effects have very significant technological applications.

The Doppler Effect

Have you ever stood by a highway and listened to the sound of a car passing by at top speed? The sound that you hear as the car approaches and the one that you hear as it moves away have a distinguishable difference to the ear. The sound has a higher pitch as the car approaches and a lower pitch as it moves away. Expressed in terms of frequencies, the frequency of the approaching sound is greater and the frequency of the receding sound is lesser. This is called the Doppler effect.

The Doppler effect is observed when both the wave source and the observer are moving. Let us first calculate both cases separately, then combine them together.

Standing source, moving observer

Let us clearly designate our parameters: v is the speed of sound. f_s and λ_s are the original frequency and wavelength emitted when the source is at rest. v_s and v_o are the speeds of the source and the observer, respectively.

As seen in Figure 10.15, a source at rest is emitting sound at frequency f_s and an observer approaching the source is traveling at speed v_o .

Let us remember the relation between the frequency f_s and wavelength λ_s emitted by a source at rest:

$$f_s = \frac{v}{\lambda_s}$$

In other words, the distance between two consecutive wavefronts is λ_s . The speed of this wave with respect to the ground is v , whereas the observer running at speed v_o observes the wave to be approaching at speed $v + v_o$. In this case, the frequency heard by the observer is

$$f_o = \frac{v + v_o}{\lambda_s}$$

If we eliminate the wavelength λ_s between these two equations and rearrange the terms, we get

$$f_o = \frac{v + v_o}{v} f_s \tag{10.20}$$

The observer will hear the sound at a higher frequency.

If the observer is moving away from the source, we take the velocity of the observer as negative. In this case, $v + v_o$ will be smaller and the sound heard by the observer will have a lower frequency.

Moving source, stationary observer

As seen in Figure 10.16b, let the source travel with a speed v_s and approach a stationary observer. If the emitted sound has frequency f_s and period $T_s = 1/f_s$ when the source is stationary, the wavelength of the sound emitted from the stationary source will be

$$\lambda_s = v T_s = \frac{v}{f_s}$$

Now, let us consider that the first of the two consecutive wavefronts is emitted. The source will have traveled a distance of $v_s T_s = v_s / f_s$ during the time that it takes for the second wavefront to be emitted. Therefore, the new distance between two consecutive fronts, in other words, the wavelength λ_o measured by the observer, will be,

$$\lambda_o = \lambda_s - v_s T_s = \lambda_s - \frac{v_s}{f_s}$$

The stationary observer will perceive the frequency of the sound approaching with wavelength λ_o , as follows:

$$f_o = \frac{v}{\lambda_o}$$

If we substitute the value of λ_o and rearrange, we find that:

$$f_o = \frac{v}{v - v_s} f_s \tag{10.21}$$

The observer will again observe the sound at a higher frequency. If the source is moving away from the observer, we will take the value v_s as negative in the formula. In that case, the sound heard by the observer will have a lower frequency.

Let us note one point here: The respective speeds of the source and observer do not appear as symmetrical in these formulas. In other words, the Doppler

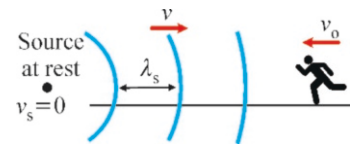


Figure 10.15: Sound source at rest and observer moving towards the source at speed v_o .

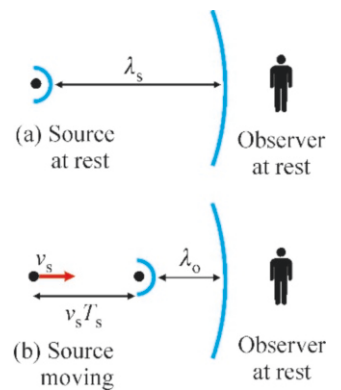


Figure 10.16: (a) The source at rest generates sound with wavelength λ_s . (b) The distance between two fronts (wavelength λ_o) decreases when the source is moving.

effect gives different results depending on whether the object or the source is approaching with the same speed. The reason for this is that the wavelength of the sound propagating in the air is different in both cases. Relative velocity does not change this fact.

Both source and observer moving

We can easily calculate this case by using the two formulas found above for only one of the observer or the source moving.

The frequency f_o heard by a stationary observer from a moving source was given by Eq. (10.21). This f_o value replaces f_s in Eq. (10.20), which we found for a moving observer. Therefore, if we apply both formulas successively, we get

$$f_o = \frac{v + v_o}{v} \frac{v}{v - v_s} f_s$$

$$f_o = \frac{v + v_o}{v - v_s} f_s \quad (\text{Doppler formula}) \quad (10.22)$$

Note the signs of the velocities v_s and v_o when using this formula: *Velocities in the approaching direction are taken as positive for both the source and the observer.* In the “away” direction, they are taken as negative.

The Doppler formula for electromagnetic waves is slightly different from this. This is because the relative velocity addition formula of the speed of light is different from that of classical physics.

The Doppler effect has a wide range of applications in science and technology. The most important one is determining the velocities of celestial bodies. The approaching or receding speed of a star moving relative to the Earth can be calculated by measuring the change in the frequency of a known color in the light emitted by that star. Measurements indeed show that stars mostly move away from each other. This means that the universe is expanding.

Another application is the device known as “radar” which is used by highway patrols to monitor the speed of vehicles. In this device, an electromagnetic wave (microwave) sent forward is reflected from a moving vehicle back into the device. The speed of the vehicle can be calculated from the difference between the frequencies of the outgoing and incoming waves.



Figure 10.17: The speed control device called *radar*, operates with the Doppler effect in traffic.

Shock Wave

What happens if a wave source travels faster than the wave it generates? We can again answer this question with wavefronts. Figure 10.18 shows the wavefronts generated at various times by a source traveling at speed v_s . These are spherical surfaces in three-dimensional space. If $v_s < v$, in other words, if the source velocity is less than the wave velocity (Figure 10.18a), the distance between consecutive fronts will decrease in the direction of motion and increase in the rear direction, as seen in the figure. In this case, the Doppler effect will be observed.

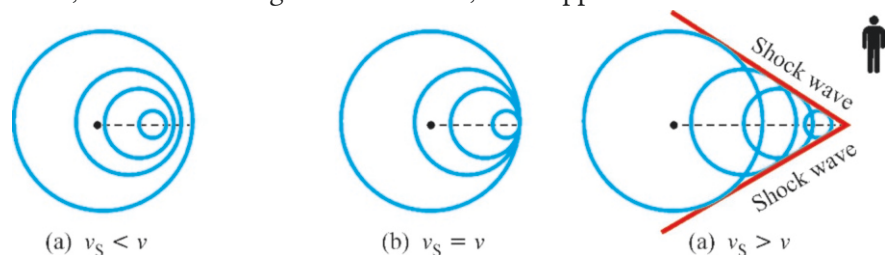


Figure 10.18: Wavefronts formed when the source velocity v_s is (a) less than, (b) equal to or (c) greater than the wave velocity v .

However, if $v_s > v$, in other words, if the source velocity is greater than the wave velocity (Figure 10.18c), the source moves by leaving behind the wavefronts that it previously emitted. Let us examine the surface of the cone tangent to all of these wavefronts. When this cone reaches a stationary observer, the observer will hear the sum total of the magnitudes of each wavefronts at once. He/she will hear a booming explosion (sonic boom). Thus, we have what is called a **shock wave**.

When modern fighter planes fly faster than the speed of sound, this conic wavefront produces a sound that resembles an explosion everywhere it passes. It is said that the “plane broke through the sound barrier.” This is actually incorrect, because a shock wave is not an instantaneous phenomena. The plane is already flying faster than the speed of sound, but the conic wavefront passes through an observer only once. As it applies the energy of all of the wavefronts at once, its impact can be large enough to break window glasses.

Shock waves are used to break kidney stones in modern medicine. Sound waves at frequencies called *ultrasound*, which are sent from outside of the body transfer their energy into dense stones and break them into small parts.



Figure 10.19: The shock wave formed when a fighter plane exceeds the sound barrier causes the air to concentrate where it passes, and a conic wavefront is observed.

Example 10.11

The siren of an ambulance can emit sound at the 440 Hz frequency. The ambulance is approaching a junction at a speed of 144 km/hour.

- What frequency is heard by an observer standing at the junction?
- What frequency is heard by the observer when the ambulance is moving away? (The speed of sound in air is $v = 340$ m/s.)

Answer

(a) We calculate f_o , the frequency heard by the observer, by using Eq. (10.22), which we found for the Doppler effect:

$$f_o = \frac{v + v_o}{v - v_s} f_s$$

The velocity of the stationary observer will be $v_o = 0$. We convert the velocity v_s of the ambulance into m/s units:

$$v_s = \frac{144 \times 1000}{3600} = 40 \text{ m/s}$$

The velocity v_s of the ambulance (source) is taken as positive when it is approaching. Accordingly, we calculate the frequency heard by the observer:

$$f_o = \frac{340 + 0}{340 - 40} \times 440 = 499 \text{ Hz}$$

(b) We take the velocity v_s as negative as the ambulance moves away:

$$f = \frac{340 + 0}{340 + 40} \times 440 = 394 \text{ Hz}.$$

Example 10.12

An orchestra in an open-top train car is playing the note A (La) (frequency 440 Hz) as it travels.

- An observer sitting in a stationary car beside the railroad tracks hears this sound as the note B-flat (Si-bemol) (frequency 466 Hz). Calculate the speed of the train.
- In what direction and at what speed should the observer drive his/her car to hear this sound as the note B (Si) (frequency 494 Hz)?

Answer

(a) We set $v_o = 0$ in The Doppler formula (Eq. 10.22) as the observer is stationary:

$$f_o = \frac{v}{v - v_s} f_s$$

We find the source velocity by substituting the frequencies

and the speed of sound:

$$466 = \frac{340}{340 - v_s} \times 440 \rightarrow v_s = 19 \text{ m/s}.$$

(b) We use the Doppler formula in which both the source and the observer are moving:

$$f_o = \frac{v + v_o}{v - v_s} f_s$$

The source velocity was found in item (a). We find the observer velocity by also substituting the frequencies:

$$494 = \frac{340 + v_o}{340 - 19} 440 \rightarrow v_o = +20 \text{ m/s}$$

The observer should drive the car towards the train, because the velocity is positive.

(Note: The experimental proof of the Doppler effect was provided by a similar orchestra playing on a train in 1845.)

Multiple-choice Questions

- The speed of a wave on a string is not dependent on which of the following?
 - The force stretching the string.
 - The density of the string.
 - The length of the string.
 - None of the above.
- A string is vibrated from one end. Which of the following is incorrect if the vibration frequency is doubled?
 - The wave speed will double.
 - The angular frequency will double.
 - The period will halve.
 - The wavelength will halve.
- Two strings of equal lengths are stretched with equal forces. Which of the following is correct?
 - The one with higher density will have a higher wave speed.
 - The one with the lower density will have a lower wave speed.
 - The one with the higher density will have a lower wave speed.
 - Their wave speeds will be equal.
- How would you hear the frequency of a sound source approaching you?
 - At a higher frequency.
 - At a lower frequency.
 - At the same frequency.
 - At double the frequency.
- Which is correct if the frequency of a wave is doubled on a string with constant tension?
 - The velocity will double.
 - The wavelength will double.
 - The wavelength will halve.
 - The period will double.
- Which is correct when two waves reach the same point at the same time?
 - Destructive interference will occur if they arrive with same phase.
 - Constructive interference will occur if they arrive with opposite phases.
 - Destructive interference will occur if they arrive with opposite phases.
 - There will be no interference.
- Which of the following is correct?
 - Sound waves are transverse waves.
 - Water waves are longitudinal waves.
 - Sound waves are longitudinal waves.
 - The wave on a string is a longitudinal wave.
- When will a shock wave occur?
 - If the source velocity is less than the speed of sound.
 - If the source velocity is equal to the speed of sound.
 - If the source velocity is greater than the speed of sound.
 - If the amplitude of the sound is high.
- Which is incorrect for two waves propagating on the same string?
 - Their frequencies may be different.
 - Their velocities may be different.
 - Their wavelengths may be different.
 - Their amplitudes may be different.
- The tension in a string is increased by a factor of 4. What will happen to the wave speed?
 - It will decrease by a factor of 4.
 - It will be halved.
 - It will be doubled.
 - It will increase by a factor of 4.
- The density of a string is increased by a factor of 4. What will happen to the wave speed?
 - It will decrease by a factor of 4.
 - It will be halved.
 - It will be doubled.
 - It will increase by a factor of 4.
- Two strings with different densities are attached end to end. Which will remain constant when a periodic wave is transmitted from one string to the other?
 - Speed
 - Wavelength
 - Frequency
 - Amplitude
- Which is incorrect when a wave propagating on a light string reaches a dense string?
 - It will be reflected with the same phase.
 - It will be reflected with the opposite phase.
 - It will be transmitted with the same phase.
 - It is impossible to tell.
- Which is correct for two waves propagating towards each other on the same string?
 - They will collide like billiard balls.
 - They will pass through each other.
 - They will merge and become a single wave.
 - It is impossible to tell.

15. Which of the following is a wave propagating in the $-x$ -direction?
 (a) $\sin 3x \cos 5t$
 (b) $\sin(3x - 5t)$
 (c) $\sin(3x + 5t)$
 (d) $\cos(3t - 5x)$
16. Which is a standing wave?
 (a) $\sin 3x \cos 5t$
 (b) $\sin(3x - 5t)$
 (c) $\sin(3x + 5t)$
 (d) $\cos(3t - 5x)$
17. A wave propagating on a string does not transmit which of the following?
 (a) Material (b) Energy (c) Momentum (d) Work
18. Which of the following is correct when there is interference between two waves?
 (a) They exchange energy.
 (b) They will affect each other's propagation.
 (c) They lose energy during the interference.
 (d) They are added algebraically.
19. Which wave property distinguishes the sound of a violin from that of a guitar?
 (a) Amplitude
 (b) Frequency
 (c) Wavelength
 (d) Harmonics
20. Which of the following situations will decrease the frequency heard by the observer due to the Doppler effect?
 (a) When the observer approaches the source.
 (b) When the source approaches the observer.
 (c) When the source and observer approach each other simultaneously.
 (d) When the source moves away from the observer.

Problems

(The speed of sound in air is to be used as 340 m/s wherever necessary.)

10.1 General Properties of Waves

10.1 The note C (Do) coming out of a musical instrument has a frequency of 262 Hz. Calculate its wavelength.
 [A: 1.3 m.]

10.2 The range of visible light, which is a type of electromagnetic wave, is between red light at the frequency 4.3×10^{14} Hz and violet light at frequency 7.5×10^{14} Hz. Taking the speed of light as $c = 3 \times 10^8$ m/s, calculate the wavelength range of visible light in units of nanometers (nm).
 [A: 400 – 700 nm.]

10.3 Sound waves above 20 kHz, which is the highest frequency audible by the human ear, are called *ultrasound*. They are used to obtain images by passing them through human skin, after which they are reflected by the internal organs. The speed of ultrasound in the body is 1540 m/s. The wavelength should be around 1 mm for high-quality imaging. What should the frequency of the ultrasound be that gives 1 mm wavelength inside of the body?
 [A: 1.54 MHz.]

10.4 Wave speed is 30 m/s when a tension of 9 N is applied between two ends of a string. How much tension should be exerted to obtain a wave speed of 40 m/s?
 [A: 16 N.]

10.5 Two strings with linear densities $\mu_1 = 0.05$ kg/m and $\mu_2 = 0.2$ kg/m and with equal length $L = 1.20$ m are attached

end to end. If a tension of $F = 50$ N is applied from the two outer ends, how much time will it take for a wave to travel from one end to the other?
 [A: 0.11 s.]

10.6 Tsunami waves in oceans have very long wavelengths. The wavelength of a tsunami wave that occurred after an earthquake in Japan was observed to be 200 km. This wave was observed to reach the coasts of Australia, which are 7000 km away, in 9 hours. (a) What is the wave speed? (b) As the wavelength is very large with respect to the depth of the ocean, what is the average ocean depth between Japan and Australia?
 [A: (a) 216 m/s, (b) 4.7 km.]

10.7 The sinusoidal wave on a string with linear density $\mu = 0.2$ kg/m is given as

$$y(x, t) = 0.7 \sin(0.4\pi x - 12t) \quad (\text{meters})$$

(a) What are the wave amplitude, wavelength, period and wave speed? (b) What is the tension on the string?
 [A: (a) $A = 0.7$ m, $\lambda = 5$ m, $T = \pi/6$ s, $v = 9.5$ m/s, (b) $T = 18$ N.]

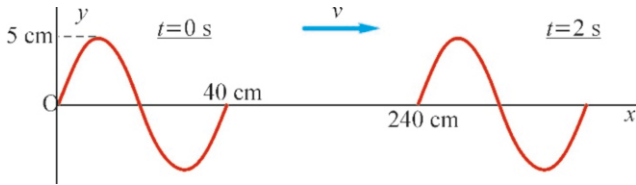
10.8 Determine the direction of propagation and speed of the wave for the following wave functions:

- (a) $y = 2 \sin(2x - 7t)$
 (b) $y = 3 \cos(3t - 8x)$
 (c) $y = 4 \cos(7x + 21t)$

[A: (a) 3.5 m/s in the $+x$ -direction, (b) 0.38 m/s in the $+x$ -direction, (c) 3 m/s in the $-x$ -direction.]

10.9 Two ends of a string with linear density $\mu=0.5$ kg are stretched with a force $F=8$ N. When the string is transversely pulled from one end by 5 cm and released, it oscillates with a frequency of 20 Hz. (a) Find the wave speed, period and wavelength. (b) Write the equation of the sinusoidal wave that is produced.

[A: (a) 4 m/s, 0.05 s, 0.2 m, (b) $y = 0.05 \sin(10\pi x - 40\pi t)$.]



Problem 10.10

10.10 The figure shows the shape of a sinusoidal wave on a string at times $t = 0$ and $t = 2$ s. Write the expression of the wave function. [A: $y = 0.05 \sin(5\pi x - 6\pi t)$.]

10.11 The distance between two successive maximum points of water waves in a pool is measured as 1.5 m. 10 maximum points pass by an observer in 5 s. (a) What is the wave speed? (b) Considering that the pool is shallow, what is the average depth of the pool? [A: (a) 3 m/s, (b) 0.9 m.]

10.2 Interference, Reflection and Transmission of Waves

10.12 Two waves propagating on a string are given as

$$y_1 = 2 \sin\left(\frac{2\pi}{3}x - 7t\right)$$

$$y_2 = 2 \sin\left(\frac{2\pi}{3}x + 7t\right)$$

(a) Determine the nodes and antinodes of the standing wave produced. (b) Calculate the vibration amplitude of the point $x=0.25$ m. [A: (a) Nodes: $x = 0, 0.75, 1.5, 2.25 \dots$ m, Antinodes: $x = 0.38, 1.13, 1.88 \dots$ m. (b) $A' = 3.5$ m.]

10.13 A tension $F=90$ N is applied to both ends of a string with length 1.20 m and linear density 0.1 kg/m. What are the fundamental frequency (1st harmonic) and 3rd harmonic frequency generated on this string? [A: 12.5 Hz, 37.5 Hz.]

10.14 The wave speed on a string with length 40 cm is 360 m/s. (a) What is the fundamental frequency? (b) Considering that the human ear can hear sounds with a maximum

frequency of 20 000 Hz, what is the highest harmonic that can be heard on this string? [A: (a) 450 Hz, (b) $n = 44$.]

10.15 On a string with length 40 cm, the frequencies of two successive harmonics are 440 Hz and 500 Hz. (a) What is the fundamental frequency? (b) What is the wave speed on the string? [A: (a) 60 Hz, (b) 48 m/s.]

10.16 A guitar string with linear density $\mu=0.001$ kg/m and length 30 cm is tuned such that its fundamental frequency is the note E (Mi, frequency 330 Hz). (a) How much is the string stretched? (b) Where should the string be pressed so that it can produce the note F (Fa, frequency 349 Hz)? [A: (a) 39 N, (b) 2 cm.]

10.17 A sound wave sent from a speaker at one end of a conference hall hits the opposite wall and is reflected back. An observer walking between these two walls notices that no sound is heard at 2 m intervals. Calculate the frequency of the sound. [A: 85 Hz.]

10.3 The Doppler effect and Shock Waves

10.18 The whistle of a train traveling at a speed of 50 m/s is able to emit sound at a frequency of 330 Hz. What frequency will a stationary observer beside the railroad tracks hear, (a) when the train is approaching, (b) when the train is moving away? [A: (a) 387 Hz, (b) 288 Hz.]

10.19 A train is approaching as its whistle blows the note C (Do, frequency 262 Hz). (a) An observer sitting in a stationary car beside the tracks hears this sound as the note D-flat (Re bemol, frequency 277 Hz). Calculate the speed of the train. (b) In what direction and at what velocity should the observer drive his/her car to hear this sound as the note D (Re, frequency 294 Hz)? [A: (a) 18 m/s, (b) 21 m/s towards the train.]

10.20 A stationary observer beside a railroad track measures the frequency of the train's whistle. He/she finds the frequency to be 400 Hz while the train is approaching and 240 Hz while it is moving away. Calculate the velocity of the train and the actual frequency of the whistle. [A: 85 m/s and 300 Hz.]

11

FLUIDS



The Niagara Falls is located on the border between Canada and USA and, with its three waterfalls, has the highest flow rate in the world.

The properties of liquids are much richer and more complex than those of solids. What are the macroscopic quantities that define a liquid without looking into the movement of each molecule? Which laws of physics specify the relations between them?

We can distinguish the three states of matter known as solid, liquid and gas by looking at their physical properties. Solids are hard, liquids flow and gases are volatile. The reason for such behavior can only be understood by looking into their microscopic structures and by examining the forces between atoms and molecules. Solid atoms are regularly positioned and tightly bonded with atomic bonds; they may only vibrate around their equilibrium positions. Liquid molecules, on the other hand, do not have a very regular structure and the intermolecular force is very weak. Their bonds are repeatedly broken and renewed. And, in gases, the molecules are located too far apart to interact with each other and, they move freely. The only force keeping gases together is the walls of the container. Liquids and gases are jointly known as **fluids**.

Can we apply the dynamic and static methods that we have learned thus far to fluids? The motion of fluids is much more complex, and it is practically impossible to work with Newton's law for so many molecules. Instead, we can reach many conclusions using energy concepts.

11.1 GENERAL PROPERTIES OF FLUIDS

Density

It is not practical to work with total mass in liquids and gases, the results obtained could be mistaken to be valid only for that specific amount of liquid. Instead, it is more useful to work with the mass of fluid per unit volume, in other words, with **density**.

If the mass inside of a small volume ΔV around a certain point of the fluid is Δm , then the density of the fluid at that point is

$$\rho = \frac{\Delta m}{\Delta V} \quad (11.1)$$

but the density may be different at another place in the fluid. The unit of density is kg/m^3 , but it is more common to use g/cm^3 in daily life. The table below lists the densities of some fluids.

Densities of some liquids and gases			
(The values are for 0°C and 1 atm pressure, unless specified otherwise.)			
material	density (kg/m^3)	material	density (kg/m^3)
Water	1 000	Air	1.20
Sea water	1 030	Carbon dioxide	1.98
Olive oil	920	Hydrogen	0.09
Ethyl alcohol	790	Oxygen	1.33
Mercury (Hg)	13 595	Water vapor (100°C)	0.80

You may find the densities of the other elements in the periodic table provided in Appendix D.

Pressure

A bicycle or automobile tire can carry very heavy loads, despite containing air. Likewise, we feel a discomforting pain in our ears when we dive deep into the sea. All fluids exert a force on the walls of their containers and on the objects contained inside of them. This force is perpendicular to the surface; fluids cannot exert a force parallel to the surface.



Figure 11.1: Pressure in daily life: Automobile tire, pressure cooker, blood pressure gauge.

If a fluid applies a force F on a surface area A , the **pressure** it exerts on that surface is,

$$p = \frac{F}{A} \quad (11.2)$$

The unit of pressure is N/m^2 and is called the Pascal (Pa):

$$1 \text{ pascal} = 1 \text{ Pa} = 1 \text{ N/m}^2$$

Other units, such as **atmosphere**, the **bar**, the **millibar**, and the **height of a column of mercury** are used in technology and meteorology:

$$1 \text{ atm} = 76 \text{ cm Hg} = 1.013 \text{ bar} = 1.013 \times 10^5 \text{ Pa}$$

$$1 \text{ bar} = 1000 \text{ millibar} = 10^5 \text{ Pa}$$

Hydrostatic Pressure

Pressure increases as you go deeper into a liquid. To calculate this, let us consider a thin layer of liquid with surface area A and a small thickness dy at a depth y , inside a liquid with density ρ (Figure 11.2). The mass of the liquid in this layer is the product of volume and density:

$$m = \rho V = \rho A dy$$

If the pressure at the top surface of this mass of liquid is P , the pressure at its bottom surface will be higher by dP . We may consider only the forces in the vertical direction, as the forces on the lateral surfaces will balance each other:

$$\begin{aligned} F_{\text{top}} + mg &= F_{\text{bottom}} \\ PA + \rho A dy g &= (P + dP)A \end{aligned}$$

By simplifying we get

$$dP = \rho g dy$$

Let us take the level $y = 0$ at the surface of the liquid. If there is an external pressure P_0 as well, acting on the surface from the outside, we take the integral from that point down to depth y :

$$\begin{aligned} \int_{P_0}^P dP &= \rho g \int_0^y dy \\ P - P_0 &= \rho g y \end{aligned}$$

Therefore, the **hydrostatic pressure** inside of a liquid varies with depth as follows:

$$\boxed{P = P_0 + \rho g y} \quad (\text{Hydrostatic pressure}) \quad (11.3)$$

y should be replaced with $-y$ for upward cases, like calculating the air pressure in the atmosphere. In obtaining this formula, we considered density and gravity to be constants and took them outside of the integral. However, the change in density and the gravitational acceleration must also be taken into consideration if the change in height is very high.

Very high pressures are exerted on submarines and divers when they dive very deep below the sea level. The surfaces of submarines are reinforced with special steel cages. Likewise, special suits prevent a diver's thorax from collapsing in.

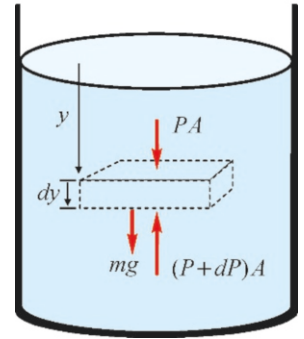


Figure 11.2: Forces acting on the top and bottom surfaces of a layer with thickness dy inside a liquid.

Pascal's Principle

In the expression (11.3), which we found for the variation of pressure with depth, notice that the surface pressure P_0 is effective at every depth. When the surface pressure changes, its impact will be felt at every depth. This is called the **Pascal's principle**:

The external pressure applied to a liquid is equally transmitted to each point of the liquid.

Hydraulic cranes used in lifting jobs in industry operate on Pascal's principle (Figure 11.3). As the force exerted on a small surface of the liquid is transmitted as pressure to another large surface, we can write Pascal's principle for both surfaces:

$$P_1 = P_2 \quad \Rightarrow \quad \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad (11.4)$$

Thus, the force on the large surface will also be larger. The hydraulic brake system of automobiles also uses this principle. With the small force we exert on the brake pedal, the brake pads compress a disk on the wheel with a much greater force.

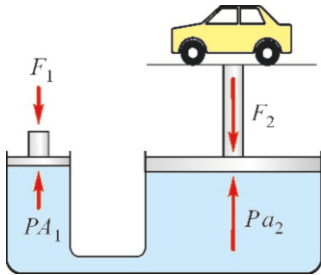


Figure 11.3: Hydraulic crane.

Measuring Pressure

The general name for an apparatus that measures pressure is **pressure gauge**, also called a **manometer**; however, gauges used to measure atmospheric pressure are called **barometers**. There are two main types of pressure gauge: closed-tube and open-tube.

A closed-tube gauge is shown in Figure 11.4a. A glass tube with one closed-end is fully filled with mercury and its open end is dipped inside of a tank of mercury. The external atmospheric pressure forces some mercury to remain inside of the column. The height of the mercury column is equal to the external atmospheric pressure P_0 :

$$P_0 = \rho_{\text{Hg}}gh$$

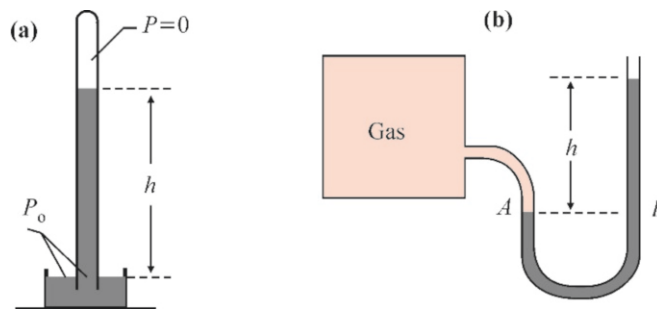


Figure 11.4: (a) Closed-tube manometer, (b) Open-tube manometer.

An open-tube manometer compares the pressures in two columns of a U-tube. As seen in Figure 11.4b, one end of a U-tube containing liquid is connected to the container whose pressure is to be measured. The other end is open. Let us examine the difference in height between the two columns. For example, let the height of the liquid on the open column be higher by h . As the hydrostatic pressure will be equal at equal heights, the pressures of points A and B should be equal:

$$P_A = P_B$$

The end A has the pressure P , which is to be measured. On the other hand, the end B is higher by h , and above it, there is atmospheric pressure P_0 . Therefore, writing this equality as

$$P = P_0 + \rho gh \quad (11.5)$$

we get the pressure P in terms of the liquid height in the column and density.

Gauge pressure

Let us return to the pressure formula in the U-tube:

$$P = P_0 + \rho g h$$

The air pressure P_0 of 1 atmosphere is present in all measurements on the surface of the Earth. We find the absolute pressure by adding the pressure that results from the level difference in the U-tube to this value.

However, in industry, the difference with respect to the atmospheric pressure is more useful than the absolute pressure. This is called the **gauge pressure**. Accordingly, the gauge pressure is found by only calculating the $\rho g h$ pressure value of the mercury column in the U-tube above:

$$P_g = \rho g h \quad (11.6)$$

Recommended pressure values for automobiles, blood pressure values, city water and gas pressure values, etc. should all be considered to be gauge pressure.

The Force and Torque on a Dam Wall

As a good application of hydrostatic pressure, let us calculate the total force acting on a dam wall and the torque that is trying to topple it.

Let us consider a dam with width L and a height of water H enclosed by it (Figure 11.5). As the pressure exerted by the water on the wall is different at each height, we can calculate the total force only by integration. For this purpose, let us consider the small force dF exerted by a strip of water of length L and height dy located between y and $y + dy$ as measured from a point O at the bottom. As the origin is selected at the bottom of the water, the depth of this strip from the surface is $(H - y)$. Accordingly, the force acting on the strip with surface $dA = L dy$ due to hydrostatic pressure is

$$dF = P dA = [\rho g(H - y)] (L dy) = \rho g L (H - y) dy$$

We can find the total force by summing, in other words, integrating the contributions of these small strips from the value $y = 0$ to the value $y = H$:

$$F = \rho g L \int_0^H (H - y) dy = \rho g L \left[Hy - \frac{y^2}{2} \right]_0^H$$

We find the total force by substituting the integral limits:

$$F = \frac{1}{2} \rho g L H^2 \quad (11.7)$$

Note that the force is proportional to the square of the water height.

The torque of the force trying to topple the dam around the point O is likewise calculated by integration. We again write the small torque $d\tau$ of the force dF in the figure about the point O:

$$d\tau = dF y = \rho g L (H - y) dy y$$

The total torque will be the integral of these small $d\tau$ contributions:

$$\tau = \int dF y = \rho g L \int_0^H (Hy - y^2) dy = \rho g L \left[H \frac{y^2}{2} - \frac{y^3}{3} \right]_0^H$$

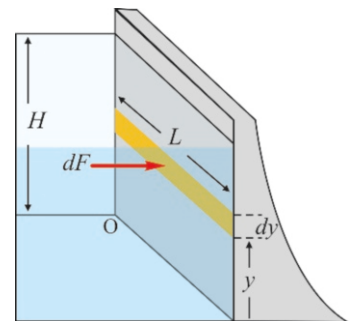


Figure 11.5: Force exerted on a strip with thickness dy on the dam's surface.

Substituting the integral limits, we find the total torque to be:

$$\tau = \frac{1}{6} \rho g L H^3 \quad (11.8)$$

Note that the total torque is proportional to the cube of the height of water. In dam constructions, the toppling torque of water is more dangerous than the force exerted by the water.

Example 11.1

- (a) What is the total mass of air in a classroom with dimensions $10 \times 20 \times 4 \text{ m}^3$?
- (b) A gold crown weighing 1.5 kg is submerged in water and it displaces an 80 cm^3 volume of water. Is the crown made of pure gold?

Answer

(a) We first take the density of air from the table on page 180 as 1.2 kg/m^3 . We then calculate the mass of air using the

volume of the classroom:

$$m = \rho V = 1.2 \times (10 \times 20 \times 4) = 960 \text{ kg}$$

It is surprising to see that the air in a typical room is so heavy.

(b) We take the density of gold from Appendix D as 19.3 g/cm^3 . We then calculate what the volume of 1.5 kg pure gold would be:

$$V = m/\rho = 1500/19.3 = 77.7 \text{ cm}^3$$

The volume to be displaced is lower, therefore the crown is not pure gold.

Example 11.2

The density of water is 1.0 g/cm^3 and the density of ethyl alcohol is 0.8 g/cm^3 . A water-ethyl alcohol mixture with volume 150 cm^3 has a mass of 132 g. Calculate the amount of water and ethyl alcohol in this mixture.

Answer

Total volume will be the sum of the volume of water V_1 and the volume of alcohol V_2 . If we indicate the mass of water

with m_1 , the mass of ethyl alcohol will be $(132 - m_1)$. We write the volumes as $V = m/\rho$ and add:

$$V = V_1 + V_2 = \frac{m_1}{\rho_1} + \frac{132 - m_1}{\rho_2}$$

$$150 = \frac{m_1}{1.0} + \frac{132 - m_1}{0.8}$$

From this equation, we first find the mass of water and then the mass of alcohol:

$$m_1 = 60 \text{ g and } m_2 = 132 - 60 = 72 \text{ g.}$$

Example 11.3

The pressure of the gas inside of a balloon is to be measured using an open-ended manometer in a place where the atmospheric pressure is 72 cm Hg. If the open-ended mercury column is 38 cm higher, what is the absolute pressure and gauge pressure of the gas?

Answer

The gauge pressure is only the pressure of the mercury col-

umn, in other words, 38 cm Hg.

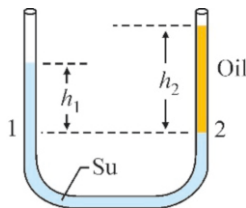
The absolute pressure is the mercury column plus the atmospheric pressure acting on it, in other words, $38 + 72 = 110 \text{ cm Hg}$.

We use the definition $1 \text{ atm} = 76 \text{ cm Hg}$ if we wish to find these pressures in terms of atmospheres:

$$\text{Gauge pressure: } P = 38/76 = 0.5 \text{ atm}$$

$$\text{Absolute pressure: } P = 110/76 = 1.4 \text{ atm}$$

Example 11.4



An unknown type of oil is added to the water column in a U-tube with two open ends. Since the height of the water is $h_1 = 36 \text{ cm}$ and the height of the oil is $h_2 = 50 \text{ cm}$ as seen in the figure,

calculate the density of the oil.

Answer

The hydrostatic pressure of the water in two columns is equal at points 1 and 2, which have the same height:

$$P_0 + \rho_1 g h_1 = P_0 + \rho_2 g h_2$$

Using this, we find the relation between the densities:

$$\rho_2 = \rho_1 \frac{h_1}{h_2}$$

We take the density of water as 1 g/cm^3 and calculate:

$$\rho_2 = 1 \times \frac{36}{50} = 0.72 \text{ g/cm}^3$$

Example 11.5

In a hydraulic crane used to lift cars, the circular piston pushing the large platform has a radius of 20 cm. The other piston pushing the liquid has a radius of 4 cm. How much force is required to lift a car with a mass of 1500 kg?

Answer

The pressures on the piston and platform are equal according

to Pascal's principle. Accordingly, the relation between the forces is

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}$$

We calculate the force by taking the area of the circles πr_1^2 and the weight of the car $W = mg$:

$$F_1 = \frac{\pi r_1^2}{\pi r_2^2} mg = \frac{4^2}{20^2} \times 15000 = 600 \text{ N}$$

Example 11.6

The wall of a dam has a height of 170 m and length of 1800 m. Find the total force acting on the dam wall and the torque trying to topple it over.

Answer

As the pressure varies with the height, we had calculated

the force by integration and found Eq. 11.7. We calculate by substituting the numerical values:

$$F = \frac{1}{2} \rho g L H^2 = \frac{1}{2} \times 1000 \times 10 \times 1800 \times 170^2 = 2.6 \times 10^{11} \text{ N}$$

Likewise, we had found the torque with respect to a point at the bottom of the wall (Eq. 11.8):

$$\begin{aligned} \tau &= \frac{1}{6} \rho g L H^3 = \frac{1}{6} \times 1000 \times 10 \times 1800 \times 170^3 \\ \tau &= 1.5 \times 10^{13} \text{ N}\cdot\text{m} \end{aligned}$$

11.2 BUOYANCY AND ARCHIMEDES' PRINCIPLE

Objects immersed in a fluid feel lighter. A balloon filled with helium gas floats. Fish go up or down in water by compressing their swim bladders. Submarines surface or dive by pumping air or water into their diving tanks. Likewise, when whales wash up on a shore, their lungs get crushed under their own weight and they die. This property is called **buoyancy**.

Fluids always exert an upward force on the submerged objects, called the **buoyant force**. Discovered by Archimedes, its value can be expressed as follows:

Archimedes' Principle

Every object immersed in a liquid will be pushed upwards by a force equivalent to the weight of the liquid displaced by that object.

Attention should be paid to the expression "liquid displaced" in this principle. The whole volume of the object will be taken into consideration if it is fully immersed in the liquid. However, as in the case of a piece of wood, if it floats while partially submerged, then only the volume inside of the liquid will be taken into consideration.

The source of buoyant force is the variation of hydrostatic pressure with depth. We can easily see this by calculating the forces exerted on the surfaces of an object shaped like a cylinder (Figure 11.6).

Let us consider a cylinder with base area A and height h , immersed inside a liquid with density ρ_0 . There is no need to consider the forces on the lateral surfaces of the cylinder, because they mutually balance each other. Let the top base of the cylinder be at depth y and the bottom base at depth $y+h$. Accordingly, the buoyant force F_B will be the difference of the forces at the bottom and top

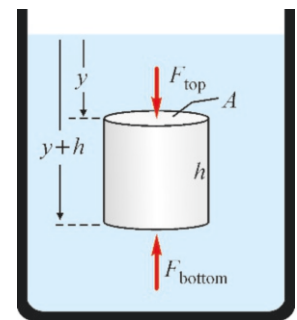


Figure 11.6: Forces acting on the top and bottom surface inside of a liquid.

surfaces. If we write the force on each surface as the product of the pressure at that depth and the surface area, we get

$$\begin{aligned} F_B &= F_{\text{bottom}} - F_{\text{top}} = P_{\text{bottom}} A - P_{\text{top}} A \\ &= \rho_0 g(y+h) A - \rho_0 g y A \\ &= \rho_0 g h A = \rho_0 g V \end{aligned}$$

The value Ah here is the volume V of the object. Therefore, the expression is found for Archimedes' buoyant force F_B :

$$F_B = \rho_0 g V \quad (\text{Archimedes' buoyant force}) \quad (11.9)$$

Archimedes' principle also applies for partially immersed bodies, but in this case, only the immersed volume is taken into consideration. Likewise, if we write the hydrostatic pressure difference between the top and bottom surfaces of the object and repeat the calculations, the volume V of the object is replaced with the volume V' of the immersed part:

$$F_B = \rho_0 g V' \quad (\text{for partially immersed bodies}) \quad (11.10)$$

The expression for the apparent weight of an object inside of a liquid can also be found from this formula. If the object is made of a material with density ρ , its real weight will be $mg = \rho Vg$. If we subtract the lightening effect of the buoyant force, we get

$$\begin{aligned} W' &= W - F_B = mg - F_B = \rho Vg - \rho_0 Vg \\ W' &= (\rho - \rho_0)Vg \quad (\text{apparent weight}) \end{aligned} \quad (11.11)$$

Example 11.7

What percent of the volume of an iceberg floating on water is under water? The density of sea water is $\rho_0 = 1.03$ and the density of ice is $\rho = 0.92 \text{ g/cm}^3$.

Answer

The buoyancy of the water balances the weight of the iceberg. If the submerged part of an iceberg with volume V has

volume V' , the buoyant force F_B will be equal to the weight of the displaced water:

$$\begin{aligned} mg &= F_B \\ \rho Vg &= \rho_0 V'g \end{aligned}$$

From here, we find the volume:

$$\frac{V'}{V} = \frac{\rho}{\rho_0} = \frac{0.92}{1.03} = 0.89$$

Example 11.8

A wooden block is observed to float half-immersed when placed in water. 60% of its volume is immersed when it is placed in oil. Find the densities of the wood and the oil.

Answer

Let us denote the densities of the wood, water and oil with ρ_1 , ρ_2 and ρ_3 , respectively. If the submerged part of the wood with volume V has volume V' , this means that a buoyant force F_B equal to the weight of water with volume V' can balance the weight mg of the wood:

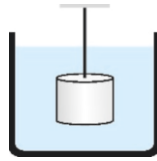
$$mg = F_B \rightarrow \rho_1 g V = \rho_2 g V'$$

We find the density of the wood by taking the density of the water as $\rho_2 = 1 \text{ g/cm}^3$:

$$\rho_1 = \frac{V'}{V} \rho_2 = \frac{1}{2} \times 1 = 0.5 \text{ g/cm}^3$$

Likewise, if the part submerged in oil has volume V'' , the weight of the displaced oil will be equal to the weight of the wood:

$$\begin{aligned} \rho_1 g V &= \rho_3 g V'' \\ \rho_3 &= \frac{V}{V''} \rho_1 = \frac{1}{0.6} \times 0.5 = 0.83 \text{ g/cm}^3 \end{aligned}$$

Example 11.9

A stone block with volume 0.01 m^3 and density $\rho=2.7 \text{ g/cm}^3$ is hanging in water as attached to a rope, as seen in the figure. Calculate the tension in the rope.

Answer

The downward weight of the stone is jointly balanced by the upwards buoyant force and the tension in the rope. We write the balance of the forces:

$$F_B + T = mg$$

The buoyant force is equal to the weight $\rho_0 Vg$ of the mass of water displaced by the volume V . From here, we calculate the tension T :

$$T = mg - F_B = \rho Vg - \rho_0 Vg = (\rho - \rho_0) Vg$$

$$T = (2.7 - 1) \times 10^3 \text{ kg/m}^3 \times 0.01 \times 10 = 170 \text{ N}.$$

11.3 SURFACE TENSION AND CAPILLARITY

Liquid surfaces that we encounter in daily life can exhibit very interesting behavior. Raindrops that fall on leaves or car windows do not spread, but rather form spherical bubbles. Likewise, liquid mercury does not stick to a surface, but collects in spherical drops (Figure 11.7).

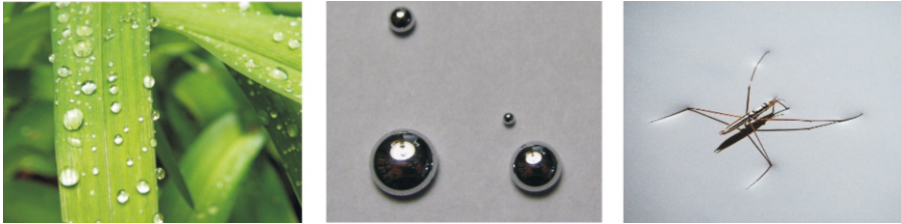


Figure 11.7: Examples of surface tension: Raindrops, mercury bubbles and an insect walking on water without sinking.

In all of these examples, the liquid surface acts like an elastic membrane. Liquid molecules bond together and try to form a stretched surface. For example, a sewing needle carefully dropped on water may float. Mosquitoes and certain insects can walk on the surface of water without sinking.

The property of liquid surfaces to resist external forces is called **surface tension**. Water molecules, although they are neutral, can still attract each other (we shall examine this in Chapter 14 as dipole interaction.) Briefly, if a water molecule is considered as a rod with one positive end and one negative end, a weak but attractive force called the **van der Waals force** occurs between the two molecules. This force generates surface tension.

Let us consider a molecule inside of a liquid and another molecule on the surface (Figure 11.8). As the forces inside of the liquid balance each other, this molecule can circulate freely throughout the liquid. However, the forces are not in equilibrium for a molecule on the surface; a net force is exerted that pulls this molecule towards the liquid and prevents it from leaving the surface.

Consider an imaginary line with length L drawn on the surface of the liquid (Figure 11.9). If the force required to separate the molecules on both sides of this line is F , then the coefficient

$$\gamma = \frac{F}{L} \quad (\text{surface tension coefficient}) \quad (11.12)$$

is called the **surface tension coefficient**. The surface tension coefficients of some liquids are as follows:

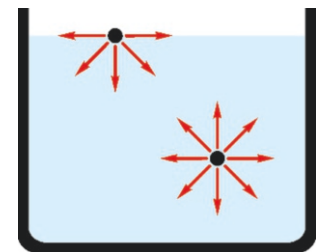


Figure 11.8: Forces acting on water molecules inside and on the surface of a liquid.

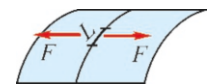


Figure 11.9: Surface tension coefficient.

Surface tension coefficients of some liquids

liquid	γ (N/m)	liquid	γ (N/m)
Water (0°)	0.076	Mercury	0.487
Water (20°)	0.072	Ethyl alcohol	0.023
Water (100°)	0.059	Acetone	0.024
Soapy water (20°)	0.025	Glycerin	0.064

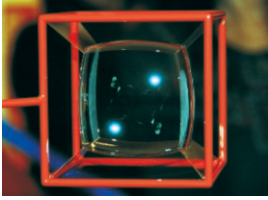


Figure 11.10: A minimal surface formed by soapy water.

In laundry, water should be able to reach the stains between the fibers of the clothing material. Pure water cannot seep into these small gaps, because the surface tension of water is high. However, the surface tension of water greatly decreases when it is heated and soap is added, and it thus becomes able to pass through fine gaps.

As the surface tension of soapy water is lower, it can spread over a larger surface without breaking and larger foams can be formed. The surface of soapy water can be stretched over a wire frame. The areas of these surfaces are minimum. Various minimal surfaces can thus be formed (Figure 11.10).

Capillarity

When a tube with two open ends is submerged in water, the water inside of the tube is observed to rise. The smaller the diameter of the tube is, the higher the liquid rises. Water can be observed to rise up to a meter high in tubes with very small diameters. This effect is called the **capillarity**. However, certain other liquids do not rise, but in contrast, sink in capillary tubes.

A liquid exhibits two different behaviors when it comes into contact with a solid surface, depending on the forces between the molecules. For example, some liquid surfaces are observed to curve upwards where they touch a glass tube. On the other hand, the surface of mercury curves downward (Figure 11.11).

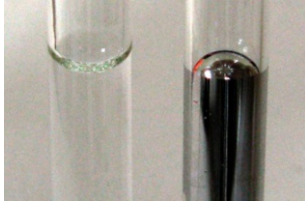


Figure 11.11: In a glass tube, the surface curves upward for water, and downward for mercury..

This behavior can be explained with the forces between molecules. The attractive force between two of the same type of molecule in a liquid is called the **cohesion force**. On the other hand, the attractive force between molecules of different types is an **adhesion force**.

In the case of a water-glass surface, the adhesion force is greater and the water is attracted to the glass; the level of water rises and these molecules attracted to the surface are replaced with other molecules from the inside. As the contact angle changes during the rise, the adhesion force steps in again and attracts new liquid molecules. This process in which the liquid rises along the glass tube continues until it is balanced by gravitational force.

The angle θ that the liquid surface makes with the solid wall is called the **contact angle**. Measured from solid surfaces, this angle is acute (less than 90°) for liquids curved upward, such as water, and obtuse for liquids curved downward, such as mercury (Figure 11.12).

In a capillary tube, the height h is expressed in terms of the contact angle θ , the surface tension coefficient γ and the tube radius r as follows:

$$h = \frac{2\gamma \cos \theta}{\rho g r} \quad (11.13)$$

Capillarity is perhaps the most vital application of surface tension. It plays a very significant role in the survival of living beings. The sap of a tree can reach

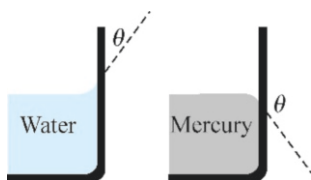


Figure 11.12: Contact angles of mercury and water with a glass tube.

very high branches above; the blood in humans and animals can reach the finest veins in the circulation system, due to capillarity. Capillarity is used in many applications in medicine and technology. For example, a capillary glass tube is used to draw blood sample; paper towels absorb water through capillary action.

11.4 VISCOSITY

Every liquid has a consistency. Liquids such as water and alcohol flow easily, while liquids such as glycerin, tar and honey are difficult to stir. As the latter shows, moving a solid object inside of a liquid is difficult. It is relatively easy to walk in water but becomes more difficult when you try to move faster.

The internal friction effect of liquids, one that resists its own fluidity and the motion of bodies inside of it, is called **viscosity**. Liquids with high consistency have higher viscosity. Gases also have internal friction, but it is much less compared to that of liquids.

The sources of viscosity are different in liquids and gases: Internal friction is caused by attractive van der Waals forces in liquids and by the collision of molecules in gases. Because of this difference, the viscosities of liquids decrease, whereas those of gases increase with temperature.

A viscosity coefficient is defined to specify the effect of internal friction: Let us consider a layer of liquid of width L between two solid horizontal plates as seen in Figure 11.14. Let the top plate be pulled with speed v as the bottom plate is kept fixed. The speed of the liquid layer will vary from one plate to the other: The liquid surface in contact with the bottom plate will remain motionless, while the surface in contact with the top plate will have speed v due to adhesion forces. If the speed v is not too high, the speed of the liquid will increase linearly from one layer to the other. This layered structure is called laminar flow. If the surface area of each plate is A , the exerted force is

$$F = \eta A \frac{v}{L} \quad (11.14)$$

and the proportionality constant is called the **viscosity coefficient**, indicated with the Greek letter η (eta). Its unit is $(\text{N}/\text{m}^2) \cdot \text{s} = \text{Pascal} \times \text{s} = (\mathbf{Pa} \cdot \mathbf{s})$. Another unit used in industry is the **poise**, denoted with P, and its value is one tenth that of the SI unit. One hundredth of a poise is called a centipoise and is indicated with cP:

$$1 \text{ Pa} \cdot \text{s} = 10 \text{ poise} = 1000 \text{ cP}$$

The viscosity coefficients of some liquids are as follows:

Viscosity coefficients of some liquids			
liquid	η (cP)	liquid	η (cP)
Water (20°)	1.00	Olive oil (20°)	84
Water (40°)	0.65	Motor oil (SAE20, 20°)	125
Water (100°)	0.28	Motor oil (SAE20, 150°)	3
Honey (20°)	10	Tar (20°)	30 000

Viscosity varies rapidly with temperature. Motor oils that are viscous at room temperature become as fluid as water when they reach the high temperature of a running engine.



Figure 11.13: The higher consistency a liquid has, the higher is its viscosity.

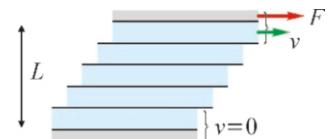


Figure 11.14: Velocity profile in laminar flow.

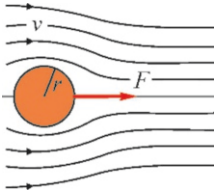


Figure 11.15: A sphere inside viscous liquid.

Stokes' Law

Solid bodies moving in fluids experience a resistant force caused by viscosity. The magnitude of this **drag force** depends on the speed and geometric shape of the object and the viscosity of the liquid. The result may be different for each geometrical shape. The formula found by the British scientist G.G. Stokes for the drag force acting on a sphere with radius r is as follows:

$$F = 6\pi\eta r v \quad (\text{Stokes' law for drag force}) \quad (11.15)$$

v is the speed of the fluid (or the speed of the object going in the opposite direction). The fact that drag force is proportional to velocity v was used in the damped harmonic oscillator that we examined in Chapter 9.

In gases, the relation between drag force and velocity can be much different. Internal friction is proportional to v in low speeds, but in airplanes traveling faster than the speed of sound (supersonic), it becomes proportional to v^2 and enormous power must be spent at such speeds.

11.5 BERNOULLI'S EQUATION

The motion of a fluid can take a wide range of appearances and is very difficult to describe. A laminar flow in a calm river, the turbulent flow of the exhaust gas of an airplane, vortexes in seas, etc. All of these types of flows are subject to completely separate analyses. In this section, we shall consider an ideal fluid and reach certain conclusions under general energy conservation laws.

First, let us give some definitions. An **ideal fluid** is a type of fluid that has no viscosity and is incompressible. Of course, all fluids have internal friction and may change volume under pressure. However, with a small margin of error, we can take water and other liquids with low viscosity as ideal.

In a liquid medium, curves on which the velocity is considered as tangent at every point are called **streamlines**. Streamlines can be made visible by adding coloring at various points in the medium (Figure 11.16). There is a **steady flow** if the streamlines do not vary with time.



Figure 11.16: Streamlines around an automobile.

The structure of streamlines provides information about the type of motion. If streamlines form smooth and soft curves, it is called a **laminar flow**. In contrast, if the curves exhibit irregular behavior like whirlpools or turbulences, it is called a **turbulent flow**.

Laminar flow can occur in low velocities. As velocity increases, turbulent flow starts after a certain value. We shall only consider laminar flow here, as the analysis of turbulent flow is complicated.

Flow Rate and Equation of Continuity

The volume of a liquid that passes through a surface with cross-section A per unit time is called the **flow rate** and is indicated with Q . Its unit is m^3/s . If the fluid passes through a cross-section A with a constant velocity v , the volume that passes through this cross-section in a time interval Δt is the volume of a cylinder with length $v \Delta t$ and base area A . Accordingly, the flow rate expression will be as follows:

$$Q = \frac{\Delta V}{\Delta t} = \frac{A v \Delta t}{\Delta t}$$

$$Q = A v \quad (\text{Flow rate}) \quad (11.16)$$

In laminar flow, streamlines do not intersect. Streamlines that surround any cross-section A stretch out like a tube inside of the liquid. Let us consider two different cross-section of such a tube (Figure 11.17). Let the velocity through cross-section A_1 be v_1 and that through A_2 be v_2 . As this fluid is ideal, in other words, incompressible, an equal volume of liquid should pass through each cross-section per unit time. Therefore, the flow rates should be equal in both cross-sections.

$$\begin{aligned} Q_1 &= Q_2 \\ A_1 v_1 &= A_2 v_2 \quad (\text{Equation of continuity}) \end{aligned} \quad (11.17)$$

This equation is known as the **equation of continuity**. We can flush water much faster through a garden hose by squeezing its end with our fingers, because the same amount of water is passing through a narrower cross-section. Therefore, it should flow faster to keep the flow rate constant.

The equation of continuity states that volume is conserved, but since this is an incompressible liquid, it actually states that mass is conserved. In other words, the liquid undergoes no loss in mass along the streamline.

Conservation of energy

Let us consider the energy of a fluid mass in a flow tube at two different points. The fluid mass passing through the cross-section A_1 located at height y_1 with velocity v_1 later passes through the cross-section A_2 located at height y_2 with velocity v_2 (Figure 11.18).

The work performed by the external forces acting on this tube will be equal to the increase in total energy. Let us remember the expression (5.17) that we found for the general conservation of energy when discussing the topic of energy in Chapter 5:

$$(K_1 + U_1) + W_{nc} = K_2 + U_2$$

Here, K is the kinetic energy, U the gravitational potential energy and W_{nc} the work performed by nonconservative forces. Let us write this law as follows for convenience:

$$W_{nc} = (K_2 - K_1) + (U_2 - U_1) \quad (11.18)$$

The nonconservative external force acting on the flow tube is caused by the pressures on both ends. The lateral forces acting on the tube do not perform any work, as they are perpendicular to the path. The force exerted by pressure P_1 on the first cross-section is equal to $P_1 A_1$ and the force exerted by pressure P_2 on the other end is $P_2 A_2$. Let the fluid mass travel a distance of L_1 under the effect of force in the first cross-section and let the mass in the second cross-section travel a distance of L_2 . Accordingly, the net work performed is

$$W_{nc} = F_1 L_1 - F_2 L_2 = P_1 A_1 L_1 - P_2 A_2 L_2$$

The work performed by the second force is negative, because it is opposite the displacement. Now, let us write the gravitational potential energy of the mass in both cross-sections. As the displaced liquid mass has the same value m in both places, we get

$$U_2 - U_1 = mgy_2 - mgy_1$$

The difference between the kinetic energies can likewise be written directly:

$$K_2 - K_1 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

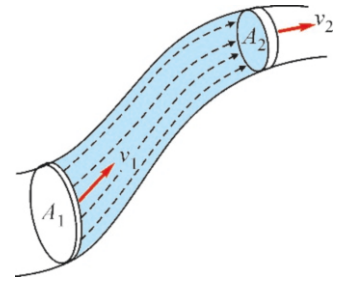


Figure 11.17: Equal amount of material passes through each cross-section in unit time.

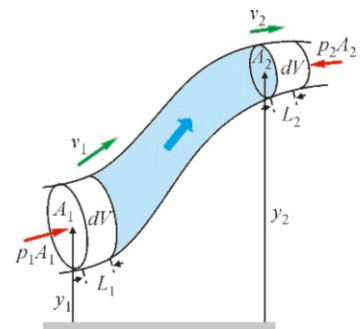


Figure 11.18: Two cross-sections of a flow tube.

Adding these terms in the (11.18) equation, we get

$$P_1 A_1 L_1 - P_2 A_2 L_2 = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 + m g y_2 - m g y_1$$

Let us write the mass m in terms of density and volume. The two cross-sections of the tube and the distances traveled may be different, but the volume will remain the same:

$$V = A_1 L_1 = A_2 L_2$$

If we also express mass in terms of density, we get

$$m = \rho V = \rho A_1 L_1 = \rho A_2 L_2$$

Therefore, if the products $A_1 L_1$ and $A_2 L_2$ are written as m/ρ , the expression above simplifies as follows:

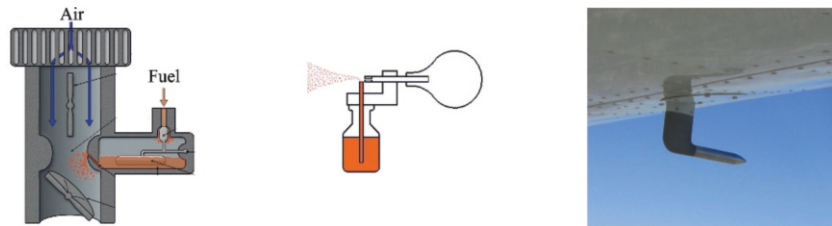
$$P_1 - P_2 = \frac{1}{2} \rho v_2^2 - \frac{1}{2} \rho v_1^2 + \rho g y_2 - \rho g y_1$$

Rearranging the terms, we find the **Bernoulli's equation**:

$$P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 \quad (\text{Bernoulli's equation}) \quad (11.19)$$

The Bernoulli's equation is the version of conservation of energy for liquids. This equation is valid for ideal, in other words, non-viscous and incompressible fluids, but it is also approximately correct for other fluids.

Figure 11.19: Applications of the Bernoulli's equation: Carburetor in gasoline engines, perfume sprays, Pitot tubes in airplanes.



Let us emphasize the conclusions that can be drawn from the Bernoulli's equation:

- **Hydrostatic pressure.** If we take $v_1 = v_2 = 0$ in the Bernoulli's equation for a mass of liquid at rest, we get

$$P_1 + \rho g y_1 = P_2 + \rho g y_2$$

If we take one end at $y_1 = 0$ and the other at $y_2 = y$, we get

$$P = P_0 + \rho g y$$

This is the hydrostatic pressure formula that we discussed earlier.

- **Venturi tube.** Let us consider two cross-sections at the same height in a liquid tube (Figure 11.20). If we set $y_1 = y_2$ in Bernoulli's equation,

$$P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2$$

According to the equation of continuity, the pressure will be lower in the narrow cross-section of the tube because the velocity is higher there. This

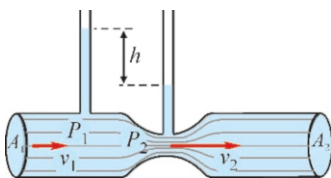


Figure 11.20: Venturi tube.

device, called the *Venturi tube*, is used in many technologies: in the carburetors of gasoline engines for mixing gasoline and air, in pesticide sprayers, in perfume sprays, etc. As seen in Figure 11.19, when air is blown from a tube passing through the open end of the liquid, the pressure will decrease there, and the liquid in the tank will rise and mix with the passing air.

The **Pitot tube**, which is a similar device, is used to measure the velocity of airplanes with respect to air (Figure 11.21). Air passes with velocity v at one end of a U-tube filled with liquid and the velocity is ensured to be zero at the other end. The velocity of the air can be calculated using Bernoulli's equation by measuring the height h of the liquid rising due to the pressure difference.

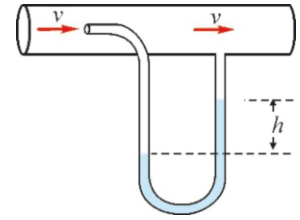


Figure 11.21: Pitot tube.

- **Torricelli's formula.** Let us consider that a hole is opened at a distance h below the surface on a closed tank containing liquid. Let us compare the top liquid surface in the container and the point of the hole in Bernoulli's equation. The velocity of the liquid can be taken as zero at the top, because the liquid surface is very large. Let the liquid surface inside of the container be at pressure P and the pressure at the hole, which is just the atmospheric pressure P_0 , because it is open to the air. Accordingly,

$$P + 0 + \rho gh = P_0 + \frac{1}{2}\rho v^2$$

$$v = \sqrt{2gh + \frac{2(P - P_0)}{\rho}} \quad (11.20)$$

We have $P = P_0$ if the top of the container is also open to atmospheric pressure, and we get

$$v = \sqrt{2gh} \quad (11.21)$$

Called **Torricelli's formula**, this relation shows that, just like a stone dropped at a height of h in free fall, a liquid will also gain the velocity $v = \sqrt{2gh}$. This is not surprising, because Bernoulli's equation is an expression of conservation of energy. Figure 11.22 shows the velocities of water coming out of holes opened at various depths.



Figure 11.22: Velocities of water at various depths according to Torricelli's formula.

- Aeronautics technology has the most striking application of Bernoulli's equation. This equation can be used to understand how a lifting force is produced on the wings of a heavy steel bird. Examining the cross-section of the wing of an aircraft in Figure 11.23, the lower part is more straight and the top part is more curved. When an aircraft enters a beam of streamlines, the air passing through the top will travel a longer distance and meet with the others at the back. This means that the air at the top has a higher speed. Therefore, according to Bernoulli's equation, the pressure at the top of the wing will be lower than that at the bottom. This pressure difference causes a net upward **lift force**. This force should not be confused with Archimedes' buoyancy.
- The Bernoulli effect is likewise observed in curved balls in football and baseball. When the ball spins as it is kicked forward, its effect on the air speed is different on sides A and B due to the rotation of the ball (Figure 11.24). Consider an observer moving with the ball: On side A, the rotation against

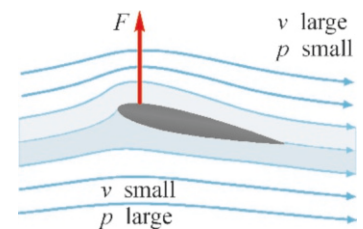


Figure 11.23: Speed and pressure on an aircraft wing.

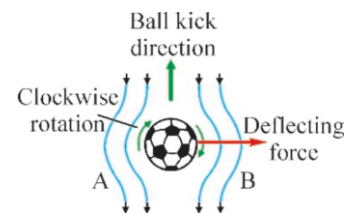


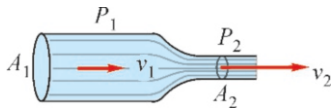
Figure 11.24: Forces on a ball.

the air decreases the air speed locally, while the effect is opposite on side B. Hence the air speed is greater at B than at A. This air speed difference causes a pressure difference on both sides. As a result of the Bernoulli effect, a net force arises that pushes the ball in the direction of lower pressure B shown in the figure.

Example 11.10

The main pipe of the city water network shown in the figure has cross-section area $A_1=4\text{ m}^2$ and another pipe connected to it has a cross-section of $A_2=2\text{ m}^2$. The water flowing through cross-section A_1 has a flow rate of $Q=24\text{ m}^3/\text{s}$.

- (a) Calculate the speed of water in both cross-sections.
- (b) If the pressure at the first cross-section is $P_1=2\text{ atm}$, what is the pressure at the second cross-section?



Answer

(a) The definition of flow rate is given in Eq. (11.16):

$$Q = A_1 v_1$$

From here, we find the speed v_1 :

$$v_1 = \frac{Q}{A_1} = \frac{24}{4} = 6\text{ m/s}$$

We use the flow rate or equation of continuity (11.17) to find the speed at the other cross-section:

$$A_1 v_1 = A_2 v_2 \rightarrow v_2 = \frac{A_1}{A_2} v_1 = \frac{4}{2} \times 6 = 12\text{ m/s}$$

(b) We write Bernoulli's equation (11.19):

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

We take $y_1 = y_2$, because the cross-sections are at the same height and solve for P_2 :

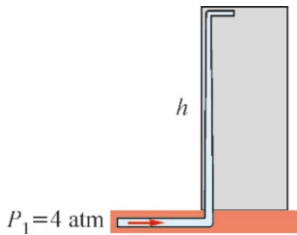
$$P_2 = P_1 - \frac{1}{2}\rho(v_2^2 - v_1^2)$$

We calculate the pressure by first converting the atmosphere unit into Pascal:

$$P_2 = 2 \times 1.013 \times 10^5 - \frac{1}{2} \times 1000 \times (12^2 - 6^2)$$

$$P_2 = 1.5 \times 10^5\text{ Pa} = 1.5\text{ atm}$$

Example 11.11



Water is pumped with a pressure of $P_1=4\text{ atm}$ into the water pipe at the entrance of a building. The water speed at the pipe with cross-section 5 cm^2 is 2 m/s at the entrance.

- (a) What is the speed and pressure of the water flowing from a tap with cross-section 1 cm^2 at a height of 4 m on the second floor?
- (b) What is the minimum pressure at the entrance such that water can rise up to the 10th floor at a height of 40 m ?

Answer

(a) We find the water speed at the second floor using the equation of continuity:

$$A_1 v_1 = A_2 v_2 \rightarrow v_2 = \frac{A_1}{A_2} v_1 = \frac{5}{1} \times 2 = 10\text{ m/s}$$

We find the pressure using Bernoulli's equation:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

We take $y_1 = 0$ at the ground level and solve for P_2 :

$$P_2 = P_1 - \frac{1}{2}\rho(v_2^2 - v_1^2) - \rho g y_2$$

We convert the pressure values into Pascal units, take the density of water as $\rho = 1000\text{ kg/m}^3$ and calculate:

$$P_2 = 4 \times 1.013 \times 10^5 - \frac{1}{2} \times 1000 \times (10^2 - 2^2) - 1000 \times 10 \times 4$$

$$P_2 = 320\,000\text{ Pa} = 3.1\text{ atm}$$

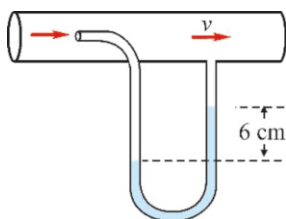
(b) The water rising up to the 10th floor means that it reaches that floor with at least zero speed. According to the equation of continuity, the velocity at the ground should also be zero. Therefore, taking $v_1=v_2=0$ and $P_2=1\text{ atm}$, Bernoulli's equation simplifies as follows:

$$P_1 + 0 + 0 = P_2 + 0 + \rho g y_2$$

Since the tap is in open air, $P_2=1\text{ atm}$. From here, we calculate P_1 :

$$P_1 = 1 \times 1.013 \times 10^5 + 1000 \times 10 \times 40 = 501\,000\text{ Pa} = 5\text{ atm}$$

Example 11.12



The mercury column of a Pitot tube on the wing of a flying aircraft has risen by 6 cm . What is the speed of the aircraft with respect to the air?

Answer

Let us compare the end points of both columns. The speed of air will be zero on one of end and v on the other. Since both ends are at the same height $y_1=y_2=y$, Bernoulli's equation is

written as follows:

$$P_1 + \frac{1}{2}\rho \times 0^2 + \rho gy = P_2 + \frac{1}{2}\rho v^2 + \rho gy$$

$$P_1 - P_2 = \frac{1}{2}\rho v^2$$

Now, we look at the U-tube. The pressure difference will be equal to the pressure of the mercury column:

$$\rho_{\text{Hg}}gh = \frac{1}{2}\rho v^2$$

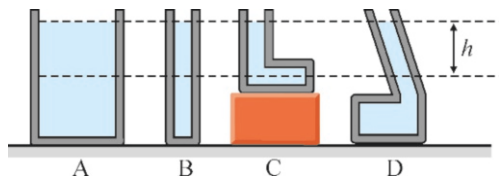
$$v = \sqrt{2gh \frac{\rho_{\text{Hg}}}{\rho}}$$

We take the density of air as 1.2 and the density of mercury as $\rho_{\text{Hg}} = 13600 \text{ kg/m}^3$ and calculate the speed:

$$v = \sqrt{20 \times 0.06 \times \frac{13600}{1.2}} = 117 \text{ m/s.}$$

Multiple-choice Questions

1. In which of the containers in the figure below is the pressure at depth h lower?
 (a) A (b) B (c) C (d) Equal



2. In which of the containers in the figure above is the pressure at the base lower?
 (a) A (b) B (c) C (d) Equal

3. In which of the containers in the figure above is the force at the base greater?
 (a) A (b) B (c) C (d) Equal

4. Which of the following are correct for hydrostatic pressure?

- I. It increases with depth.
- II. It decreases with depth.
- III. It increases with the density of the liquid.
- IV. It decreases with the density of the liquid.

- (a) I (b) I & II (c) I & III (d) I & IV

5. Which of the following is incorrect for the buoyant force of liquids?

- (a) It increases with depth.
- (b) It increases with the density of the liquid.
- (c) It increases with the volume of the object.
- (d) It increases with the gravitational acceleration.

6. Will a boat sink more in lake water or salty sea water?

- (a) Lake
- (b) Sea
- (c) Equal
- (d) It is impossible to tell.

7. Lead is denser than copper. When two equal masses of lead and copper are submerged in water, which one will experience a larger buoyant force?

- (a) Lead
- (b) Copper
- (c) Equal
- (d) It is impossible to tell.

8. Lead is denser than copper. When two equal volumes of lead and copper are submerged in water, which one will experience a larger buoyant force?

- (a) Lead
- (b) Copper
- (c) Equal
- (d) It is impossible to tell.

9. Archimedes' buoyancy force is equal to which of the following?

- (a) The weight of the submerged part of the object.
- (b) The weight of the floating part of the object.
- (c) The weight of the water that it displaces.
- (d) None of the above.

10. An iron sphere attached to a wooden block floats when placed on water. In which case will the buoyancy of the water be greater?

- (a) When the iron is on the top and out of the water.
- (b) When the iron is at the bottom and under the water.
- (c) Buoyancy is the same.
- (d) It is impossible to tell.

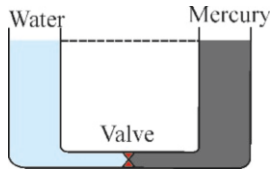
11. A piece of ice is floating in a glass of water. What will happen to the level of water when the ice fully melts?

- (a) It will increase.
- (b) It will decrease.
- (c) It will remain the same.
- (d) It is impossible to tell.

12. A man inside a boat in a pool steps out of the boat into the pool. What will happen to the water level of the pool?

- (a) It will increase.
- (b) It will decrease.
- (c) It will remain the same.
- (d) It is impossible to tell.

13. A metal block floats half-submerged in a container full of mercury. How will the level at which it floats in the mercury be affected when it is taken to the surface of the Moon?
- The level will be higher.
 - The level will be lower.
 - The level will stay the same.
 - It is impossible to tell.



14. Two immiscible liquids (water and mercury) are at equal heights on two sides of a U-tube when the central valve is closed. On which side will the level rise when the valve is opened?
- On the water side.
 - On the mercury side.
 - The two sides will remain equal.
 - It is impossible to tell.
15. A bird lands on a man sitting in a boat in a pool. What will happen to the level of the water?
- It will rise.
 - It will fall.
 - It will remain the same.
 - It is impossible to tell.
16. The viscosity force acting on a sphere moving at speed v inside of a liquid depends on which of the following?

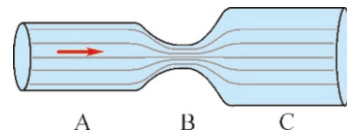
- The radius of the sphere.
- The speed of the sphere.
- The viscosity of the liquid.
- All of the above.

17. Two spheres with equal radius, one made of lead and one made of wood, are dropped from the same height. Assume that the drag force of the air on both is the same. Which one reaches the ground first?

- The lead one.
- The wooden one.
- Both will reach the ground at the same time.
- It is impossible to tell.

18. In which cross-section of the pipe in the following figure is the speed greater?

- A
- B
- C
- Equal



19. In which cross-section of the pipe in the figure above is the pressure greater?

- A
- B
- C
- Equal

20. In which cross-section of the pipe in the figure above is the flow rate greater?

- A
- B
- C
- Equal

Problems

11.1 General Properties of Fluids

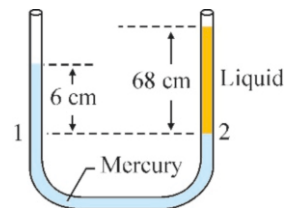
11.1 In a place where atmospheric pressure is 1.2 atm, an open-ended gauge is used to measure the pressure of a gas in a balloon. If the mercury column of the gauge indicates 40 cm, what are the absolute pressure and gauge pressure of the gas? [A: 1.73 and 0.53 atm.]

11.2 Air density is 1.3 kg/m^3 at sea level. What would the thickness of the Earth's atmosphere be if the density of air did not decrease with height, remaining constant? (*Hint*: The pressure on the surface is still 1 atm.) [A: 7.8 km.]

11.3 (a) What is the pressure at a depth of 1200 m below the ocean? Calculate in terms of Pascal and atmosphere. (b) What amount of force should the lens of a research camera with an area of 1 cm^2 be able to withstand at such a depth?

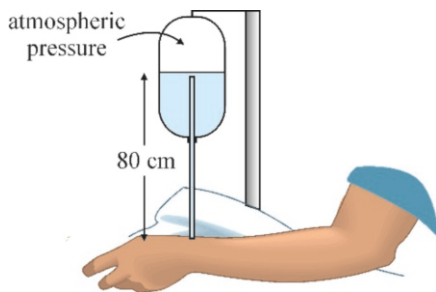
The density of sea water is $\rho = 1030 \text{ kg/m}^3$.

[A: (a) $1.25 \times 10^7 \text{ Pa}$, 123 atm, (b) 1250 N.]



Problem 11.4

11.4 An unknown liquid is added to the mercury column in a U-tube with two open ends. Since the height of the mercury is $h_1 = 6 \text{ cm}$ and the height of the unknown liquid is $h_2 = 68 \text{ cm}$, as seen in the figure, calculate the density of the liquid. [A: 1.2 g/cm^3 .]



Problem 11.5

11.5 The pressure in veins is called *blood pressure*. This is the gauge pressure that is present in addition to the atmospheric pressure. When a patient is fed with serum, the serum pressure at the entry point into the body should be greater than the blood pressure. The serum inside of the bottle shown in the figure has a density of 1040 kg/m^3 . The liquid is able to enter the blood only when the serum bottle is hung at a height of 80 cm. Calculate the blood pressure (gauge pressure) of the patient in terms of Pascal and cm Hg units.

[A: 8320 Pa, 6.2 cm Hg.]

11.6 In a hydraulic crane used to lift vehicles, the cylinder pushing the platform has a radius of 30 cm. The piston pushing the liquid has a radius of 3 cm. How much force is required to lift a 5-ton truck?

[A: 500 N.]

11.7 The wall of a dam has a height of 210 m and a length of 1100 m. Find the total force acting on the dam wall and the torque that is trying to topple it over.

[A: $2.4 \times 10^{11} \text{ N}$ and $1.7 \times 10^{13} \text{ N}\cdot\text{m}$.]

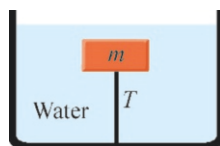
11.2 Archimedes' Buoyancy

11.8 What percentage of the volume of a wooden block with density 0.7 g/cm^3 will sink when placed in an oil with a density of 0.92 g/cm^3 ?

[A: 76 %.]

11.9 An aluminum block with a density of 2.7 g/cm^3 floats as partially submerged when placed in mercury with a density of 13.6 g/cm^3 . Water is poured on the mercury to make the aluminum block become fully submerged. What percentage of the block is underwater?

[A: 87 %.]



Problem 11.10

11.10 A wooden block with a mass of 2 kg and a density of 400 kg/m^3 is submerged by being tied to a rope at the bottom of a container of water, as seen in the figure. Calculate the tension in the rope.

[A: 30 N.]

11.11 What is the minimum volume of an ice layer that can carry a car with a mass of 1000 kg on a lake? The density of ice is 0.92 g/cm^3 . (Hint: Consider that the ice floats as fully submerged at the limit condition.)

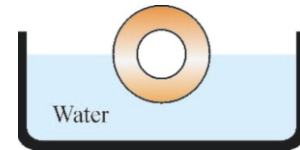
[A: 12.5 m^3 .]



Problem 11.12

11.12 The scale in the figure shows 2.4 kg when a container of water is placed on it. A piece of iron with a mass of 780 g and a density of 7.8 g/cm^3 is tied to a rope and suspended inside of the water. What will the scale show?

[A: 2.5 kg.]



Problem 11.13

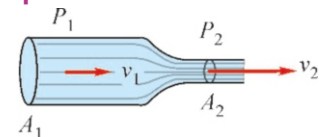
11.13 An empty spherical shell made of plastic has an internal radius of 5 cm and an external radius of 6 cm. This object floats half-submerged when dropped into the water. Calculate the density of the plastic.

[A: 1.2 g/cm^3 .]

11.14 A block suspended on a spring scale shows 1.4, 1.9 and 2.5 kg, respectively, when suspended in water, oil and an unknown liquid. As the density of water is 1000 kg/m^3 and the density of oil is 900 kg/m^3 , what is the density of the unknown liquid?

[A: 780 kg/m^3 .]

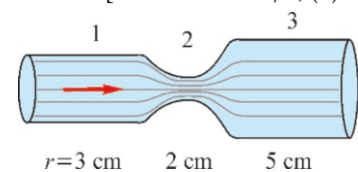
11.5 Bernoulli's Equation



Problem 11.15

11.15 The main pipe of a city's water network, shown in the figure, has a cross-sectional area $A_1=3 \text{ m}^2$ and another pipe connected to it has $A_2=1 \text{ m}^2$. The water flowing through cross-section A_1 has a flow rate of $Q=15 \text{ m}^3/\text{s}$. (a) Calculate the flow speed at each cross-section. (b) If the pressure is $P_1=8 \text{ atm}$ at the first cross-section, what will the pressure be at the second?

[A: 5 and 15 m/s, (b) 7 atm.]



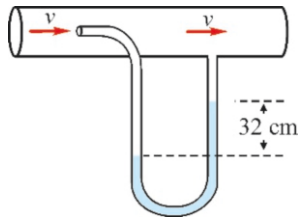
Problem 11.16

11.16 The radii of the three separate cross-sections of the pipe in the figure are $r_1=3$, $r_2=2$ and $r_3=5 \text{ cm}$. Considering that the flow velocity of the water at the first cross-section is 2 m/s, calculate (a) the velocities in the other two cross-sections and (b) the flow rate of the water.

[A: (a) $v_2 = 4.5$, $v_3 = 0.72 \text{ m/s}$, (b) $0.006 \text{ m}^3/\text{s}$.]

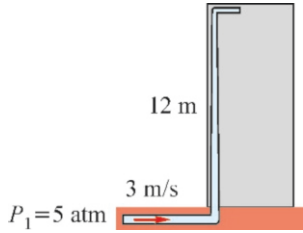
11.17 The flow rate of water in a fire hose is constant at $1 \text{ m}^3/\text{s}$. Water is to be sprayed on a fire located at a height of 40 m using a metal nozzle attached to the end of the hose. What should the maximum radius of the nozzle be such that the water can reach the fire?

[A: 0.11 m.]

**Problem 11.18**

11.18 The Pitot tube in the wing of an aircraft contains alcohol. What is the speed of the aircraft with respect to the air if the alcohol column rises by 32 cm? The density of alcohol is 800 kg/m^3 and the density of air is 1.2 kg/m^3 .

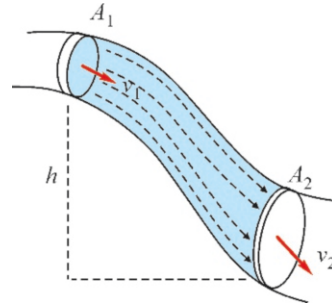
[A: 65 m/s.]

**Problem 11.19**

11.19 Water is pumped with pressure $P_1=5 \text{ atm}$ and a speed

of 3 m/s into a pipe with a cross-section area of 5 cm^2 at the entrance to a building. What are the speed and pressure of the water flowing from a tap with a cross-section of 1.5 cm^2 at a height of 12 m on the 3rd floor?

[A: 10 m/s and $3.41 \times 10^5 \text{ Pa} = 3.37 \text{ atm}$.]

**Problem 11.20**

11.20 Water is flowing at a speed of 3 m/s from the cross-section of a pipe with an area $A_1=4 \text{ cm}^2$ located at a height of $h=12 \text{ m}$. If the pressure at the top cross-section is 1.2 atm , what is the pressure at the bottom cross-section with an area $A_2=5 \text{ cm}^2$?

[A: $2.4 \times 10^5 \text{ Pa} = 2.4 \text{ atm}$.]

12

TEMPERATURE AND HEAT



Temperature in shade can rise up to 50°C at the Sharqiya Sands in Oman. Only camels can survive such high temperatures.

What is the meaning of *temperature* in terms of atoms? How does temperature change lead to changes in physical properties?

Up until now we have consistently examined the motion of a small number of particles. However, this gets difficult in systems with an enormous number of particles. Consider a glass of water. There are approximately 10^{24} molecules in this glass. In principle, we can write Newton's laws for each molecule and find the solutions. However, solving that great number equations will take years, even with the fastest computers.

Suppose that we solved that many equations and found the positions and velocities of each molecule. What purpose would that much information serve? On the other hand, 3-5 quantities are sufficient to specify the state of a glass of water: Temperature, volume, pressure, etc. These are quantities that specify the system at hand as a whole, in other words, macroscopic quantities.

The branch of science that macroscopically examines the interactions of a system with its surroundings is called **Thermodynamics**. We will learn the basic concepts of thermodynamics in this chapter and apply them in the next chapter.

12.1 THERMODYNAMIC EQUILIBRIUM AND TEMPERATURE

Quantities that specify a system as a whole are called **state variables**. These variables include temperature T , pressure P , volume V , mass m , and internal energy E , which we will discuss later. No matter how complex the internal structure of a macroscopic system is, it is possible to describe the system as a whole using a limited number of state variables.

Consider a cup of tea. If we leave the tea as it is, it will start to cool down and reach room temperature after a while. If we mix it using a spoon, the rotational motion of the liquid will stop after a while. Likewise, if we put some sugar in tea and mix it, the concentration of sugar will distribute evenly in the tea after a while.

So, *if the state variables of a system do not vary over time and have the same value throughout the system, this state is called **thermodynamic equilibrium**.*

Temperature

When we touch an object, we feel it to be “hot” or “cold.” Also, we know that, if we put two samples of water of different temperature into a container and wait, their temperatures will become equal.

As the most basic thermodynamic quantity, temperature is the quantitative measure of thermodynamic equilibrium. Let us take two objects A and B and put them in contact with each other, insulating them from the external environment (Figure 12.1). *These two objects will have equal temperature once they reach thermodynamic equilibrium.*

This definition specifies temperature macroscopically. Microscopic definition, however, is the subject of the branch of physics called Statistical Physics. Briefly stated, *temperature is a measure of the average kinetic energy of molecules that constitute a system.*

Temperature Measurement – Thermometer

As the temperature of an object changes, another one of its quantities will inevitably change as well. For example, a heated rod of iron will expand, the pressure of a heated gas will increase and the electrical resistance of a heated wire will increase. These changes help us to measure temperature and define a unit for it.

The instrument used to measure temperature is called a **thermometer**. The most commonly known type is the **mercury thermometer**. The liquid mercury inside of a chamber located at the bottom expands in proportion with the temperature and rises in a column. Temperature can be calculated from the height of the mercury column. (Today, dyed alcohol is used instead, because mercury is a toxic substance.)

Figure 12.1: Two objects insulated from the environment will have equal temperature when they reach equilibrium.

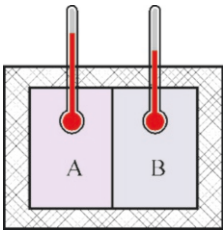
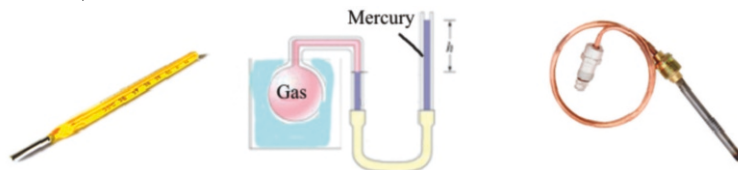


Figure 12.2: Types of thermometer: mercury thermometer, constant volume gas thermometer, thermocouple.



Another type of thermometer is the **constant-volume gas thermometer**. In this instrument, a gas inside of a glass chamber is trapped with mercury filling a U-tube (Figure 12.2). When the gas inside of the chamber is placed inside of a liquid and heated, its volume and pressure increases. However, the mercury

column is adjusted so that the volume remains constant and only the pressure increases. The temperature can again be measured from the height of the mercury column.

Yet another type of thermometer called a **thermocouple**, is used in technology and laboratories. Thermocouple has two ends made of two different metals (copper, iron or tungsten) or alloys (constantan, alumel or nisil). A voltage difference occurs between these two ends when one is kept at a standard temperature (0°) and the other one is brought to a different temperature. The temperature is determined by measuring this voltage, which is proportional to temperature.

Temperature Scales – Celsius, Fahrenheit, Kelvin

To define a temperature scale, we must first decide on two reference temperatures that can easily be realized by anyone. A temperature scale is formed by dividing the interval between these two temperatures into equal parts. There are three commonly used temperature scales in everyday life, technology and science: the Celsius ($^\circ\text{C}$), the Fahrenheit ($^\circ\text{F}$), and the Kelvin (K) scales.

In the Celsius scale, the freezing point of water under 1 atmosphere pressure is taken as zero degrees Celsius (0°C) and the boiling point of water as 100°C . The distance between these two points is divided into 100 equal parts.

In the Fahrenheit scale, the freezing point of water under 1 atmosphere pressure is taken as 32 degrees Fahrenheit (32°F) and the boiling point as 212°F . The distance between these two points is divided into 180 equal parts.

The conversion formula between the Celsius and Fahrenheit scales is:

$$T_C = \frac{5}{9} (T_F - 32^\circ\text{F}) \quad (12.1)$$

The Celsius and Fahrenheit scales both require two constant points.

The **absolute temperature scale**, or the **Kelvin scale**, which requires only one fixed point, was adopted in modern science. The unit of temperature in the SI unit system is Kelvin and is indicated as K (the $^\circ$ sign is not used). The fixed point of this scale is the point called the **triple point of water**, at which all three states (water-ice-steam) of water coexist. The property of the triple point is that it occurs at a single value of temperature and pressure.

In the Kelvin scale, the temperature of the triple point is fixed as follows:

$$T_K = 273.16 \text{ K} \quad (\text{Triple point temperature}) \quad (12.2)$$

The relation between the Kelvin and Celsius scales is as follows:

$$T_K = T_C + 273.15 \quad (12.3)$$

The difference in the last digit between these two formulas may seem odd, but it is, in fact, correct, because the triple point of water is actually at temperature 0.01°C . We will take it as zero in practice in this course:

$$0^\circ\text{C} \approx 273 \text{ K} \quad 100^\circ\text{C} \approx 373 \text{ K} \quad (\text{in this course}) \quad (12.4)$$

The zero point of the Kelvin scale, in other words, the temperature -273.16°C , is the absolute zero of the universe. At this temperature, all motion stops in the universe and lower temperatures are meaningless.

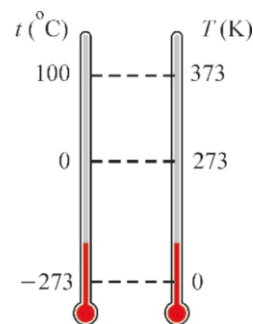


Figure 12.3: Celsius and Kelvin scales.

12.2 HEAT

A pot of water starts to get warmer when put over the fire on a gas stove. Here, the combustion energy of gas is transferred into the water. This energy can be given to the water because the burning gas is hotter than water. Therefore, the main reason for this energy transfer is temperature difference.

The type of energy that a system gives or receives due to a temperature difference with its environment is called **heat**. Heat is indicated with Q .

The unit of heat is the Joule (J), because it is a measure of energy. The **calorie** (cal) unit used in industry has the following value in terms of joules:

$$1 \text{ cal} = 4.186 \text{ J}$$

The expression “calorie” used for food in daily life is actually the kilo-calorie (kcal) unit.

We must emphasize a point here: Heat is always something that is given or taken. A system will not have heat as one of its properties, in other words, it is wrong to say “the heat of the water”; it is proper to say “taken heat” or “given heat.” In terms of thermodynamic language, heat is not a state variable.

Specific Heat

Objects use the heat that they receive in various ways: The temperature of the object may increase, it may start to boil or expand, etc. Let us consider these respectively.

Every object has a different rate of temperature change upon heating. A coefficient called the specific heat is defined to indicate this feature. If Q is the amount of heat required to increase the temperature of the mass m of an object by ΔT , then the coefficient

$$c = \frac{Q}{m \Delta T}$$

is called the **specific heat** of the object.

$\Delta T = T_2 - T_1$ is the temperature increase here. In other words, specific heat is the amount of heat required to increase the temperature of 1 unit of mass per 1 unit of temperature. The unit of specific heat is J/kg·K.

Based on this definition, the amount of heat required for a ΔT increase in temperature is expressed as follows:

$$Q = mc \Delta T \quad (12.5)$$

This formula is used in the technique called calorimetry. To find the final temperature when different liquids are mixed, the algebraic sum of received and given heats is set to zero.

The specific heats of various materials are given in the following table. Specific heat varies slightly with temperature, but can be taken as constant for small ΔT variations.

	Specific heats				
	c (J/kg·K)	c (cal/g·°C)		c (J/kg·K)	c (cal/g·°C)
Copper	390	0.092	Water	4186	1.000
Aluminum	900	0.215	Ice	2100	0.5
Gold	129	0.030	Water vapor	2010	0.48
Silver	234	0.056	Ethyl alcohol	2400	0.58
Iron	448	0.107	Glass	837	0.200

Note that the highest specific heat in this table belongs to water. So much energy is required to heat or cool water that it prevents sudden temperature variations in neighboring environments. In other words, water makes the surrounding climate temperate.

Here, we defined specific heat per mass; molar specific heat, which is more convenient for gases, will be defined later. In defining specific heat for gases, separate specific heats are defined depending on the type of heating. The constant-volume specific heat c_V is defined if heating takes place under constant volume, and the constant-pressure specific heat c_P is defined if it takes place under constant pressure.

Latent Heat

If we continue to heat water at 100° C it will start to boil. The system will continue to receive heat during this change of state, but surprisingly, its temperature will not change. The received heat is used to convert water at 100° C into vapor at 100° C. This is called a **change of phase**.

Another phase change is observed during the melting of ice. If we give heat to ice at 0° C, it will start to melt and turn into water at 0° C. We say that there is a *latent* heat if no temperature change occurs during a phase change.

The heat received or given by a unit mass of a system during a change of phase is called the **latent heat**. Latent heat is indicated with L . According to this definition, the heat received or given by an object with mass m is

$$Q = mL \quad (L : \text{Latent heat}) \quad (12.6)$$

L will have a positive sign if the object is vaporizing or melting, and a negative one if it is condensing or freezing. The heat received by the system is always considered to be positive. Although many types of latent heats are defined in thermodynamics, here we will only consider the latent heat of fusion (melting) and the latent heat of vaporization. Some latent heats of fusion and vaporization are provided in the following table:



Figure 12.4: Temperature always remains constant when water boils.

	Latent heats of fusion and vaporization					
	Fusion point (°C)	Heat of fusion		Boiling point (°C)	Heat of vaporization	
		(cal/g)	(kJ/kg)		(cal/g)	(kJ/kg)
Water	0	80	333	100	540	2260
Ethyl alcohol	-114	25	104	78	204	850
Lead	327	6	25	1750	208	870
Silver	961	21	88	2193	558	2300
Tungsten	3410	44	184	5900	1150	4800

Example 12.1

(Note: The specific heats required in the problems in this chapter will be taken from the tables above.)

- (a) Convert the temperatures 100 °F and -30 °F into °C.
 (b) Convert the temperatures 100 °C and -30 °F into Kelvin units.

Answer

(a) We subtract 32 from the Fahrenheit value, divide by 9 and

multiply by 5, obtaining the following:

$$T_C = \frac{5}{9} (T_F - 32^\circ\text{F})$$

$$\text{For } T_F = 100^\circ\text{F}, \quad T_C = 38^\circ\text{C}$$

$$\text{For } T_F = -30^\circ\text{F}, \quad T_C = -34^\circ\text{C}$$

(b) Celsius temperatures are converted into Kelvin by adding 273:

$$\text{For } T_C = 100^\circ\text{C}, \quad T_K = 100 + 273 = 373\text{ K}$$

$$\text{For } T_C = -30^\circ\text{C}, \quad T_K = 243\text{ K}$$

Example 12.2

How much heat is required to increase the temperature of iron with a mass of 250 g from 20 °C to 80 °C?

Answer

We use the formula (12.5):

$$Q = mc \Delta T$$

It is more convenient to make such calorimetry calculations in calorie and gram units. At the end of the calculation, we can take 1 cal = 4.18 J and convert it into the joule unit.

The specific heat of iron is given as $c = 448\text{ J/kg} = 0.11\text{ cal/g}$ in the table. We calculate by substituting the numerical values:

$$Q = 250 \times 0.11 \times (80 - 20) = 1650\text{ cal}$$

$$Q = 1650 \times 4.18 = 6900\text{ J}$$

Example 12.3

200 g -water at a temperature of 20 °C is mixed with 100 g -alcohol at a temperature of 30 °C. What will the final temperature of the mixture be?

Answer

We take the specific heat of water from the table as $c_1 = 1\text{ cal/g}$ and that of alcohol as $c_2 = 0.58\text{ cal/g}$. If we show the final temperature with T , the algebraic sum of the heats received

by the water and the alcohol should be zero:

$$Q_1 + Q_2 = 0$$

$$m_1 c_1 (T - T_1) + m_2 c_2 (T - T_2) = 0$$

Notice that one of the received heats (Q_2) will be negative. We calculate the final temperature by substituting the numbers:

$$200 \times 1 \times (T - 20) + 100 \times 0.58 \times (T - 30) = 0$$

$$T = 22^\circ\text{C}$$

Example 12.4

How much heat is required to turn a block of ice with a mass of 50 g at a temperature of -30 °C into water at 20 °C?

Answer

In this problem, latent heat is also required during the melting of the ice. From the table, we take the latent heat of fusion as $L = 80\text{ cal/g}$ and the specific heat as $c_1 = 0.5$ for ice and $c_2 = 1\text{ cal/g}$ for water.

Let us write the stages of the whole process:

Ice at -30 °C → ice at 0 °C → water at 0 °C →
 → water at 20 °C.

We write the heats required at each stage and add them up:

$$Q = mc_1[0 - (-30)] + mL + mc_2(T - 0)$$

We calculate the result by substituting the numerical values:

$$Q = 50 \times [0.5 \times 30 + 80 + 1 \times 20]$$

$$Q = 5750\text{ cal}$$

Example 12.5

Some amount of water vapor at a temperature of 110°C is mixed with a block of ice with a mass of 30 g at a temperature of -20°C in an insulated container. How much vapor was used, considering that the final temperature of the mixture is 20°C ? (The mass of the container can be neglected.)

Answer

As they all turn into water in the end, the ice and the water vapor have undergone phase changes to reach their final states. From the table, we take the values for latent heat of fusion as $L_1=80$ cal/g, the latent heat of vaporization as $L_2=540$ cal/g,

the specific heat of ice as $c_1=0.5$, of water as $c=1$ and of vapor as $c_2=0.48$ cal/g.

The heat received by the ice with mass m_1 is set to be equal to the heat given by the vapor with mass m_2 :

$$\begin{aligned} Q_1 &= Q_2 \\ m_1 c_1 [0 - (-20)] + m_1 L_1 + m_1 c (20 - 0) &= \\ &= m_2 c_2 (110 - 100) + m_2 L_2 + m_2 c (100 - 20) \end{aligned}$$

We find the amount of vapor by substituting the numbers:

$$\begin{aligned} 30 \times [0.5 \times 20 + 80 + 1 \times 20] &= \\ m_2 [0.48 \times 10 + 540 + 1 \times 80] & \\ m_2 &= 5.3 \text{ g.} \end{aligned}$$

Example 12.6

100 g of ice at a temperature of -10°C is put inside 500 g water at a temperature of 0°C in an insulated container. What will the final state be?

Answer

The fact that the amount of ice increases in this problem may seem odd at first glance, but it is true. Ice (at -10°C) will receive heat from water because it is colder. Water at 0°C can only give heat by turning some of it into ice.

Let us assume that the final temperature will be 0°C , with water and ice coexisting. We set the heat received by the ice

with mass m_1 as equal to the heat given by some m_2 amount of water turning into ice. (If, at the end of the calculation, this value of m_2 turns out to be higher than the total amount of water, it means that the final temperature will be less than zero. In that case, we will use a different route.) There follows:

$$m_1 c_1 [0 - (-10)] = m_2 L$$

We calculate by taking the specific heat of ice as $c_1=0.5$ cal/g and the latent heat of fusion as 80 cal/g:

$$100 \times 0.5 \times 10 = m_2 \times 80 \rightarrow m_2 = 6.3 \text{ g.}$$

According to this result, there will be $100 + 6.3 = 106.3$ g of ice and $500 - 6.3 = 493.7$ g of water at a temperature of 0°C in the container.

12.3 THERMAL EXPANSION

We know from our daily lives that heated objects increase their volume, or expand. We have seen that the mercury heated inside of a mercury thermometer expands when heated, cables in electrical transmission lines sag on hot days, heated balloons expand, etc. Expansion may sometimes produce unintended results (Figure 12.5). A margin is left in the construction of building, bridges and roads in order to account for this.

If an object with length L_0 at a reference temperature T_0 , has the length L at another temperature T , its linear expansion $\Delta L = L - L_0$ will be proportional to the increase in temperature $\Delta T = T - T_0$ and to its initial length L_0 :

$$\Delta L = \alpha L_0 \Delta T$$

Here, α is the **coefficient of linear expansion** and depends on the type of material. Its unit is $(1/^\circ\text{C})$ or $(1/\text{K})$. If we write this formula in terms of the final length L of the object, we get

$$\begin{aligned} L &= L_0 + \Delta L \\ L &= L_0 [1 + \alpha(T - T_0)] \end{aligned} \quad (12.7)$$

The coefficient α is taken as a constant in calculations, although it varies slightly with temperature.

A similar behavior is observed in volume expansion. The increase in volume ΔV caused by temperature increase ΔT is

$$\Delta V = \beta V_0 \Delta T \quad (12.8)$$



Figure 12.5: Railroad tracks that have expanded due to heat on the Delaware river (1935).

and the constant β is called the **coefficient of volume expansion**. Rigid bodies have the same expansion property in three dimensions, therefore the relation between the α and β coefficients is as follows:

$$\beta \approx 3\alpha \quad (12.9)$$

The expansion coefficients of some materials are given in the following table:

Coefficients of linear and volume expansion of materials at 20 °C			
Solids	α (°C) ⁻¹	Liquids and gases	β (°C) ⁻¹
Copper	17×10^{-6}	Water	2.1×10^{-4}
Iron or steel	12×10^{-6}	Ethyl alcohol	11×10^{-4}
Aluminum	23×10^{-6}	Mercury	1.8×10^{-4}
Concrete	12×10^{-6}	Gasoline	9.5×10^{-4}
Glass	9×10^{-6}	Air	34×10^{-4}

The most striking feature in this table is the fact that the expansion coefficients of iron, steel and concrete are equal. The fact that buildings made of steel and concrete can withstand very high temperature changes without cracking is only possible with this property of concrete.

Examining the table, the expansion coefficients of iron and concrete may seem to be small (on the order of one ten thousandths), but this may result in considerable expansion. In a climate with a $[-20, +40 \text{ °C}]$ temperature range, such a difference in length means 1 cm per 10 meters. This is taken into account in the construction of buildings, bridges and railroads with margins being left.

Also note that, in the table, the expansion coefficient of alcohol is much greater than those of water and mercury; this is why alcohol is preferred in thermometers. Gasoline is also observed to expand as much as alcohol. For this reason, an expansion margin should always be left in the tank of vehicles when filling with gasoline.

Anomalous behavior of water: The expansion of water exhibits an unusual behavior in the temperature range $[0, 4 \text{ °C}]$. The volume of water at 0 °C starts to decrease when heated. This anomalous behavior continues until 4 °C. Then, it starts to expand with temperature. In other words, water is densest at 4 °C. This property allows fish and other living beings to stay alive under ice in frozen lakes. This unusual behavior of water is caused by the special structure of H₂O molecules.



Figure 12.6: Expansion margin in a bridge.

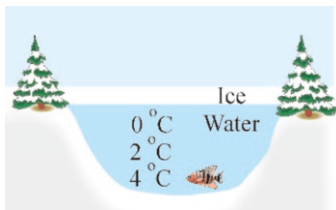


Figure 12.7: This is how life goes on in lakes during winter.

Example 12.7

A steel bridge with a length of 1200 m at 0 °C will expand by how much in length at 50 °C?

Answer

We use the formula (12.3), giving the increase in length:

$$\Delta L = \alpha L_0 \Delta T$$

The coefficient of expansion of steel is given as $\alpha = 12 \times 10^{-6}$ in the table. We substitute the values:

$$\Delta L = 12 \times 10^{-6} \times 1200 \times (50 - 0)$$

$$\Delta L = 0.72 \text{ m} = 72 \text{ cm}$$

Example 12.8

A copper vessel with a volume of 5 liters at 20 °C is completely

filled with water and heated on a stove. How much water will spill at a temperature of 95 °C?

Answer

When two objects expand together, it is more convenient to work with $(\alpha_2 - \alpha_1)$, the difference of their expansion coefficients.

Accordingly, if we use ΔV_1 to indicate the increase in the boiler's volume and ΔV_2 for the increase in the water's volume, according to the formula (12.8), we get

$$\Delta V_2 - \Delta V_1 = \beta_2 V_0 \Delta T - \beta_1 V_0 \Delta T$$

The coefficient of volume expansion is $\beta \approx 3\alpha$ in terms of linear expansion. From here, the expression for the difference in volumes becomes:

$$\Delta V_2 - \Delta V_1 = 3(\alpha_2 - \alpha_1)V_0 \Delta T$$

We find α coefficients for copper and water from the table and substitute to calculate the result:

$$\begin{aligned} \Delta V_2 - \Delta V_1 &= 3 \times (210 - 17) \times 10^{-6} \times 5 \times 10^{-3} \times (95 - 20) \\ \Delta V_2 - \Delta V_1 &= 0.00022 \text{ m}^3 = 220 \text{ cm}^3 \end{aligned}$$

Example 12.9

A simple pendulum used to measure time is made of a mass attached to the end of a steel wire. The period of the pendulum correctly shows 1 s when the temperature is 0°C . Calculate how much time the clock will lose or gain in 1 day in a hot country where the average temperature is 40°C ?

Answer

(a) We had calculated the period of a simple pendulum in Chapter 9 (Eq. 9.23). However, let us use t here to show the period so that it will not get confused with the temperature T :

$$t = 2\pi \sqrt{L/g}$$

Let us take the ratio of periods t_0 and t at temperatures 0°C and 40°C , respectively, and use the thermal expansion formula:

$$\frac{t}{t_0} = \sqrt{\frac{L}{L_0}} = \sqrt{\frac{L_0(1 + \alpha \Delta T)}{L_0}} = \sqrt{1 + \alpha \Delta T}$$

We take $\alpha = 12 \times 10^{-6}$ for steel and $t_0 = 1$ s and calculate the new period for the temperature difference $\Delta T = 40^\circ\text{C}$:

$$t - t_0 = \sqrt{1 + 12 \times 10^{-6} \times 40} - 1 = 24 \times 10^{-6} \text{ s}$$

This difference is calculated for one day:

$$24 \times 3600 \times (t - t_0) = 20 \text{ s}$$

The pendulum lags by 20 s per day.

Example 12.10

A steel ruler measures correctly at a temperature of 0°C . This ruler is used to measure the length of an aluminum rod at a temperature of 0°C and it reads as 2.800 m. (a) How much will the ruler measure the length of the aluminum rod when the temperature of both the steel ruler and the aluminum rod is increased to 400°C ? (b) How much has the length of the aluminum rod really expanded?

Answer

(a) This problem can be solved in various ways: We find the amount of expansion of the steel and aluminum and then find their proportion. However, the easier method is to work with the coefficient difference $(\alpha_2 - \alpha_1)$ for jointly expanding objects:

Of the two objects with the same length $L_0 = 2.8$ m, if the coefficient of expansion of the steel ruler is $\alpha_1 L_0 \Delta T$ and that

of the aluminum rod is $\alpha_2 L_0 \Delta T$, then the difference in expansion will be $(\alpha_2 - \alpha_1)L_0 \Delta T$. We can reach the same result by considering that the steel ruler is not expanding, but the aluminum rod is expanding with a coefficient of $(\alpha_2 - \alpha_1)$. However, this time, the marks of the steel ruler will remain at the same place and correctly show the relative expansion of the aluminum.

Therefore, we calculate the relative expansion of aluminum using the coefficient $(\alpha_2 - \alpha_1)$:

$$L = L_0 [1 + (\alpha_2 - \alpha_1) \Delta T]$$

We look up the expansion coefficients from the table and calculate:

$$L = 2.800 \times [1 + (23 - 12) \times 10^{-6} \times 400] = 2.812 \text{ m}$$

The expanded ruler shows the length of the expanded rod as 2.812 m.

(b) We calculate the absolute expansion of the aluminum:

$$L = L_0 [1 + \alpha_2 \Delta T] = 2.8 [1 + 23 \times 10^{-6} \times 400] = 2.826 \text{ m}.$$

12.4 HEAT TRANSFER

The conduction of heat from one medium to another can take place through three different mechanisms: conduction, convection and radiation. In conduction, heat is conducted without any displacement of matter. In convection, heat is transferred by the displacement of material. And in radiation, heat is conducted through electromagnetic waves.

Now let us examine in turn these mechanisms:

Conduction

If we hold a copper rod at one end and place the other end on the heat of a stove, we will soon feel heat in our hand. This is because, heat is transmitted through *conduction* from one end to the other end of the rod. We have to examine this at the atomic level to understand how it occurs.

There are two separate mechanisms for heat transmission at the atomic level:

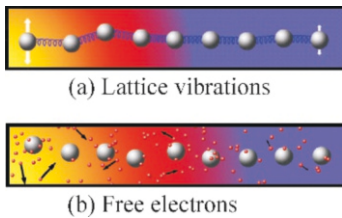


Figure 12.8:

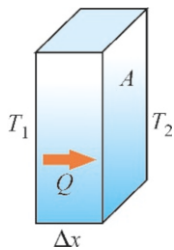


Figure 12.9: Heat transfer through conduction.

- **Lattice vibrations** (Figure 12.8a): Copper atoms may only oscillate about their fixed positions. However, their vibration amplitude, and thus their energy, increases with the heat that they receive. They transmit some of this energy to their neighboring atoms. A vibration wave thus reaches one end of the copper from the other end.
- **Free electrons** (Figure 12.8b): As we will discuss when addressing electricity (Section 14.1), some of the electrons in the atoms of conductors, such as copper and steel, become free in a solid medium. They therefore carry energy by freely traveling from one end of the copper to the other. (This is why metals that are good electric conductors are also good heat conductors.)

The amount of heat transferred by a medium depends on many factors: the thickness of the medium, temperature difference, etc. To examine this, let us consider a medium with thickness Δx and cross-section area A (Figure 12.9). Let one of the surfaces of this medium be kept at temperature T_1 and the other at temperature T_2 . Accordingly, the amount of heat Q transferred from one region into the other in time t is

$$\frac{Q}{t} = kA \frac{\Delta T}{\Delta x} \quad (12.10)$$

The coefficient k here depends on the type of material and is called the **thermal conductivity**. Its unit is $\text{watt}/\text{m}\cdot^\circ\text{C}$. The ratio of the temperature difference $\Delta T = T_2 - T_1$ to the thickness Δx in this formula is called the **temperature gradient** $\Delta T/\Delta x$.

Thermal conductivities of some materials are given in the following table:

Thermal conductivities of some materials			
	k (W/m $\cdot^\circ\text{C}$)		k (W/m $\cdot^\circ\text{C}$)
Lead	35	Air	0.024
Iron	80	Wood	0.1
Aluminum	238	Water	0.6
Copper	397	Glass	0.8
Silver	427	Concrete	0.8

The larger the coefficient k is, the better the heat is transferred. We want heat conductivity to be low or high when selecting a material for any job in technology or daily life. A material with good heat conductivity is required for heating food on a stove. For this reason, we use copper and aluminum pots. On the other hand, poor conducting materials should be used to insulate houses against the cold and prevent heat losses. Therefore, glass (or, even better, double glazing with a layer of air between two glass panes) is used.

As seen in the table, metals such as copper, silver and aluminum, that are good electrical conductors are also good heat conductors. This is because the free electrons in the structure of metals also conduct energy while conducting electrical charge.

The table shows that air is a poor conductor. This is true for still air. Hence, the air trapped in double glazing or the material called styrofoam severely decreases the heat conductivity of that material. The *super insulators* recently developed in the aeronautics industry are also spreading into our daily life (Figure 12.10).

Convection

We cool off with the cool air blown by a fan or an air conditioner on a hot day. Here, heat is transferred by the motion of air molecules. Heat transfer through the displacement of matter is called **convection**.

Convection can occur in two ways. The air heated around a stove or heater will expand to have less density, and thus, start to rise. It is replaced by the cooler air above, which also soon gets heated and rises. A convection current is thus formed automatically. This is called *natural convection*.

The second type of convection is *forced convection*. The cool air generated by a fan or air conditioner is pushed in the desired direction by a fan motor. Although still air is a poor conductor, it turns into a good conductor through convection.

The most important example of convection is the currents formed in the atmosphere and the oceans. The natural convections of air and water are two of the most important factors that make all kinds of life possible.

The detailed examination of convection is complex, and the transferred heat depends on many factors, such as type of material, velocity distribution, pressure and temperature conditions in the environment, etc. It is examined in detail in the branch of science called Fluid Mechanics.

Radiation

The third heat transfer mechanism is the type of heat transferred through electromagnetic waves. Its most important feature is its ability to transfer heat in vacuum, without the presence of material in the medium. The solar rays that pass through the empty space between the Earth and the Sun, which has no conducting medium, to heat the Earth and are the source of all life.

All bodies radiate at every temperature. However, we can only see that part that is in the visible light spectrum. Pieces of coal in a grill radiate by emitting red light. When the coals cool down, the radiation continues as infrared waves, but we cannot see them. Today, binoculars and cameras that operate with infrared rays can be used to view the radiation of all objects.

The relation between radiated energy and heat is given by the **Stefan-Boltzmann law**:

$$\frac{Q}{t} = \sigma A \epsilon T^4 \quad (\text{Stefan-Boltzmann law}) \quad (12.11)$$

Here, A is the surface area of the object and T is the temperature in Kelvin units. As for the other two coefficients, $\sigma = 5.7 \times 10^{-8} \text{ W/m}\cdot\text{K}^4$, the *Stefan-Boltzmann constant* is the same for all objects. The other dimensionless coefficient ϵ is called *emissivity*. This is a parameter that can take values within the range $[0, 1]$ and depends on the type of radiating material.

The transfer of radiated energy between bodies with different temperatures is obtained using this formula. If a body with temperature T_1 is kept in an



Figure 12.10: The superconductor aerogel developed by NASA.

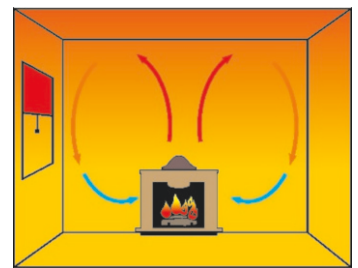


Figure 12.11: Air convection in a heated room.



Figure 12.12: Radiation of hot coal.



Figure 12.13: A cat in the night through infrared binoculars.

environment with temperature T_2 , the radiated power is

$$\frac{Q}{t} = \sigma A \epsilon (T_1^4 - T_2^4) \quad (12.12)$$

It is remarkable that the radiated energy is proportional to the fourth power of temperature. Heat losses through radiation increase to a tremendous degree at high temperatures.

Heat transfer is a very extensive topic. It will be sufficient in this course to know only the basic concepts.

12.5 IDEAL GAS

Gases have a special place in thermodynamics, because they can easily exchange heat and energy with their environment over a wide range of temperature and pressure. For this reason, they play a role in many fields of industry, through the use of gas fuels, gas turbines, steam boilers, etc. In this section, we will discuss the basic concepts that describe the properties of gases.

Avogadro's Number and Mole

A constant called **Avogadro's Number** is used to specify the amount of material in terms of atomic scale. The value of this number is:

$$N_A = 6.022 \times 10^{23} \quad (\text{Avogadro's number}) \quad (12.13)$$

The amount of a material containing Avogadro's number of atoms (or molecules) is called **1 mole** (symbol: mol). In other words, 1 mole of carbon contains N_A number of carbon atoms, and 3 mole of water contains $3N_A$ number of H_2O molecules.

The number of moles in a given amount of gas is indicated with n . Accordingly, if the given number atoms (or molecules) of a substance is N , the amount of moles n that it contains is:

$$n = \frac{N}{N_A}$$

The total mass of 1 mole of atoms (or molecules) of a substance is called the **molar mass** and is indicated with M . If the given mass of a substance is m , the amount of moles n that it contains is found as follows:

$$n = \frac{m}{M}$$

For example, to find the number of moles in 100 grams of oxygen (O_2), we divide it by the molar mass of oxygen $M = 32 \text{ g}$:

$$n = \frac{m}{M} = \frac{100 \text{ g}}{32 \text{ g}} = 3.125 \text{ mol}$$

The molar masses of some materials are given in the following table:

Molar masses of some materials			
Gases	M (gram/mole)	Solids and liquids	M (gram/mole)
Hydrogen (H_2)	2	Water (H_2O)	18
Helium (He)	4	Aluminum (Al)	27
Oxygen (O_2)	32	Iron (Fe)	56
Nitrogen (N_2)	28	Copper (Cu)	64
Argon (Ar)	40	Silver (Ag)	108
Carbon dioxide (CO_2)	44	Mercury (Hg)	201
Air	29	Lead (Pb)	207

Ideal Gas Law

It is impossible to exactly determine the microscopic behavior of a gas, because it contains approximately Avogadro's number of molecules. However, working with the **ideal gas assumption** gives us very good results that are close to experimental reality. As a definition, the ideal gas is one in which the intermolecular interactions are neglected. For this, *the gas must have low pressure and high temperature*.

The equation that determines the relation between the mass, temperature, volume and pressure of a gas in any state is called the **equation of state**. The equation of state for ideal gas, also called the **ideal gas law**, is as follows:

$$PV = nRT \quad (\text{Ideal gas law}) \quad (12.14)$$

Here, P is the pressure, V is the volume, n is the number of moles and T is the temperature in *Kelvin units*. The proportionality constant indicated with R is called the **universal gas constant** and its value is

$$R = 8.31 \text{ J/mol}\cdot\text{K} \quad (\text{universal gas constant}) \quad (12.15)$$

The constant R is also expressed as follows for use in practical calculations in which the pressure is given in atmosphere units and the volume in liter units:

$$R = 0.082 \text{ liter}\cdot\text{atm/mol}\cdot\text{K} \quad (12.16)$$

If the mass m of the gas is given, taking its mole number as $n = m/M$, the ideal gas law can also be expressed as follows:

$$PV = \frac{m}{M} RT \quad (12.17)$$

If we wish to write the density ρ of the gas, we form the ratio $\rho = \text{mass/volume} = m/V$ in the last equation:

$$\rho = \frac{m}{V} = \frac{MP}{RT} \quad (12.18)$$

The ideal gas law allows us to see which quantity will vary when the state of a gas changes. For example, if the volume V of the gas is kept constant, the pressure will increase with the temperature. Likewise, when the pressure is kept constant, the volume of the gas will increase with the temperature.

When the temperature is kept constant, we get what is called *isothermal curves* (Figure 12.14). In isothermal changes, volume and pressure change inversely: One of them decreases when the other increases.

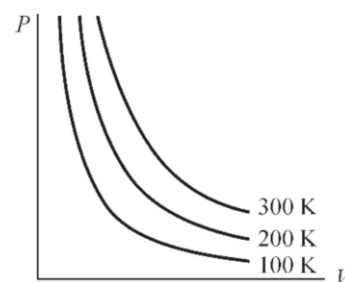


Figure 12.14: Isothermal curves.

Example 12.11

(The table on Page 211 will be used for the molar masses required in these problems.)

- (a) The molar mass of copper is 64 g. How many moles does 100 g of copper contain?
 (b) How many molecules does 100 g of oxygen contain?

Answer

(a) The number of moles is the number of M molar masses

in the mass m :

$$n = \frac{m}{M} = \frac{100}{64} = 1.6 \text{ moles.}$$

(b) A mole of mass contains N_A Avogadro's number of molecules. From the given mass, we calculate the number of moles n and then the number of molecules N . The molar mass of oxygen is 32 g, and we find the number of molecules as follows:

$$N = n N_A = \frac{m}{M} N_A = \frac{100}{32} \times 6.02 \times 10^{23} = 19 \times 10^{23}$$

Example 12.12

A container with a volume of 8 liters has 10 g of hydrogen gas (H_2) at a temperature of 27 °C.

- (a) How many moles of hydrogen are there in the container?
 (b) What is the pressure of the gas?

Answer

(a) We divide the given mass with the molar mass 2 g of H_2 :

$$n = \frac{m}{M} = \frac{10}{2} = 5 \text{ moles.}$$

(b) We solve the ideal gas law (12.14) for the pressure P :

$$PV = nRT \rightarrow P = \frac{nRT}{V}$$

Taking the constant R in units of liters \times atmosphere will be very convenient in these calculations:

$$R = 0.082 \text{ liter}\cdot\text{atm}/\text{mol}\cdot\text{K}$$

We substitute the number of moles and the other data we found in the previous item into the ideal gas law and calculate the pressure. Do not forget to convert temperature into kelvins.

$$P = \frac{nRT}{V} = \frac{5 \times 0.082 \times (273 + 27)}{8} = 15 \text{ atm}$$

Example 12.13

(a) How much volume will 1 mole of ideal gas hold at standard temperature and pressure (STP)?

(b) What is the density of 1 mole of oxygen (O_2) under STP conditions?

Answer

(a) The reference point called STP (standard T and P) for gases refers to 0 °C temperature and 1 atm pressure. In such a case, we calculate the volume of 1 mole of gas using the ideal gas law:

$$V = \frac{nRT}{P} = \frac{1 \times 0.082 \times 273}{1} = 22.4 \text{ liters}$$

All ideal gases have the same volume of 22.4 liters at STP.

(b) We use Eq. (12.17), which gives the ideal gas law in terms of masses:

$$PV = \frac{m}{M} RT$$

In this equation, we form density as $\rho = m/V$:

$$\rho = \frac{m}{V} = \frac{MP}{RT}$$

We take the STP temperature as 273 K and the pressure as 1 atm and use the molar mass of oxygen, which is 32 g. Using the constant R in units of liters \times atmospheres here, the other units simplify, and we get the density in terms of grams/liter:

$$\rho = \frac{32 \times 1}{0.082 \times 273} = 1.43 \text{ g/liter}$$

Example 12.14

Air is present under STP conditions in a room with volume $6 \times 10 \times 4 \text{ m}^3$.

- (a) How many moles of air is present in the room?
 (b) How many moles of air is left in the room if it is heated up to 27 °C?

Answer

(a) We solve the ideal gas law for the number of moles n :

$$PV = nRT \rightarrow n = \frac{PV}{RT}$$

Let us calculate in terms of the SI units this time. The volume of the room is $6 \times 10 \times 4 = 240 \text{ m}^3$ and, in STP conditions, 1 atmosphere of pressure equals 1 atm = $1.013 \times 10^5 \text{ Pa}$ in

terms of pascals, and we take the universal gas constant as $R = 8.31 \text{ J/mol}\cdot\text{K}$:

$$n = \frac{PV}{RT} = \frac{1.013 \times 10^5 \times 240}{8.31 \times 273} = 10700 \text{ mol}$$

(b) Using ratios is very convenient in problems that involve state changes of a gas. We write the ideal gas law for each temperature and compare them. As the pressures and volumes are equal, we get

$$\frac{P_2 V_2}{P_1 V_1} = \frac{R n_2 T_2}{R n_1 T_1} \rightarrow n_2 = \frac{T_1}{T_2} n_1$$

We substitute the values and calculate as follows:

$$n_2 = \frac{273}{300} \times 10700 = 9740 \text{ mol}$$

Example 12.15

The cylinder of a diesel motor with a volume of 4 lt contains air at a temperature of 27 °C and a pressure of 1 atm. The piston is pushed to decrease the volume of the air by a factor of 16 and increase the pressure by a factor of 40.

- (a) How many moles of air does the cylinder contain?
- (b) What will be the final temperature of the air?

Answer

(a) We calculate the number of moles from the ideal gas law:

$$n = \frac{PV}{RT} = \frac{1 \times 4}{0.082 \times 300} = 0.16 \text{ moles.}$$

Example 12.16

A cylinder contains an ideal gas at a temperature of 27 °C and a pressure of 5 atm. The gas is subject to the following processes after the same initial state each time:

- (a) Its volume is increased by a factor of 3 by heating under constant pressure. What will its final temperature be?
- (b) This time its pressure is increased by a factor of 4 by heating under constant volume. What will its final temperature be?
- (c) It is heated such that both its pressure and volume increase by factors of 3. What will its final temperature be?
- (d) Its absolute temperature is doubled by heating under constant volume. What will its final pressure be?

Answer

(a) It is sufficient to establish the ratios in this problem. If we write the ideal gas law twice for each process, only the changing variables will remain in the formulas.

The ratio is as follows for heating under constant pressure:

(b) If we indicate the initial variables with (P_1, V_1, T_1) and the final variables with (P_2, V_2, T_2) , we can write the ideal gas law for each state and divide on both sides:

$$\frac{P_2 V_2}{P_1 V_1} = \frac{nRT_2}{nRT_1} = \frac{T_2}{T_1}$$

We solve this expression for the final temperature T_2 :

$$T_2 = \frac{P_2 V_2}{P_1 V_1} T_1$$

We calculate the final temperature by substituting the numerical values:

$$T_2 = \frac{40}{1} \times \frac{1}{16} \times 300 = 750 \text{ K}$$

$$\frac{PV_2}{PV_1} = \frac{nRT_2}{nRT_1} \rightarrow \frac{V_2}{V_1} = \frac{T_2}{T_1}$$

From here, we calculate the final temperature:

$$T_2 = \frac{V_2}{V_1} T_1 = 3 \times T_1 = 3 \times 300 = 900 \text{ K.}$$

(b) The ratio of the change under constant volume is found the same way:

$$\frac{P_2}{P_1} = \frac{T_2}{T_1}$$

$$T_2 = \frac{P_2}{P_1} T_1 = 4 \times 300 = 1200 \text{ K.}$$

(c) The ratio is as follows when both the pressure and the volume change:

$$\frac{P_2 V_2}{P_1 V_1} = \frac{T_2}{T_1}$$

$$T_2 = \frac{P_2 V_2}{P_1 V_1} T_1 = 3 \times 3 \times 300 = 2700 \text{ K.}$$

(d) The ratio for heating under constant volume was found in item (b):

$$\frac{P_2}{P_1} = \frac{T_2}{T_1} \rightarrow P_2 = \frac{T_2}{T_1} P_1 = 2 \times 5 = 10 \text{ atm.}$$

Multiple-choice Questions

1. How many °C is a temperature of 373 K?
(a) 0 °C (b) 73 °C (c) 100 °C (d) 173 °C
2. Which of the following is not a unit of temperature?
(a) Fahrenheit
(b) Kelvin
(c) Centigrade
(d) Celsius
3. How will a pendulum clock made of metal wire show the time when temperature increases?
(a) It will lag.
(b) It will run too fast.
(c) It will not change.
(d) It is impossible to tell.
4. The specific heat of iron is approximately twice the specific heat of silver. How will their temperatures increase if the same amount of heat is given to pieces of iron and silver with the same mass?
(a) The iron will heat up more.
(b) The silver will heat up more.
(c) They will heat up equally.
(d) It is impossible to tell.
5. Two quantities of waters with equal mass at temperatures of 20 °C and 50 °C are mixed. What will the final temperature be?
(a) 25 °C (b) 30 °C (c) 35 °C (d) 40 °C
6. The absolute temperature of a gas in a container is doubled under constant volume. What will its pressure be?

- (a) $2P$ (b) $4P$ (c) $P/2$ (d) $P/4$
7. The absolute temperature of a gas in a container is doubled under constant pressure. What will its volume be?
(a) $2V$ (b) $4V$ (c) $V/2$ (d) $V/4$
8. The volume of a gas in a container is doubled under constant temperature. What will its pressure be?
(a) $2P$ (b) $4P$ (c) $P/2$ (d) $P/4$
9. The pressure of a gas in a container is doubled and its volume is halved. What will its temperature be?
(a) $2T$ (b) $4T$ (c) T (d) $T/2$
10. A gas at a pressure of 1 atm in a container of volume V has its temperature increased from 0°C to 273°C at constant pressure. What will its volume be?
(a) V (b) $2V$ (c) $V/2$ (d) $4V$
11. Which of the following is incorrect?
(a) The volume of a heated ice will decrease.
(b) Water heated at 10°C will expand.
(c) Heated water vapor will expand.
(d) Water heated at 2°C will expand.
12. If thermal expansion coefficients are sorted in increasing order, which sorting will be correct?
(a) Solid \rightarrow liquid \rightarrow gas.
(b) Liquid \rightarrow solid \rightarrow gas.
(c) Solid \rightarrow gas \rightarrow liquid.
(d) Gas \rightarrow liquid \rightarrow solid.
13. How many grams of substance are there in 2 moles of hydrogen (H_2)?
(a) 1 g (b) 2 g (c) 4 g (d) 10 g
14. How many moles of hydrogen gas (H_2) are there in 10 g of hydrogen?
(a) 1 (b) 2 (c) 5 (d) 10
15. How many atoms are there in 16 g of oxygen (O_2) in terms of Avogadro's number N_A ?
(a) N_A (b) $2N_A$ (c) $3N_A$ (d) $8N_A$
16. Which of the following is not the ideal gas law?
(a) $\frac{PV}{nT} = R$ (b) $P = \frac{nRT}{V}$ (c) $V = \frac{P}{nRT}$ (d) $V = \frac{nRT}{P}$
17. Which of the following is correct when ice at 0°C is heated?
I. Its temperature will increase.
II. It will melt under constant temperature.
III. Its volume will decrease.
IV. Its volume will increase.
(a) I & II (b) I & IV (c) II & III (d) II & IV
18. Which of the following is correct if a system is in thermodynamic equilibrium with its surroundings?
(a) The temperatures are equal.
(b) The pressures are equal.
(c) The volumes are equal.
(d) All of the above.
19. Under which conditions do real gases approach the ideal gas model?
(a) Low pressure and low temperature.
(b) Low temperature and high pressure.
(c) High temperature and low pressure.
(d) High temperature and high pressure.
20. Which of the following are types of latent heat?
I. Specific heat.
II. Heat of fusion.
III. Heat of vaporization.
(a) I (b) I & II (c) II & III (d) I & III

Problems

12.2 Heat

(Note: In the problems of this section the containers are assumed to be insulated and have zero specific heats.)

12.1 How much heat is required to turn 100 g of water at 30°C into steam at 120°C ? [A: 63 kcal.]

12.2 600 g of water at a temperature of 90°C is added to 400 g of water at a temperature of 20°C in a container. What will the final temperature be? (The container is thermally insulated and its specific heat is negligible.) [A: 62°C .]

12.3 An aluminum block of 200 g at a temperature of 300°C is put into 300 g of water at a temperature of 20°C in a container. What will the final temperature be? [A: 55°C .]

12.4 Some amount of ice at a temperature of -40°C is put together with 10 g of steam vapor at a temperature of 120°C in an insulated container. How much ice was used if the final temperature of the mixture is 30°C ? [A: 48 g.]

12.5 A 600 g piece of iron at a temperature of 500°C is put into 100 g of water at a temperature of 30°C in a container. How much of the water will evaporate? [A: 35 g.]

12.6 A 200 g piece of iron at a temperature of 400°C is put together with 50 g of ice at a temperature of -10°C in an insulated container. What will the final state be?

[A: Water and iron at 60°C .]

12.7 400 g of vapor at a temperature of 150°C is mixed with water with a mass of 100 g at a temperature of 100°C in an insulated container. What will the final state be?

[A: 82 g of water and 418 g of vapor at 100°C .]

12.3 Thermal Expansion

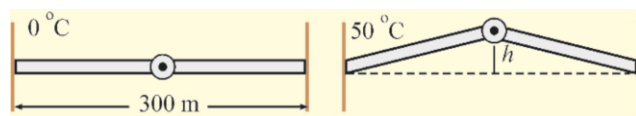
(Note: *Expansion coefficients that may be needed in the problems of this section shall be taken from the table on Page 206.*)

12.8 A steel tower with a height of 800 m at 0°C will expand by how much at a temperature of 40°C ? [A: 38 cm.]

12.9 A glass tube with a base area of 1 cm^2 and a height of 1 m is fully filled with mercury at 0°C . How much mercury will spill when the temperature is 150°C ? [A: 2.3 cm^3 .]

12.10 A physical pendulum made of an aluminum rod of length 1 m is used to measure time. The period of the pendulum is 1 s at a temperature of 0°C . How much error will accrue in the pendulum's measurement in one day when the temperature becomes 50°C ? (The period of a physical pendulum of length L is $T = 2\pi\sqrt{2L/3g}$.)

[A: It will lag by 50 s.]



Problem 12.11

12.11 A steel bridge consists of two parts, each with a length of 150 m and hinged to each other at the center. The bridge can stretch flat in the horizontal position at 0°C . By how much will the central point rise when the temperature is 50°C ? [A: 5.2 m.]

12.12 A steel measuring tape measures correctly at 0°C . The length of a copper pipe is measured by the measuring tape as 25 m at 0°C . (a) What will the copper pipe's measurement be when both the measuring tape and the pipe are heated up to 250°C ? (b) What is the real length of the pipe at that temperature? [A: (a) 25.03 m, (b) 25.11 m.]

12.4 Ideal Gas

12.13 The molar mass of the hydrogen atom (H) is 1 g, the molar mass of the oxygen atom (O) is 16 g and the molar mass of the carbon atom (C) is 12 g. (a) How many moles are there in 100 g of water (H_2O)? (b) What is the weight in

grams of 5 moles of carbon dioxide (CO_2)?

[A: (a) 5.6 moles, (b) 220 g.]

12.14 A container with a volume of 20 liters has 160 g of oxygen (O_2) gas at 25 atm of pressure. (a) How many moles of oxygen are there in the container? (b) What is the temperature of the gas? [A: (a) 5 moles, (b) 1220 K.]

12.15 An oven with a volume of 1 m^3 contains air at STP conditions (0°C and 1 atm). (a) How many moles of air are present in the oven? (b) How many moles of air are left in the oven if it is heated up to a 200°C temperature? [A: (a) 45 moles, (b) 26 moles.]

12.16 An oxygen tube used in industry contains oxygen gas compressed at 100 atm and at 27°C temperature in a volume of 200 liters. (a) What is the mass of the oxygen inside of the tube? (b) It is observed that the pressure of the tube decreases to 80 atm in the next week due to a leak in its valve. How much oxygen remains in the tube? (Assume that the temperature remains constant.) [A: (a) 14.6 kg, (b) 11 kg.]

12.17 The *Loschmidt constant* used in chemistry is defined as the number of gas molecules under STP conditions in 1 cm^3 volume. Calculate the Loschmidt constant. [A: 2.7×10^{19} .]

12.18 The cylinder of a gasoline engine with a volume of 3 lt contains air-gasoline mixture at a temperature of 27°C and a pressure of 1 atm. The piston is pushed to decrease the volume of the mixture by a factor of 8 and increase the pressure by a factor of 12. What will the final temperature of the mixture be? [A: 447°C .]

12.19 A cylinder contains 10 liters of an ideal gas at a temperature of 27°C . The gas is subject to the following processes each time after the same initial state:

- Its pressure is increased by a factor of 5 by heating under constant volume. What will its final temperature be?
- Its volume is increased by a factor of 8 by heating under constant pressure. What will its final temperature be?
- It is heated such that both its pressure and volume double. What will its final temperature be?
- Its *absolute temperature* is increased by a factor of 4 by heating under constant pressure. What will the final volume be? [A: (a) 1500 K, (b) 2400 K, (c) 1200 K, (d) 40 liters.]

12.20 On a day when air pressure is 1 atm and the temperature is 27° , an air bubble with a volume of 1 cm^3 breaks off from the bottom of a lake with 25 m depth. What will its volume be when it reaches the surface? (*Hint: Calculate rates with the $P = P_0 + \rho gh$ formula that we found in Chapter 11 for the hydrostatic pressure in liquids.*) [A: 3.5 cm^3 .]

13

THE LAWS OF THERMODYNAMICS



The heat engine that started the industrial revolution: The steam train.

Steam expands and performs work by pushing a piston. From where does the steam draw the energy to perform this work? Is there an internal source that can be used either as work or as heat? (Photo: Christoph Ehlen).

In the previous chapter, we learned the basic concepts of thermodynamics. Now, we are ready to discuss what a system can do and what changes it can undergo with the heat that it receives. Thermodynamics is the science that deals with systems that exchange heat and work with their environment. It is a phenomenological (experimental) science; it makes no assumptions about the structure of matter. It derives macroscopic results for the properties of matter based on two general laws. It is a remarkable and complete classical theory.

The first law of thermodynamics makes a bilan of energy for both heat and work exchanged between the system and its environment. But it does not indicate the direction in which the system will develop. The second law of thermodynamics establishes which processes, among those allowed by the first law, can really occur.

13.1 WORK DONE BY A GAS

We have previously discussed the fact that a system could exchange energy with its surroundings in the form of heat. Another way to exchange energy is to perform work.

Let us consider a gas inside of a cylinder. The top surface of this cylinder is a movable piston, as seen in Figure 13.1. When we heat this gas, it will expand by pushing the piston. It will thus perform work. Let us calculate the amount of this work.

Let the gas inside of the cylinder have pressure P and volume V . The total force exerted by the gas at this pressure on the piston with surface area A will be $F = PA$. If the piston is displaced by dx as a result of this force, the small amount of work performed by the gas along this displacement will be

$$dW = F dx = PA dx = P dV$$

Here, $dV = A dx$ is the increase in the volume of the gas. If the gas expands ($dV > 0$), the work it performs is positive; indeed, it expands by pushing the piston upwards. However, if the piston is pushed downwards by an external force, the volume decreases ($dV < 0$), hence the work performed by the gas is negative. (Of course, the work done by the external force is positive, but we are interested in *the work performed by the gas*.)

If, during a finite expansion, the volume of the gas goes from V_1 to V_2 , the total work performed by the gas will be the limit sum of these small works, in other words, its integral:

$$W = \int_{V_1}^{V_2} P dV \quad (\text{Work performed by a gas}) \quad (13.1)$$

The pressure P inside of the integral cannot be taken outside of the integral as it is a function of both the volume V and the temperature T .

It will be instructive to examine the work performed by the gas on a P - V diagram. As shown in Figure 13.2, recalling the definition of an integral, the area of the region under the curve will be the work W .

The path taken by the gas when going from V_1 to V_2 during expansion is important. The work will vary depending on the path. The work performed by the gas can be calculated as follows for the processes indicated in Figure 13.3:

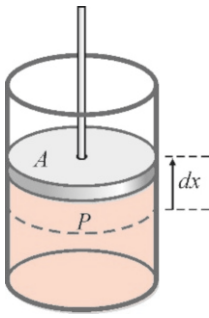


Figure 13.1: The pressure of the gas will push the piston with a force of $F = PA$.

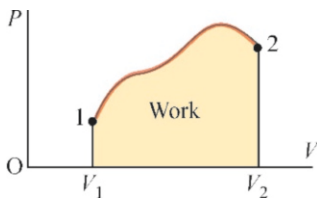


Figure 13.2: In the P - V diagram, the area under the curve is the work W performed by the gas.

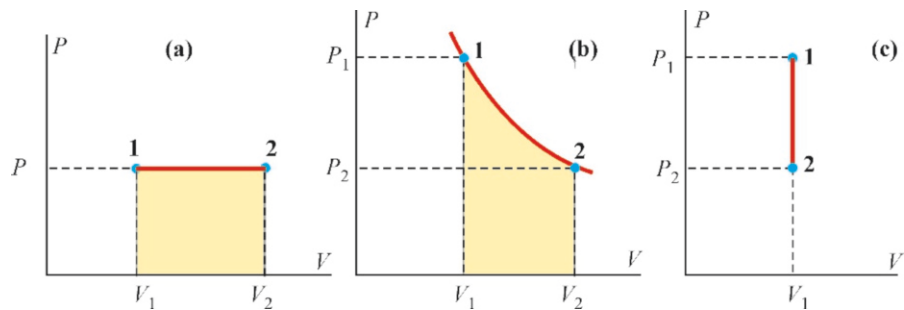


Figure 13.3: The work performed by the gas (a) At constant pressure, (b) At constant temperature (isothermal), (c) At constant volume.

(a) Work under constant pressure (Figure 13.3a) : It is sufficient to calculate the area of the rectangle as the pressure P is constant along the path shown in the figure. Then, P can be taken outside of the integral in Eq. (13.1):

$$W = P \int_{V_1}^{V_2} dV = P(V_2 - V_1) \quad (\text{Work under constant pressure}) \quad (13.2)$$

(b) Work at constant temperature (isothermal) (Figure 13.3b) : If we write the ideal gas equation as $P = nRT/V$, the isothermal curves in the P - V diagram will decrease as $P \sim 1/V$. Let us use the integral to calculate the work performed on such a curve. If we write the pressure P inside of the integral as $P = nRT/V$, we get

$$W = \int_{V_1}^{V_2} P dV = nRT \int_{V_1}^{V_2} \frac{dV}{V}$$

The integral of this expression is the logarithm function:

$$W = nRT \ln \frac{V_2}{V_1} \quad (\text{Work at constant temperature}) \quad (13.3)$$

(c) Work at constant volume (Figure 13.3c) : In this case the work is zero: $W=0$. At constant volume ($dV=0$), the area under a vertical line is zero.

Example 13.1

A container with 3 liters of volume contains a gas at 27 °C temperature and 2 atm pressure. Starting each time from the same initial state, calculate the work performed by this gas during the following processes:

- Its temperature is increased by a factor of 5 at constant volume.
- Its volume is increased by a factor of 4 at constant pressure.
- Its volume is increased by a factor of 2 at constant temperature.

Answer

(a) The definition of work is $dW=PdV$. If the volume is constant, then $dV=0$ and the work performed will be zero: $W=0$.

(b) Since the pressure is constant, integration is not needed:

$$W = P dV = P(V_2 - V_1)$$

The final volume is $V_2 = 4V_1 = 4 \times 3 = 12$ L. Using the SI units (pascal and cubic meters), we get

$$W = 2 \times 1.013 \times 10^5 \times (12 - 3) \times 10^{-3} = 1800 \text{ J} = 1.8 \text{ kJ}$$

(c) We had found the isothermal work in Eq. (13.3):

$$W = nRT \ln \frac{V_2}{V_1}$$

Taking the final volume as $V_2=2V_1$, we get $\ln 2V_1/V_1 = \ln 2$:

$$W = nRT \ln 2$$

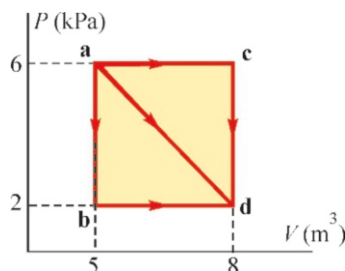
The number of moles n here could be calculated separately, but this is not necessary, because we use the left-hand side rather than the right-hand side of the ideal gas equation $PV = nRT$:

$$W = P_1 V_1 \ln 2$$

We substitute $\ln 2 = 0.69$ and the other values:

$$W = 2 \times 1.013 \times 10^5 \times 3 \times 10^{-3} \times 0.69 = 420 \text{ J}$$

Example 13.2



Calculate the work performed by the gas during all three processes (abd , acd , ad) shown in the P - V diagram above.

Answer

Path abd : We calculate in two steps:

$$W_{abd} = W_{ab} + W_{bd}$$

Along the path ab , work will be zero, as the volume is constant : $W_{ab}=0$. The other work is found using the formula $W=P(V_2 - V_1)$ as it is performed under constant pressure:

$$W_{abd} = 0 + P_b (V_d - V_b) = 2 \times 10^5 \times (8 - 5) = 600 \text{ kJ}$$

Path acd : We again calculate in two steps and no work is performed along the path taken at constant volume:

$$W_{acd} = W_{ac} + W_{cd} = P_a (V_c - V_a) + 0$$

$$W_{acd} = 6 \times 10^5 \times (8 - 5) = 1800 \text{ kJ}$$

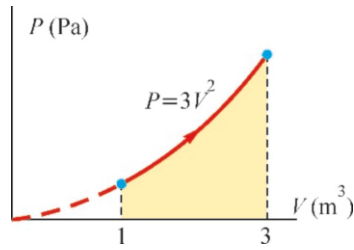
Path ad : We know the work is the area under the curve in the P - V diagram. Accordingly, the work performed by W_{ad} is the area under the diagonal. So, we simply add the area of the triangle to the work W_{bd} that we had calculated previously:

$$W_{ad} = W_{bd} + \frac{(P_a - P_b)(V_d - V_b)}{2}$$

$$W_{ad} = 600 \text{ kJ} + \frac{1}{2} \times (6 - 2) \times 10^5 (8 - 5) = 1200 \text{ kJ}.$$

Example 13.3

A gas expands from volume $V_1=1\text{ m}^3$ to volume $V_2=3\text{ m}^3$ along the parabola curve $P = 3V^2$ (Pa) shown in the figure. Calculate the work performed by the gas.

**Answer**

The work performed is found by integration if the pressure varies:

$$W = \int_1^3 P dV$$

We substitute the equation of the curve as $P=3V^2$ and integrate:

$$W = \int_1^3 3V^2 dV = \left| V^3 \right|_1^3.$$

$$W = 3^3 - 1^3 = 26\text{ J}.$$

13.2 INTERNAL ENERGY – FIRST LAW OF THERMODYNAMICS

Consider heating a gas in a container at constant volume. The gas will perform no work, as the volume is constant: $W=P dV=0$. So, where does the heat energy that we give to the system go? This energy will, of course, be received by the molecules of the gas. The kinetic energies of the molecules will increase as the gas is heated. The total energy, and therefore, the temperature of the gas will increase.

Let us consider another case: This time, let us insulate a container closed with a piston such that the system cannot exchange heat with the environment. Then, let us start to compress this gas. The gas will perform negative work, in other words, it will receive work from the outside, because its volume is decreasing. However, it cannot give it as heat outside, because the gas is insulated. Therefore, where does this work go? Again, the work will be used to increase the energies of the gas molecules. The gas molecules will thus move faster and their kinetic energies will increase.

These two cases show that a gas can convert the heat or work that it receives into an internal energy. This is defined as internal energy in thermodynamics:

The total energy of molecules (or atoms) constituting a gas in a stationary reference frame is called **internal energy**, and it is indicated with E .

At this stage, we do not yet know how to calculate internal energy. The first law tells us how to do it:

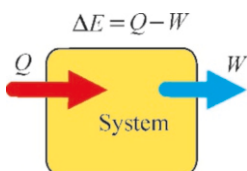


Figure 13.4: The First Law: The difference between received heat and work done equals the increase in internal energy.

First Law of Thermodynamics

In any process, the difference between the heat Q received and the work W performed by a system is independent of the process and equal to the increase in its internal energy ΔE :

$$\Delta E = Q - W \quad (13.4)$$

Let us emphasize the important points of the first law:

- The first law is a kind of energy bilan: It states that the difference between the received heat and the work done is spent on increasing the internal energy.
- Note the difference between the signs of Q and W : The heat received by the system and the work done by the system are accepted as positive. Therefore, the energy increase is included as their difference.

- Internal energy, like pressure P , temperature T or volume V , is a quantity that characterizes a system; in other words, it is a **state variable**. But the heat Q and work W are path-dependent, therefore it is wrong to say “the heat of a system” or the “work inside of a system.” But it is correct to say the “internal energy of a system.” Regardless of how it is obtained, this internal energy is now independent of the path taken.
- **Two special cases:**
 - If the system changes state without performing work ($W = 0$), then according to the first law, we get

$$\Delta E = Q$$

In other words, all of the received heat goes into increasing the internal energy.

- If the system performs work but does not receive heat, then $Q = 0$ and, according to the law, we get

$$\Delta E = -W$$

Hence, the system performs this work by drawing it from its internal energy.

- This is a law that applies not only to gases, but to all systems. All kinds of tools, machinery, etc., operate by obeying this law. For example, claims about perpetual motion machines that operate without receiving any energy are not taken seriously in science, as they violate the first law (Figure 13.5).
- However, the first law does not tell us the direction in which the change occurs. For example, let us bring together two objects, one hot and the other cold. One of these objects will give heat Q to the other, but which one? The first law says nothing and allows for heat transfer in both directions, as long as the heat received by one is Q and the other is $-Q$, in other words, as long as the net heat is zero. However, we know that, in nature, the hot body always gives heat to the cold one. It is the second law that that will tell us the direction towards which the events will develop.

13.3 APPLICATIONS OF THE FIRST LAW

The first law determines the relation between heat, work and internal energy. We can calculate many processes using this relation.

Internal Energy of an Ideal Gas

First, we must slightly change the definition of specific heat for gases. We had defined specific heats as being per mass in section 12.2:

$$Q = mc \Delta T$$

We can use the same definition in terms of the number of moles: $Q = nc\Delta T$. This is more useful for gases, as we shall see later.

However, the temperature increase may take place under constant pressure or constant volume, and therefore a separate specific heat is defined for each one as follows:

$$Q = \begin{cases} nc_v \Delta T & \text{(Heating at constant volume)} \\ nc_p \Delta T & \text{(Heating at constant pressure)} \end{cases} \quad (13.5)$$



Figure 13.5: The perpetual motion machine suggested by Robert Boyle. In this machine, water *will not* flow and circulate by itself, because the point at which water exits the hose should be below the surface of the water in the container.

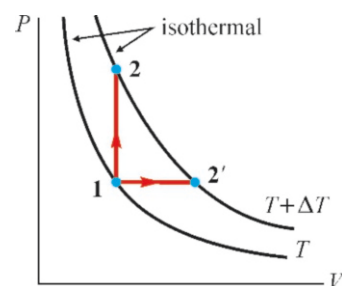


Figure 13.6: 1→2: Heating at constant volume. 1→3: heating under constant pressure.

Now let us consider that we increase the temperature of a gas by ΔT at constant volume. The work performed during this process will be zero, because the volume is constant. If we take $W = 0$ in the first law,

$$\Delta E = Q = nc_v \Delta T$$

Integrating this expression from $T = 0$ to T , we get

$$E - E_0 = nc_v (T - 0)$$

If we decide that the internal energy of the gas at absolute zero temperature $T = 0$ is $E_0 = 0$, then we find the internal energy of a gas at temperature T as follows:

$$E = nc_v T \quad (\text{Internal energy of an ideal gas}) \quad (13.6)$$

The increase in the internal energy would still have been $nc_v \Delta T$ if we had performed this heating under constant pressure. This is because internal energy is a state variable and the value that it takes at that temperature is independent of how it is reached.

The relation between c_v and c_p

Now, let us consider that we heat the same gas by ΔT under constant pressure. It will both heat and perform work during this process. We can write the heat directly:

$$Q = nc_p \Delta T$$

If the volume of the gas changes from V_1 to V_2 under constant pressure P , the work can be written as follows:

$$W = P(V_2 - V_1)$$

Using the ideal gas equation $PV = nRT$ here, we get

$$W = PV_2 - PV_1 = nRT_2 - nRT_1 = nR \Delta T$$

Now, let us use these heat and work expressions in the first law:

$$\begin{aligned} \Delta E &= Q - W \\ nc_v \Delta T &= nc_p \Delta T - nR \Delta T \end{aligned}$$

Simplifying, we find the relation between the specific heats:

$$c_p = c_v + R \quad (\text{Specific heats of an ideal gas}) \quad (13.7)$$

This expression is used to find one of the specific heats if the other is known. We will see how to find the specific heat c_v . Here are the results without proof:

Specific heats of gases		
	c_v	c_p
Monatomic gases (He, Ne, Ar, Kr ...)	$\frac{3}{2} R$	$\frac{5}{2} R$
Diatomic gases (H ₂ , O ₂ , N ₂ ...)	$\frac{5}{2} R$	$\frac{7}{2} R$

Specific heat expressions are more complex for polyatomic gases.

Adiabatic Process

The process that a system undergoes without any heat exchange ($Q = 0$) is called an **adiabatic process**. In such a case, there is only exchange of work with the environment. Let us consider the gas in a container (Figure 13.7). No heat will be exchanged if the container is insulated and all changes of state will be adiabatic. Likewise, let us compress the gas in a cylinder by suddenly compressing the piston. The gas does not have time to exchange heat with the environment, as it takes place in a short period of time.

We can use the first law to calculate the final temperature, pressure and volume of an ideal gas at the end of an adiabatic process. Consider that the temperature of a gas changes by a small amount dT during an adiabatic process. If we take $Q = 0$ in the first law and write $P dV$ for work and $nc_v dT$ for the increase in internal energy, we get

$$\begin{aligned}\Delta E &= 0 - W \\ nc_v dT &= -P dV\end{aligned}$$

Let us take the pressure as $P = nRT/V$ from the ideal gas equation:

$$\begin{aligned}nc_v dT &= -\frac{nRT}{V} dV \\ \frac{dT}{T} + \frac{R}{c_v} \frac{dV}{V} &= 0\end{aligned}$$

In this expression, let us substitute R with the formula $R = c_p - c_v$ that we found above for specific heats:

$$\frac{dT}{T} + \left(\frac{c_p}{c_v} - 1\right) \frac{dV}{V} = 0$$

If we define the ratio of the specific heats as a dimensionless constant $\gamma = c_p/c_v$, we always get $\gamma > 1$:

$$\frac{dT}{T} + (\gamma - 1) \frac{dV}{V} = 0$$

Taking the indefinite integral of both sides of this expression, we get

$$\ln T + (\gamma - 1) \ln V = \text{constant}$$

Using the properties $\ln a + \ln b = \ln(a \cdot b)$ and $c \ln d = \ln d^c$ of the logarithm, the result is as follows:

$$T V^{\gamma-1} = \text{constant}$$

If we wish to find the relation between the volumes, we substitute the temperature in this formula as $T = PV/nR$ from the ideal gas equation and simplify:

$$P V^\gamma = \text{constant}$$

It is meaningful to compare this relation between pressure and volume with isothermal curves on the P - V diagram. As seen in Figure 13.8, the adiabatic curve decreases more rapidly compared to the isothermal curve.

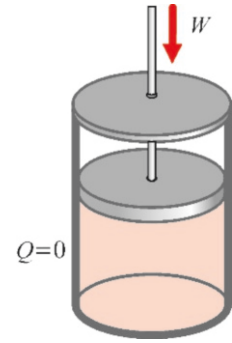


Figure 13.7: No heat is exchanged in an adiabatic process: $Q = 0$.

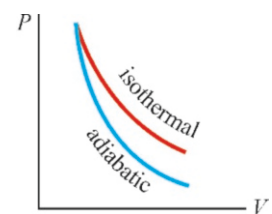


Figure 13.8: Comparison of adiabatic and isothermal curves.

A similar formula can be found for (P, T) if necessary. According to these results, these products remain constant at each stage of the adiabatic process, in particular, for the initial and final values:

$$\begin{aligned} P_1 V_1^\gamma &= P_2 V_2^\gamma \\ P_1^{1-\gamma} T_1^\gamma &= P_2^{1-\gamma} T_2^\gamma \quad (\text{Adiabatic process}) \\ T_1 V_1^{\gamma-1} &= T_2 V_2^{\gamma-1} \end{aligned} \quad (13.8)$$

It is sufficient only to keep the first of these formulas in mind; the others can be obtained from the ideal gas equation $PV = nRT$.

Work in an adiabatic process. As $Q=0$ in an adiabatic process, we get $\Delta E = -W$ from the first law. Using Eq. (13.6) for the internal energy, we get

$$\begin{aligned} \Delta E &= nc_v \Delta T \\ W &= -\Delta E = -nc_v \Delta T \end{aligned} \quad (13.9)$$

The work performed by the gas is fully provided by the internal energy.

Example 13.4

A diatomic gas of 5 moles is heated by giving it 600 J under constant pressure. ($c_v = 5R/2$, $c_p = 7R/2$.)

- What will the temperature increase ΔT be?
- What will the increase in internal energy be?
- How much work will the gas perform?

Answer

(a) It may seem that the information provided in this problem is insufficient, but it is actually sufficient for the requested calculations. We use the formula for temperature increase

under constant pressure:

$$Q = nc_p \Delta T$$

We substitute the values and calculate ΔT as follows:

$$\Delta T = \frac{Q}{nc_p} = \frac{600}{5 \times \frac{7}{2} \times 8.31} = 4.1 \text{ K}$$

(b) The energy increase will be $\Delta E = nc_v \Delta T$ regardless of the process being at constant pressure or volume:

$$\Delta E = n \frac{5}{2} R \Delta T = 5 \times \frac{5}{2} \times 8.31 \times 4.1 = 426 \text{ J}$$

(c) The work is calculated from the first law when the heat and the increase in internal energy are known:

$$\Delta E = Q - W \rightarrow W = Q - \Delta E = 600 - 426 = 174 \text{ J.}$$

Example 13.5

A vessel contains 4 moles of a monatomic gas at 0°C . From the same initial state:

- The temperature of the gas is increased to 50°C at constant volume. Calculate the given heat, the work performed by the gas and the increase in internal energy.
- The temperature of the gas is increased to 50°C under constant pressure. Calculate the given heat, the work performed by the gas and the increase in internal energy.

Answer (a) The heat received by the gas at constant volume is calculated using Eq. (13.5) and the $c_v = \frac{3}{2}R$ value in the table:

$$Q = nc_v \Delta T = 4 \times \frac{3}{2} \times 8.31 \times (50 - 0) = 2500 \text{ J.}$$

Work is zero, because the volume is constant: $W = 0$. Then we find the increase in internal energy using the first law:

$$\Delta E = Q - W = Q - 0 = 2500 \text{ J.}$$

(b) The heat taken under constant pressure is likewise calculated by taking $c_p = \frac{5}{2}R$:

$$Q = nc_p \Delta T = 4 \times \frac{5}{2} \times 8.31 \times (50 - 0) = 4200 \text{ J.}$$

We do not have the information to calculate the work directly. However, if we first calculate the increase in internal energy, we can then find the work by using the first law. The increase in internal energy is found using the formula (13.3):

$$\Delta E = n c_v \Delta T = 4 \times \frac{3}{2} R \times (50 - 0) = 2500 \text{ J}$$

We calculate the work from the first law using Q and ΔE :

$$\Delta E = Q - W \rightarrow W = Q - \Delta E = 4200 - 2500 = 1700 \text{ J.}$$

Example 13.6

The volume of a 1 g of water at 100°C in temperature and under 1 atm of pressure expands to 1670 cm^3 when converted

into steam at 100°C in temperature. As the heat of vaporization of water is $L=540 \text{ cal/g}$, calculate the increase in internal energy.

Answer

In this problem, we will first calculate heat and work and then find the increase in the internal energy using the first law. Since no temperature change is involved, there is only the latent heat of vaporization during the change of phase:

$$Q = mL = 1 \times 540 = 540 \text{ cal} = 540 \times 4.18 = 2260 \text{ J}$$

The 1 cm^3 volume of 1 g of water expands when it is turned

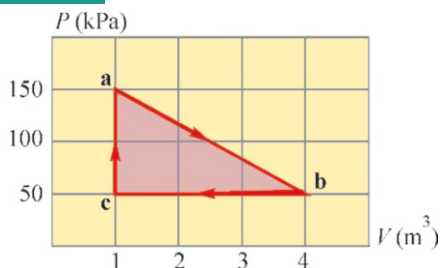
into vapor. We calculate the work under 1 atm at constant pressure:

$$W = P(V_2 - V_1) = 1.013 \times 10^5 \times (1670 - 1) \times 10^{-6}$$

$$W = 170 \text{ J}$$

We find the increase in the internal energy using the values of Q and W :

$$\Delta E = Q - W = 2260 - 170 = 2100 \text{ J.}$$

Example 13.7

A gas completes a cycle through the 3-step process ($a \rightarrow b \rightarrow c \rightarrow a$) shown in the figure. What is the net heat received by the gas?

Answer

This problem may seem complicated at first glance, but it becomes simple when you think about it. When the gas starts at point **a** and undergoes any process, it will have the same internal energy after it returns to point **a**, in other words, the

increase in internal energy will be zero: $\Delta E = 0$. Accordingly, the net heat received during a cycle will be spent on the net work done by the gas:

$$\Delta E = Q_{abca} - W_{abca} = 0 \rightarrow Q_{abca} = W_{abca}$$

We do not have sufficient information to directly calculate the heat, but we can calculate the work from the P - V diagram. The work performed at each step of the cycle is the area under that curve. However, the work will be negative if V decreases in the direction moved. Accordingly, the work performed along the path **a** \rightarrow **b** will be positive, the work performed along the path **b** \rightarrow **c** will be negative and the work performed along the path **c** \rightarrow **a** will be zero. As a result, the net work will be equal to the area of the red triangle shown in the figure.

$$Q_{abca} = W_{abca} = \text{Area of the triangle } ABC$$

$$Q_{abca} = \frac{1}{2} \times (4 - 1) \times (150 - 50) \times 10^3 = 150\,000 \text{ J}$$

$$Q = 150 \text{ kJ.}$$

Example 13.8

A cylinder with 5 liters of volume contains a diatomic ideal gas at 27°C in temperature and 1 atm in pressure with $\gamma = 1.4$. The piston is suddenly pushed to compress the volume of the gas to 1 L before it can exchange any heat.

(a) What will the final temperature and pressure be?

(b) How much work will the gas perform?

Answer

(a) This is an adiabatic process, because there is no heat exchange ($Q = 0$). We first find the pressure using Eqs. (13.8):

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

From here, we calculate the final pressure:

$$P_2 = \left(\frac{V_1}{V_2}\right)^\gamma P_1 = \left(\frac{5}{1}\right)^{1.4} \times 1 = 9.5 \text{ atm}$$

In order to find the final temperature, we write the ideal gas

equation twice and find their proportion:

$$\frac{P_2 V_2}{P_1 V_1} = \frac{T_2}{T_1} \rightarrow T_2 = \frac{P_2 V_2}{P_1 V_1} T_1$$

$$T_2 = \frac{9.5 \times 1}{1 \times 5} \times 300 = 570 \text{ K}$$

(b) Adiabatic work is equal to the decrease in the internal energy (Eqs. 13.9):

$$W = -\Delta E = -nc_v \Delta T = -nc_v (T_2 - T_1)$$

Rather than calculating the number of moles n here, we find the pressures and volumes using the formula $nRT = PV$:

$$W = \frac{c_v}{R} (nRT_1 - nRT_2) = \frac{c_v}{R} (P_1 V_1 - P_2 V_2)$$

The value $c_v = \frac{5}{2}R$ is taken from page 222 for diatomic gases:

$$W = \frac{5}{2} (P_1 V_1 - P_2 V_2)$$

From here, we calculate the work performed by the gas:

$$W = \frac{5}{2} \times (1 \times 5 - 9.5 \times 1) \times 1.013 \times 10^5 \times 10^{-3} = -1140 \text{ J.}$$

13.4 KINETIC CALCULATION OF PRESSURE

What is the origin of the air pressure of a tire that can withstand the weight of a car? The answer at the microscopic scale is that the collisions of those small molecules with the rubber wall keep such a heavy car up.

There is a *kinetic theory of gases* that describes the thermodynamic properties of a gas, such as temperature and pressure, in terms of the microscopic motions of its atoms or molecules. Let us calculate the pressure of a gas according to the kinetic theory in order to show how calculations are made in this theory.

We will make some assumptions to keep the calculation simple:

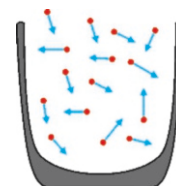


Figure 13.9: The pressure inside the automobile tire is a result of the collision of the molecules with the rubber wall.

- Gas molecules will be assumed to be monatomic. Therefore, they will act as point particles, and thus only perform translational motion and not rotation or vibration.
- The collisions among the molecules will be ignored.
- The molecules will be assumed to make elastic collisions with the wall.

Let us remember the concepts of impulse and momentum in mechanics: The momentum of an object with mass m subject to force F in the time interval Δt changes as follows:

$$\vec{F} \Delta t = m \vec{v}_2 - m \vec{v}_1$$

Now let us consider that a molecule with velocity \vec{v} inside of a cubic volume of side length L collides with a wall that is perpendicular to the x -axis (Figure 13.10). As a result of the elastic collision, the component v_y of the molecule parallel to the wall does not change and the horizontal component v_x gets reflected back. Therefore, the force on this molecule by the wall is only in the x -direction (Figure 13.11):

$$F \Delta t = m_0 v_x - m_0 (-v_x) = 2m_0 v_x$$

We are using m_0 here for the mass of a molecule to avoid any confusion.

This molecule, after colliding with the wall, rebounds toward the opposite wall, subsequently returning and colliding once more. Therefore, the time Δt between two collisions is the time of one collision. As the length of one side is L , we calculate the time required to travel back and forth along the path $2L$:

$$\Delta t = \frac{2L}{v_x}$$

Using Δt from the equation above, we write the force F as follows:

$$F = \frac{2m_0 v_x}{\Delta t} = \frac{m_0 v_x^2}{L}$$

This is the force acting on the molecule. According to the action-reaction law, the molecule will push the wall with an equal and opposite force. We will ignore the sign, because we are only concerned with the magnitude of the force here.

We have found the force exerted by one molecule on the wall. If there are N molecules in the gas, the force exerted by each one of them on the wall will have the same form. Therefore, the sum required to find the force of N molecules can be written as follows:

$$F_{\text{total}} = \frac{m_0}{L} (v_{1x}^2 + v_{2x}^2 + \dots + v_{Nx}^2)$$

Let us examine the sum in the brackets separately: If we divide this sum by the number of molecules we will find the mean (or, the average) $\langle v_x^2 \rangle$ of the v_x^2 velocities:

$$\langle v_x^2 \rangle = \frac{v_{1x}^2 + v_{2x}^2 + \dots + v_{Nx}^2}{N}$$

In this expression, we want to see the mean of the velocity itself, rather than the mean of the component $\langle v_x^2 \rangle$. As each velocity vector can be written as $v^2 = v_x^2 + v_y^2 + v_z^2$, the mean of both sides will be

$$\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle$$

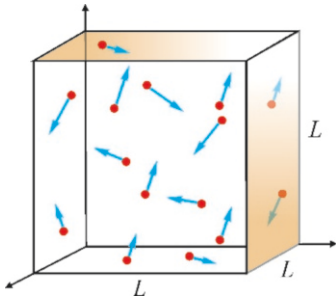


Figure 13.10: Gas molecules in a cubic volume.

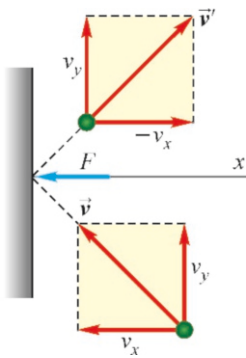


Figure 13.11: The collision of a molecule with a wall.

Ignoring the effect of gravity, the mean velocities of the molecules will be equal in all three directions due to symmetry:

$$\begin{aligned}\langle v_x^2 \rangle &= \langle v_y^2 \rangle = \langle v_z^2 \rangle \\ \langle v^2 \rangle &= 3\langle v_x^2 \rangle\end{aligned}$$

We can now write the force above in terms of this mean velocity:

$$F_{\text{total}} = \frac{Nm_0}{L} \frac{\langle v^2 \rangle}{3}$$

From here, we can calculate the pressure: As the surface area of the side is $A = L^2$ and the volume is $V = L^3$, the pressure is expressed as follows:

$$P = \frac{F_{\text{total}}}{A} = \frac{Nm_0}{3L^3} \langle v^2 \rangle = \frac{Nm_0}{3V} \langle v^2 \rangle \quad (13.10)$$

Many important conclusions can be drawn from this expression:

- **Internal energy and specific heat:** Let us move the volume to the left-hand side of this expression, and form the kinetic energy of one molecule:

$$PV = \frac{2}{3} \underbrace{N \frac{1}{2} m_0 \langle v^2 \rangle}_E$$

The product of the mean kinetic energy of one molecule, $m_0 \langle v^2 \rangle / 2$, and the number N is the total kinetic energy of the gas. At the start, we had assumed that the molecules did not interact and only engaged in elastic collisions with the wall. Therefore, according to the definition, the total kinetic energy will be the internal energy E of this gas:

$$PV = \frac{2}{3} E$$

We write the left-hand side as $PV = nRT$ from the ideal gas law:

$$\begin{aligned}nRT &= \frac{2}{3} E \\ E &= \frac{3}{2} nRT \quad (\text{Internal energy of a monatomic gas}) \quad (13.11)\end{aligned}$$

We had previously found the expression for the internal energy in terms of the specific heat c_v , in Eq. (13.6):

$$E = nc_v T$$

This result shows that the temperature is a measure of the total energy of a body. We find the specific heat c_v of an ideal gas at constant volume by comparing the last two expressions:

$$c_v = \frac{3}{2} R \quad (\text{Specific heat of a monatomic gas}) \quad (13.12)$$

We also know that $c_p = c_v + R = \frac{5}{2} R$. In conclusion, the heat received by, work performed by and the internal energy increase of ideal gases in all kinds of processes can thus be calculated using these formulas for E, c_v, c_p .

- **Root-mean-square speed:** Let us return to the expression (13.10):

$$PV = \frac{Nm_0}{3} \langle v^2 \rangle$$

The Nm_0 product on the right-hand side of the equation is the product of the number of molecules with the mass of one molecule, in other words, the total mass m of the gas. We can write this in terms of the number of moles n and the molecular mass M , as $m = nM$. Using the ideal gas equation again on the left-hand side, we get

$$\begin{aligned} nRT &= \frac{1}{3} nM \langle v^2 \rangle \\ \langle v^2 \rangle &= \frac{3RT}{M} \end{aligned}$$

We find the **root-mean-square (rms) speed** formula by taking the root of this expression:

$$v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3RT}{M}} \quad (\text{Root-mean-square speed}) \quad (13.13)$$

This formula is also valid for polyatomic gas molecules. According to the formula, the mean speed of gases with lower molecular mass (hydrogen, helium) will be higher. Therefore, light gases are very scarce in the atmosphere, because they have already escaped gravity.

Example 13.9

What will the root-mean-square speed of oxygen gas (O_2) be at the temperature at which the root-mean-square speed of Helium gas (He) is 1500 m/s? Molar masses are $M_{\text{O}_2}=32$ g, $M_{\text{He}}=4$ g.

Answer

We write Eq. (13.13) for root-mean-square speed for both

gases and form their ratio:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}} \rightarrow \frac{v_{\text{rms},2}}{v_{\text{rms},1}} = \sqrt{\frac{M_1}{M_2}}$$

We find the speed by substituting the molar masses:

$$v_{\text{rms},2} = \sqrt{\frac{M_1}{M_2}} v_{\text{rms},1} = \sqrt{\frac{4}{32}} \times 1500 = 530 \text{ m/s}$$

Example 13.10

A vessel contains 2 moles of neon (Ne) gas, which has molar mass 20 g, at 27°C in temperature.

(a) What is the total kinetic energy of the gas molecules?

(b) What is the root-mean-square speed of one molecule?

Answer

(a) The gas is monatomic, in other words, it has no rotational or vibrational energy; its internal energy consists only of the total kinetic energy. We use the formula (13.11) to calculate

the internal energy:

$$E = \frac{3}{2} nRT = \frac{3}{2} \times 2 \times 8.31 \times 300 = 7500 \text{ J}$$

(b) We use Eq. (13.13) for the root-mean-square speed:

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

We calculate by substituting the molar mass of neon and the temperature:

$$v_{\text{rms}} = \sqrt{\frac{3 \times 8.31 \times 300}{0.020}} = 610 \text{ m/s.}$$

13.5 HEAT ENGINES

Converting energy into work is the basis of technology. Hydroelectric power plants convert the potential energy of water, wind turbines convert the kinetic energy of air, and gasoline motors convert combustion energy into work.

A **heat engine** is the general name for machines that convert some of the heat that they receive into work. The steam engine, the internal combustion engine and the diesel motor are heat engines.

Regardless of their details, every heat engine performs W work with the heat Q_H that it receives from a *hot source* at temperature T_H and gives the remaining heat Q_C to a *cold source* at temperature T_C (Figure 13.12).

Heat engines operate in **cycles**, in other words, they return to their initial state after each process of receiving heat and performing work. For example, a steam engine performs W amount of work with the heat Q_H of the steam that it receives from the boiler and then releases the heat Q_C into the external atmosphere with the expanded steam. In the end, this cycle returns to the point at which it starts.

If we write the first law for one cycle, the change in the internal energy of the system is zero:

$$\begin{aligned}\Delta E &= Q - W = (Q_H + Q_C) - W = 0 \\ W &= Q_H + Q_C\end{aligned}$$

Q_C is actually a negative quantity here. If we write the absolute values of the heats, we find that the work is always equal to the difference:

$$W = |Q_H| - |Q_C|$$

Efficiency

The efficiency of a heat engine is the ratio of the net work performed to the spent heat:

$$\begin{aligned}e &= \frac{W}{Q_H} = \frac{|Q_H| - |Q_C|}{|Q_H|} \\ e &= 1 - \left| \frac{Q_C}{Q_H} \right| \quad (\text{Efficiency of a heat engine}) \quad (13.14)\end{aligned}$$

Efficiency is a number within the range $[0, 1]$, for example, if $e = 0.3$, then the efficiency is 30 %.

Now, let us review the operating principles and efficiencies of some of the most common heat engines.

Internal Combustion Engine – Otto Cycle

A cycle consists of four processes in an internal combustion engine, also known as the gasoline engine or the four-stroke engine. Let us examine these processes by following them on the adjacent P - V diagram 13.13 and on Figure 13.14 below:

A gasoline-air mixture is pumped into the cylinder through the left valve being opened when the engine is at point a . Known as intake, this process is not included in the cycle. This gasoline-air mixture is suddenly compressed after point a along the path ab . Known as the compression stroke, this process is

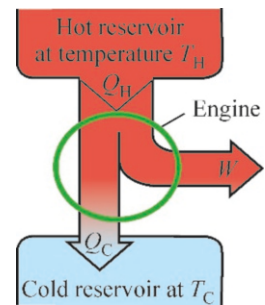


Figure 13.12: Heat engine diagram.

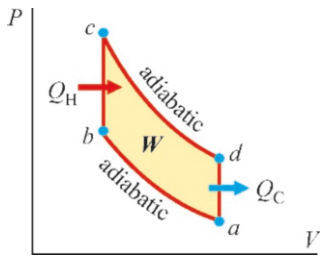


Figure 13.13: Otto cycle.

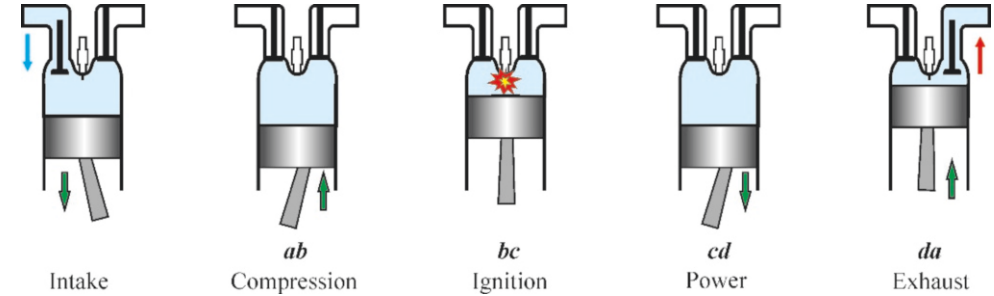


Figure 13.14: Four-stroke engine. Each figure corresponds to a different process in the P - V diagram.

Let us calculate the efficiency of this cycle, also known as the Otto cycle, which consists of two adiabatic and two constant-volume processes. It is sufficient to calculate heat along the paths bc and da , as there is no heat exchange in adiabatic processes:

$$bc \text{ process at constant volume: } Q_H = nc_v(T_c - T_b)$$

$$da \text{ process at constant volume: } Q_C = nc_v(T_a - T_d)$$

Observing from the diagram that $T_c > T_b$ and $T_a < T_d$, it can be understood that the heat Q_H is positive and the heat Q_C is negative.

Using the definition of efficiency,

$$e = \frac{Q_H + Q_C}{Q_H} = \frac{(T_c - T_b) + (T_a - T_d)}{T_c - T_b}$$

The four temperatures here can actually be reduced to a single parameter. Let us use V to indicate the value $V_b = V_c$ at which the gas is compressed at a maximum in the Otto cycle. If we use rV to indicate the volumes $V_a = V_d$ at which the gas expands, r is defined as the *compression ratio* of the motor. Now, if we use the formulas (13.8) that we found earlier for the adiabatic processes ab and cd , we get

$$\text{The process } ab: \quad T_a (rV)^{\gamma-1} = T_b V^{\gamma-1}$$

$$\text{The process } cd: \quad T_d (rV)^{\gamma-1} = T_c V^{\gamma-1}$$

If we eliminate the common term $V^{\gamma-1}$ in these equations and substitute the temperatures T , the formula simplifies as follows:

$$e = \frac{T_d r^{\gamma-1} - T_a r^{\gamma-1} + T_a - T_d}{T_d r^{\gamma-1} - T_c r^{\gamma-1}}$$

$$e = 1 - \frac{1}{r^{\gamma-1}} \quad (\text{Efficiency of the Otto cycle}) \quad (13.15)$$

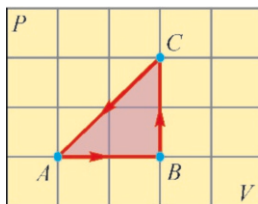
Here, $\gamma = c_p/c_v$ and is approximately $\gamma=1.4$ for air. This result is surprising. In the gasoline engine, regardless of the type of fuel or dimensions that you use, the efficiency depends only on the compression ratio r of the engine. Of course, it is not possible to achieve high compression with certain fuels, as they burn halfway through. Modern engines using high-octane gas have compression ratios around $r = 10$. If we take $r = 10$ and $\gamma = 1.4$ for air, the efficiency is found to be $e = 0.6$. In reality, the efficiency of a gasoline engine is around 25 %, due to friction and other effects until the energy is transmitted to the wheels.



Figure 13.15: The Jaguar V12 engine has a record compression ratio of $r=14$.

Example 13.11

Determine the signs of heat (Q), work (W) and internal energy increase (ΔE) in each step of the ABC cycle shown in the figure, and complete the table.



	Q	W	ΔE
$A \rightarrow B$			
$B \rightarrow C$			
$C \rightarrow A$			

Answer

This problem constitutes the basis before examining cycles. First, let us remember some of the general principles:

- In the P - V diagram, temperature increases with distance to the origin.
- Internal energy always increases with T : $\Delta E=nc_v \Delta T$.
- Heat is positive if the temperature increases. It can be in either of two forms: $Q=nc_v \Delta T$ or $nc_p \Delta T$
- The work $dW=P dV$ by the gas is positive if it expands.

Now, let us analyze each step along these principles:

The AB process:

Temperature T and volume V increase along the path. Therefore, Q and ΔE and W are all positive.

The BC process:

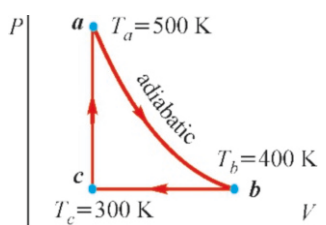
Temperature T increases, but volume V remains constant along the path. Therefore, Q and ΔE are positive but $W = 0$.

The CA process:

Temperature T and volume V decrease along the path. Therefore, Q and ΔE and W are all negative. As a result, the table is completed as follows:

	Q	W	ΔE
$A \rightarrow B$	+	+	+
$B \rightarrow C$	+	0	+
$C \rightarrow A$	-	-	-

Example 13.12



A gas with $\gamma=c_p/c_v=1.4$ undergoes a 3-step cycle, as shown in the figure. It expands adiabatically along the path ab , is compressed under constant pressure along the path bc and is heated at constant volume along the path ca . $T_a=500$ K, $T_b=400$ K and $T_c=300$ K. Calculate the efficiency of the cycle.

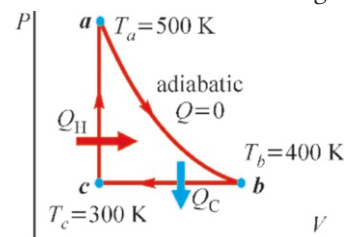
Answer

Calculating the efficiency of a cycle may seem complicated at first glance, but it becomes easy when given some thought. Efficiency was defined in Eq. (13.14) as the ratio of the net work W to the positive heat:

$$\epsilon = \frac{W}{Q_H} = 1 - \frac{|Q_C|}{|Q_H|}$$

Therefore, we will calculate the steps of the cycle at which the gas receives heat and the steps at which it gives heat, and use their ratio in the efficiency formula. In general, the result will depend only on temperature differences.

In this problem, no heat exchange will take place in step ab , as it is adiabatic. When the gas is compressed in step bc , it is forced to give off heat in order to keep the pressure constant. Therefore, $Q_C=Q_{bc}$. In step ca , its pressure is increased by heating at constant volume. Therefore, $Q_H=Q_{ca}$. In short, the situation is as shown in the figure below:



Now let us calculate the heats Q_H and Q_C :

$$Q_H = Q_{ca} = nc_v \Delta T = nc_v (T_a - T_c)$$

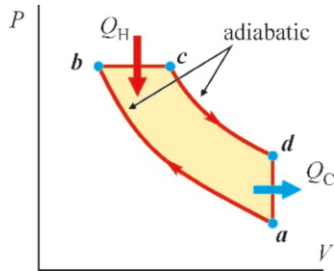
$$Q_C = Q_{bc} = nc_p \Delta T = nc_p (T_c - T_b)$$

The ratio of these heats is used in the efficiency formula:

$$\epsilon = 1 - \frac{|nc_p(T_c - T_b)|}{nc_v(T_a - T_c)} = 1 - \gamma \frac{T_b - T_c}{T_a - T_c}$$

Example 13.13

Diesel cycle. In a diesel engine, air compressed at high temperature is combusted by injected oil. The exploded air-oil mixture expands and pushes the piston. The stages of the cycle are as follows:



ab : adiabatic compression

bc : expansion under constant pressure (explosion)

cd : adiabatic expansion (work by pushing the gas piston)

da : cooling at constant volume (discharge from the exhaust)

We calculate the efficiency by substituting the data:

$$\epsilon = 1 - 1.4 \times \frac{400 - 300}{500 - 300} = 0.30 = 30 \%$$

Calculate the efficiency of the Diesel cycle shown in the figure in which $T_a=300$ K, $T_b=1000$ K, $T_c=1600$ K and $T_d=800$ K. ($\gamma=c_p/c_v=1.4$ for air.)

Answer

It is easy to see where the heats Q_H and Q_C are received and given in this cycle. No heat is exchanged in the adiabatic steps. The gas receives heat along the path *bc* because it is heated, and gets cooler and gives off heat along the path *da*. Therefore, we get $Q_H = Q_{bc}$ and $Q_C = Q_{da}$. Let us calculate these:

$$\text{Under constant pressure : } Q_H = Q_{bc} = nc_p(T_c - T_b)$$

$$\text{Under constant volume : } Q_C = Q_{da} = nc_v(T_a - T_d)$$

The ratio of these heats is used in the efficiency formula:

$$\epsilon = 1 - \frac{|nc_v(T_a - T_d)|}{nc_p(T_c - T_b)} = 1 - \frac{1}{\gamma} \frac{T_d - T_a}{T_c - T_b}$$

We calculate the efficiency by substituting the data:

$$\epsilon = 1 - \frac{1}{1.4} \times \frac{800 - 300}{1600 - 1000} = 0.40 = 40 \%$$

13.6 SECOND LAW OF THERMODYNAMICS – THE CARNOT CYCLE

When two bodies, one hot and the other cold, are brought together, heat flows from the hot body to the cold one, and never the other way around. A drop of ink dropped in water spreads out, but an ink that has been spread out does not spontaneously gather at one point. Likewise, if we remove the wall between two different types of gas in two adjacent containers, the two gases get mixed, but they do not separate spontaneously.

Although all of these processes are possible in terms of energy, in other words, although they comply with the first law, they are not observed in nature. There is a direction of development preferred by nature in thermodynamic events. The second law of thermodynamics specifies this direction.

There are many varied expressions of the second law in macroscopic and microscopic scales. Its microscopic expression includes concepts such as entropy, disorder, phase space, etc., and is the subject of the branch of physics called Statistical Mechanics. We shall examine the macroscopic expression based on heat engines here.

Second Law of Thermodynamics

It is not possible for a cyclic heat engine to convert all of the heat that it receives from a reservoir into work.

Let us emphasize the important points of the second law:

- If a machine had converted all of the energy it received into work, according to the definition of efficiency (Eq. 13.14), we would have $Q_C=0$ and efficiency would be 100 %. This is possible according to the first law, because the heat

received is equal to the performed work. However, the second law states that no heat engine can operate at 100 % efficiency. In other words, it specifies the direction towards which natural events develop.

- According to the second law, a gasoline engine cannot operate without discharging the burnt gas, or a steam engine without discharging the expanded steam. If it could not discharge this heat into the environment, the machine would eventually heat up and become inoperable.
- If there is no way to give heat to the cold source, then work cannot be done either. For example, the engine of a ship could, in principle, be able to take heat from the sea water and turn some of it into work. However, the machine would not operate, because it would not be able to return the remaining heat to the sea after the work is done.

The Carnot Cycle

The second law states that it is impossible to build a 100 % efficient heat engine. So, what is the most efficient engine that can be built and what is its efficiency? Scientists and engineers have pursued ways to increase the efficiency of the heat engines ever since the invention of the steam engine. Eventually, a French engineer named Sadi Carnot (1796-1832) demonstrated what it takes to make the most efficient heat engine. Carnot proved that it is not possible to build a heat engine more efficient than the **Carnot cycle**, which is named after him. The Carnot cycle is an ideal cycle, in other words, it has no application in real life.

The Carnot cycle consists of two adiabatic and two isothermal processes. As seen in Figure 13.16, starting from point *a* at the high temperature T_H , expansion at constant temperature (isothermal) takes place along the path *ab*, after which cooling takes place through adiabatic expansion until the temperature T_C is reached along the path *bc*, and then isothermal compression takes place along the path *cd*, until, finally, the cycle is completed with adiabatic compression along the path *da*.

In order to calculate the efficiency of the Carnot cycle, let us examine where heat is received and where it is given off. As $Q = 0$ in adiabatic processes, heat exchange occurs only during the isothermal processes. The gas receives heat during the *ab* process, because it expands, and it gives off heat to the environment during the *cd* process. Therefore, the heat received from the hot source along the path *ab* will be equal to Q_H and the heat given off to the cold source along the path *cd* will be Q_C .

As $\Delta T = 0$ in isothermal processes, we get $\Delta E = 0$. Therefore, according to the first law $\Delta E = Q - W$, we get $Q = W$ in isothermal processes. We can therefore directly calculate work using Eq. (13.3), rather than the heat:

$$\text{The isothermal process } ab: \quad Q_H = W_{ab} = nRT_H \ln \frac{V_b}{V_a}$$

$$\text{The isothermal process } cd: \quad Q_C = W_{cd} = nRT_C \ln \frac{V_d}{V_c}$$

We use these values in the definition of efficiency. Since $V_d < V_c$ in the last expression, the heat Q_C will be negative. We make the logarithm positive as $\ln(V_d/V_c) = -\ln(V_c/V_d)$:

$$e = 1 - \frac{|Q_C|}{|Q_H|} = 1 - \frac{T_C \ln(V_c/V_d)}{T_H \ln(V_b/V_a)}$$

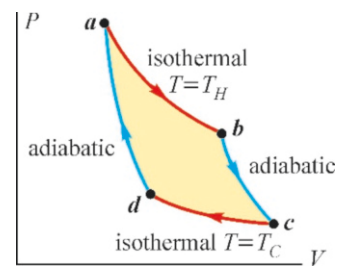


Figure 13.16: The Carnot cycle.

Now let us use the formula $TV^{\gamma-1} = \text{constant}$ for the paths bc and da :

$$\begin{aligned}T_H V_b^{\gamma-1} &= T_C V_c^{\gamma-1} \\T_H V_a^{\gamma-1} &= T_C V_d^{\gamma-1}\end{aligned}$$

Dividing both sides of these two equations, we get

$$\frac{V_b^{\gamma-1}}{V_a^{\gamma-1}} = \frac{V_c^{\gamma-1}}{V_d^{\gamma-1}} \quad \longrightarrow \quad \frac{V_b}{V_a} = \frac{V_c}{V_d}$$

The final expression for efficiency is:

$$e = 1 - \frac{T_C}{T_H} \quad (\text{Efficiency of the Carnot Cycle}) \quad (13.16)$$

It is remarkable that efficiency depends only on the ratio of temperatures of the hot and cold sources. In the Carnot cycle, the smaller the ratio T_C/T_H gets, in other words, the higher the temperature difference gets, the higher the efficiency will be. Carnot proved that no other cycle can have a higher efficiency.

Multiple-choice Questions

- Which is the first law of thermodynamics?
 - $\Delta E = Q + W$
 - $\Delta E = Q - W$
 - $\Delta W = E - Q$
 - $\Delta Q = E - W$
- Which is correct for the second law of thermodynamics?
 - Received heat can be fully converted into work.
 - The difference between received heat and given heat is converted into work.
 - The sum of received heat and given heat is converted into work.
 - It is not possible to fully convert received heat into work.
- Which is correct for the adiabatic process?
 - Gas performs no work.
 - Gas exchanges no heat.
 - Internal energy does not increase.
 - Internal energy does not change.
- Which is correct for the isothermal process?
 - The volume remains constant.
 - The pressure remains constant.
 - The temperature remains constant.
 - Gas exchanges no heat.
- Which of the following is the formula for the work performed by a gas?
 - $P dV$
 - $V dP$
 - V/dP
 - dP/V
- Which of the following is the efficiency of a heat engine?
 - Ratio of the work performed to the heat received by the gas.
 - Ratio of the work performed to the heat given by the gas.
 - Ratio of the work done to the total heat.
 - Ratio of the work performed to the internal energy.
- Which of the following is correct?
 - A gas performs work if it is heating.
 - A gas performs work if it is cooling.
 - A gas performs work if it is expanding.
 - A gas performs work if its pressure is increasing.
- The volume of a gas is increased by 2 m^3 at a constant pressure of 10 Pa . How much work will the gas perform?
 - 5 J
 - 1/5 J
 - 12 J
 - 20 J
- A gas with 3 m^3 in volume and 2 atm in pressure is heated at constant volume to increase its pressure to 10 atm . How much work will the gas perform?
 - 0
 - 18 J
 - 12 J
 - 30 J
- A gas receives 50 J of heat and performs 30 J of work. What is the change in internal energy?
 - 20 J
 - 20 J
 - 80 J
 - 80 J

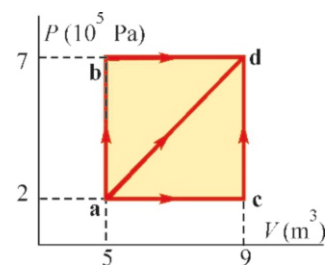
11. The internal energy of an ideal gas changes with which of the following?
 (a) Temperature.
 (b) Volume.
 (c) Pressure.
 (d) All of the above.
12. What is the relation between c_v , the specific heat at constant volume, and c_p , the specific heat at constant pressure of an ideal gas?
 (a) $c_p=c_v$ (b) $c_p=2c_v$ (c) $c_p=c_v+R$ (d) $c_v=c_p+R$
13. What is the origin of the pressure of a gas at the microscopic scale?
 (a) Collisions of atoms with container walls.
 (b) Collisions between atoms.
 (c) The gravitational force on atoms.
 (d) The kinetic energy of atoms.
14. The root-mean-square speed of the molecules of a gas is independent of which of the following?
 (a) Temperature
 (b) Molar mass
 (c) The gas constant R
 (d) Pressure
15. The efficiency of the gasoline engine is dependent on which of the following?
 (a) Temperature difference
 (b) Pressure difference
 (c) Compression ratio
 (d) Amount of the air-gasoline mixture
16. Which of the following are correct?
 I. It is not possible to build a heat engine with 100 % efficiency.
 II. There can be no heat engine more efficient than the Carnot cycle.
 III. The Carnot cycle has 100 % efficiency.
 IV. The Carnot cycle is used in industry. .
 (a) I & II (b) I & III (c) II & IV (d) I & IV
17. The molecules of which of the three gases, hydrogen (H_2), oxygen (O_2) and carbon dioxide (CO_2) at the same temperature have higher root-mean-square speed?
 (a) H_2 (b) O_2 (c) CO_2 (d) Equal
18. A gas is compressed at constant temperature. Which is correct?
 (a) It will receive heat.
 (b) It will give off heat.
 (c) Its internal energy will increase.
 (d) Its internal energy will decrease.
19. In which process is it necessary to give higher heat to increase the temperature of a gas by the same amount?
 (a) At constant volume.
 (b) At constant pressure.
 (c) Adiabatic.
 (d) Isothermal.
20. When a gas expands adiabatically, what is the source of the work that it performs?
 (a) The heat it receives.
 (b) The heat it gives.
 (c) The increase in internal energy
 (d) The decrease in internal energy.

Problems

13.1 Work Done by a Gas

13.1 A cylinder with 20 liters of volume contains 3 moles of gas at a temperature of 27°C . Calculate the work performed by this gas during the following consecutive processes: (a) Its temperature is increased by a factor of 4 at constant pressure. (b) Later, its pressure is increased by a factor of 4 at constant volume. (c) Later, its pressure is increased by a factor of 2 at constant temperature. [A: (a) 220 J, (b) 0, (c) -51 J .]

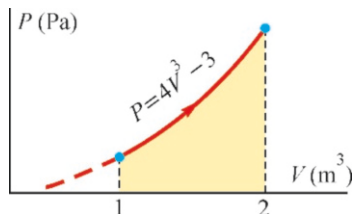
13.2 A cylinder contains 5 moles of gas at a temperature of 27°C . 2000 J of work is externally performed on this gas under constant pressure. What will the final temperature of the gas be? [A: 348 K.]



Problem 13.3

13.3 Calculate the work performed by the gas during all three processes (abd , acd , ad) shown in the P - V diagram above.

[A: $W_{abd} = 2800\text{ kJ}$, $W_{acd} = 800\text{ kJ}$, $W_{ad} = 1800\text{ kJ}$.]



Problem 13.4

13.4 A gas expands from volume $V_1=1\text{ m}^3$ to volume $V_2=2\text{ m}^3$ along the curve $P=4V^3-3$ (Pa) shown in the figure above. Calculate the work performed by the gas. [A: 12 J.]

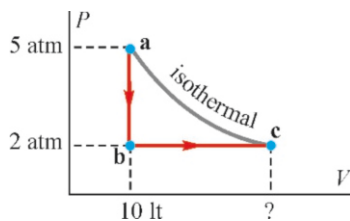
13.2-3-4 Internal Energy - First Law of Thermodynamics and Applications

13.5 As a gas is heated by being given 3000 J of energy, it simultaneously performs 500 J of work. What is the change in the internal energy of the gas? [A: $\Delta E = +2500\text{ J}$]

13.6 A monatomic gas of 4 moles is heated by giving it 800 J under constant pressure. ($c_v=3R/2$, $c_p=5R/2$.) (a) What will the temperature increase ΔT be? (b) What will the increase in internal energy be? (c) How much work will the gas perform? [A: (a) 9.6 K, (b) 480 J, (c) 320 J.]

13.7 A vessel contains 3 moles of a diatomic gas at a temperature of 27°C ($c_v=5R/2$, $c_p=7R/2$.) (a) The temperature of the gas is increased to 127°C at constant volume. Calculate the given heat, the work performed by the gas and the increase in internal energy. (b) The temperature of the gas is increased to the same 127°C at constant pressure. Calculate the given heat, the work performed by the gas and the increase in internal energy.

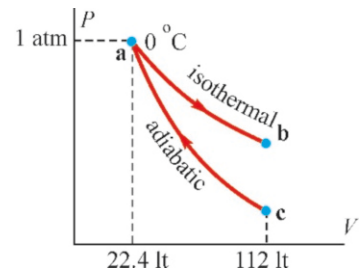
[A: (a) $Q = \Delta E = 6.2\text{ kJ}$, $W = 0$,
(b) $Q = 8.7\text{ kJ}$, $\Delta E = 6.2\text{ kJ}$, $W = 2.5\text{ kJ}$.]



Problem 13.8

13.8 A vessel contains $V_a=10\text{ L}$ of ideal gas at pressure $P_a=5\text{ atm}$. This gas is first cooled at constant volume along the path ab shown in the figure above until its pressure reaches $P_b=2\text{ atm}$. Then, it is again brought to temperature T_a by expansion under constant pressure. (a) What is the volume V_c ? (b) What is the total work performed by the gas? (c) What is the total increase in internal energy? (d) How much heat does the gas give or receive?

[A: (a) 25 L, (b) $W = 3000\text{ J}$, (c) $\Delta E = 0$, (d) it receives $Q = 3000\text{ J}$.]



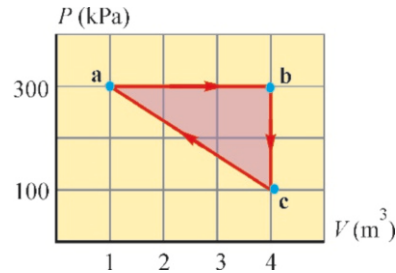
Problem 13.9

13.9 We had previously seen that 1 mole of monatomic ideal gas at STP conditions (0°C and 1 atm) has a volume of 22.4 L (point *a* in the figure above). The volume of this gas is increased by a factor of 5 in two different ways: The process ab is isothermal ($T=\text{constant}$) and the process ac is adiabatic ($\Delta Q=0$). Calculate and compare the work performed by the gas in both cases. ($c_v=3R/2$, $\gamma=1.4$.)

[A: Isothermal 3700 J, adiabatic 1600 J, isothermal work is always greater.]

13.10 A cylinder contains 3 moles of diatomic gas at a temperature of 27°C ($c_v=5R/2$). 1000 J of work is externally performed on this gas under constant pressure. (a) What will the final temperature of the gas be? (b) What will the internal energy be? (c) How much heat has the gas received or given off? (Notice that the work is negative.)

[A: (a) 260 K, (b) $\Delta E = -2500\text{ J}$, (c) $Q = -3500\text{ J}$ has been given off.]



Problem 13.11

13.11 A gas completes a cycle through the 3-step process ($a \rightarrow b \rightarrow c \rightarrow a$) shown in the figure. What is the net heat received by the gas? [A: 300 kJ.]

13.12 A cylinder with 4 liters of volume contains a monatomic ideal gas at 27°C in temperature and 3 atm in pressure ($\gamma = 1.67$). The piston is suddenly pushed to compress the volume of the gas to 1 L before it can exchange any heat. (a) What will the final temperature and pressure of the gas be? (b) How much work will the gas perform?

[A: (a) 750 K, 30 atm, (b) -2.7 kJ .]

13.4 Kinetic Calculation of Pressure

13.13 The molecules of an oxygen gas in a cylinder collide 3×10^{23} times in 1 s with the surface of a piston with a cross-section area of 100 cm^2 . If the root-mean-square speed of molecules is 500 m/s, (a) What is the force exerted by each molecule on the piston in 1 s? (b) How much pressure does

the gas exert on the piston? (The mass of the oxygen molecule is 5.3×10^{-26} kg.) [A: (a) 5.3×10^{-23} N, (b) 1600 Pa.]

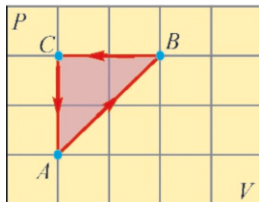
13.14 What are the root-mean-square speeds of oxygen gas (O_2), which has molar mass 32 g, and of nitrogen gas (N_2), which has a molar mass 28 g, at the temperature at which the root-mean-square speed of argon gas (Ar), which has a molar mass 40 g, is 200 m/s? [A: 224 and 239 m/s.]

13.15 A container with 5 L of volume contains 3 moles of helium gas (He) at 127°C in temperature ($M_H=4$ g). (a) What is the total kinetic energy of the gas molecules? (b) What is the root-mean-square speed of molecules? [A: (a) 15 kJ, (b) 1580 m/s.]

13.5 Heat Engines

13.16 (a) A heat engine receives 5000 J in heat from a hot source and releases 2000 J in heat to the environment. What is the efficiency of the machine? (b) A heat engine operating with 20% efficiency performs 300 J of work in one cycle. How much heat does this machine release into the environment? [A: (a) 60 %, (b) 1200 J.]

13.17 What is the minimum value of the compression ratio r such that a gasoline engine operating with the Otto cycle can have 50 % efficiency? [A: $r = 5.7$.]

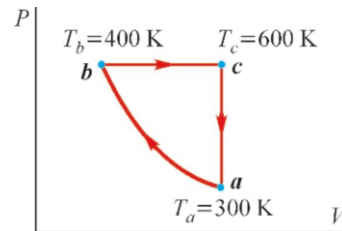


Problem 13.18

13.18 Determine the sign of heat (Q), work (W) and internal energy increase (ΔE) in each step of the cycle ABC shown

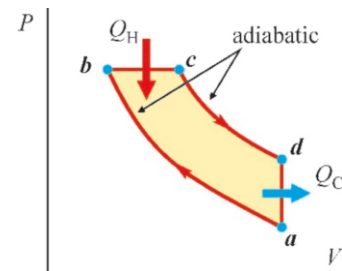
	Q	W	ΔE
$A \rightarrow B$			
$B \rightarrow C$			
$C \rightarrow A$			

on the left in the figure above and mark them in the table on the right.



Problem 13.19

13.19 A gas with $\gamma=c_p/c_v=1.67$ undergoes a 3-step cycle as shown in the figure above. It is compressed adiabatically along the path ab , then expanded at constant pressure along the path bc , and finally cooled at constant volume along the path ca . $T_a=300$ K, $T_b=400$ K and $T_c=600$ K. Calculate the efficiency of the cycle. [A: 10 %.]



Problem 13.20

13.20 Diesel cycle. Calculate the efficiency of the Diesel cycle shown in the figure in which $T_a=300$ K, $T_b=900$ K, $T_c=1400$ K and $T_d=700$ K. ($\gamma=c_p/c_v=1.4$ for air) [A: 43 %.]

14

THE ELECTRIC FIELD



Lightning is the most striking manifestation of electricity in nature. Negatively charged electrons accumulated in clouds suddenly flow to the Earth's surface and transfer large amounts of energy.

Gravitation was the first force to be discovered among the fundamental forces of nature. Electricity and magnetism only came to be understood later, around the 1600s. But electric and magnetic forces shape natural events and play a major role, perhaps the most significant role for living things and for technology. The nervous system in the human body operates with electric currents and the oxygen exchange in blood takes place with electric potential differences. Light bulbs, radios, televisions and other appliances facilitate domestic life; electric motors and electronic components in technology are included in the structure of all kinds of instruments and machinery.

Gravitational force may hold our solar system and much larger galaxies together. But it is the electric force that keeps atoms and molecules together to form living bodies and organisms. With this new force, a new property of matter, called the **electric charge**, arises in two forms, positive and negative.

Then, moving charges produce another fundamental force, called **magnetism**. The sciences of electricity and magnetism first developed independently, and were later united into a single theory, called **electromagnetism**. We will first discuss the basic concepts and laws of electricity in this chapter and then those of magnetism in subsequent chapters.

14.1 ELECTRIC CHARGE



Figure 14.1: Amber is fossilized tree resin.



Figure 14.2: A comb rubbed with a piece of cloth attracts small pieces of paper.

The existence of electric forces has been observed since ancient times. Fossilized pieces of pine resin, known as amber, could attract small objects. The word “electric” was derived from the word *electron*, which means amber in Greek. It was later observed that other materials made of glass, rubber, plastic or fur could gain the same property through friction.

You may perform such experiments at home too. Take a plastic comb or pen, rub it with a piece of cloth and then hold it close to small pieces of paper. You will observe that the comb attracts the small pieces.

However, if you rub a second plastic comb the same way and bring it close to the first one, you will observe that they *repel* each other. (You may have to suspend the comb with a rope to observe this effect.) Likewise, if you apply friction to a glass rod and bring it close to the pen, you will observe that, this time, they *attract* each other. This attractive-repulsive property proves that electric force has a different structure than gravity, because gravitational force can only be attractive.

Each force in nature shows its effect through an intrinsic quantity in matter. After gravitational force was observed, it was understood that it acted on a property of matter called *mass*. Likewise, electric forces should also arise from another quantity inherent in matter, this one called the **electric charge**. Let us emphasize the properties that this electric charge should have:

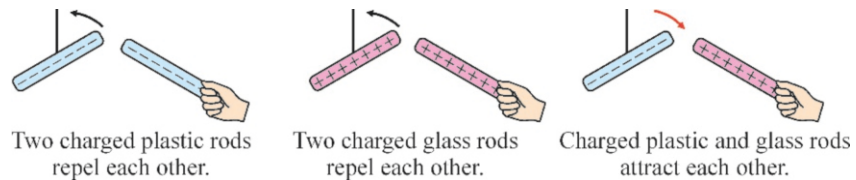


Figure 14.3: Two different types of electric charge.

- **There are two types of electric charge with opposite signs.** Only in this way can one accommodate the attractive and repulsive forces that arise in electricity. These were called positive and negative charges. Among the elementary particles that were discovered later, protons (p^+) were assumed to be positively charged and electrons (e^-) to be negatively charged. We must emphasize that this is only an assumption. Today, scientists consider that this assumption was unfortunate, and that it would have been more practical to assume electrons as positive. Indeed, in the majority of electrical phenomena, it is usually electrons that move in matter and positively charged protons and ions that are stationary, as they are heavier. As we shall discuss later, the electric current is actually a flow of electrons in the inverse direction.
- **Electric charge is conserved.** In other words, *the algebraic sum of the electric charges in an isolated system is constant*. For example, if -3 units of charge are added to an object with +5 units of charge, the object’s net charge will be +2 units. Objects accepted as neutral actually have both positive and negative charges and seem neutral because they are in equal amount. If some negative charge is taken away from such a neutral object, the object will be positively charged.

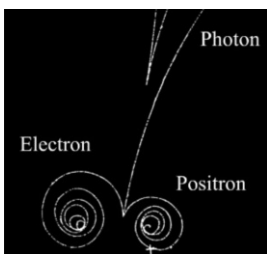


Figure 14.4: Electron-positron pair production at Lawrence Berkeley National Lab.

In modern physics, charge conservation is not only algebraic, but also valid on a much more fundamental scale. Electron-positron pairs ($e^- e^+$) can be produced in vacuum in experiments conducted with elementary particles

(Figure 14.4). As the charge of the positron is equal and opposite to the electron, the net charge remains zero and charge conservation is respected.

- **Electric charge of objects is always a multiple of an elementary charge.** It was observed that electric charge does not change continuously, but varies as multiples of a minimum charge. In the terminology of modern physics, it is said that electric charge is **quantized**. This elementary charge is indicated with e . The electron charge ($-e$) and the proton charge ($+e$) are exactly equal and opposite.

In today's modern physics, this is actually not exactly correct. Particles called **quarks**, considered to be the building blocks of elementary particles, are thought to be charged as $e/3$ or $2e/3$, in other words, as fractions of the electron charge. However, these quarks have not yet been observed directly.

- **Unit of electric charge.** In the SI system, the unit of electric charge is the **Coulomb (C)**. The value of the elementary charge e is as follows:

$$1 e = 1.6 \times 10^{-19} \text{ C} \quad (\text{elementary charge}) \quad (14.1)$$

A Coulomb is a very large unit, and fractions are used in practice:

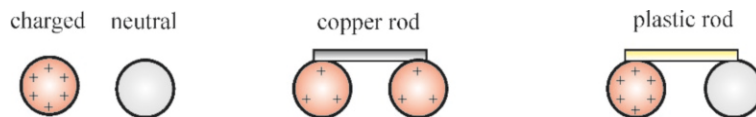
$$1 \text{ mC (milli Coulomb)} = 10^{-3} \text{ C}$$

$$1 \mu\text{C (micro Coulomb)} = 10^{-6} \text{ C}$$

- You should also know that the fundamental electrical unit in the SI system is not the Coulomb. Instead, the **ampere (A)**, the unit of current, is chosen as the fundamental unit. The reason for this is that, in practice, it is easier to measure current than charge. We shall define the ampere in Chapter 20.

Conductors and Insulators – Structure of the Atom

Let us take an object charged through friction and place it next to a neutral object. When we connect these two objects with a copper rod, we observe that some of the electric charge is transferred to the neutral object. However, when we connect these two objects with a plastic rod, we observe that no charge is transferred and the neutral object remains neutral.



This experiment shows that metals such as copper and iron easily conduct electricity and substances such as glass do not conduct electricity at all. All materials show either **conductor** or **insulator** properties in terms of electrical conductivity. All metals are conductors. On the other hands, insulators may become conductors under very high charge accumulations.

Other than these, there are **semiconductor** materials developed in laboratories and used in electronic devices. Also, many elements can switch to a **superconductor** state at very low temperatures. These two types of materials cannot be explained with classical electromagnetism and must be analyzed with modern quantum theory.

The reason why a material is a good or bad conductor can be understood by examining the structure of the atom. As explained by modern quantum theory, the

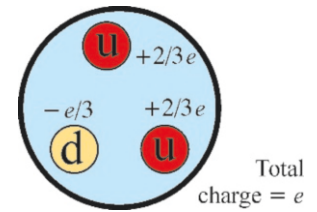


Figure 14.5: According to modern physics, a proton consists of 3 quarks with fractional charges.

Figure 14.6: Example of a conductor and an insulator: The copper rod conducts the charge and the plastic does not.

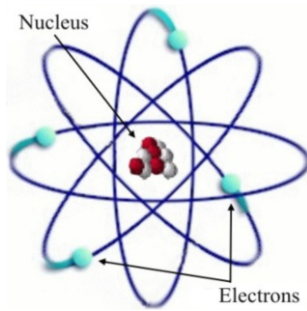


Figure 14.7: Atomic structure.

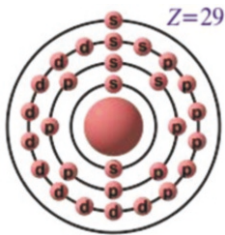


Figure 14.8: The single electron outside closed orbits in copper atom, makes it a conductor.

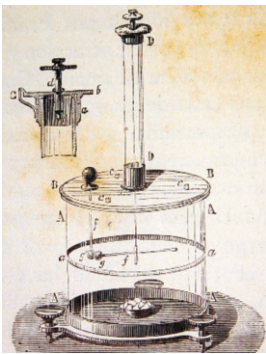


Figure 14.9: The torsion balance used by Coulomb.

structure of all elements consists of a positively charged **nucleus** with electrons rotating around it in orbits (Figure 14.7). The nucleus contains positively charged protons (p^+) and neutral neutrons (n).

In a sense, this resembles the Sun and the planets rotating around it, however, this resemblance is not exactly correct. Electrons can rotate only at certain radii and the number of electrons that can inhabit each orbit is limited. For example, the innermost orbit can only hold a maximum of two electrons, while the next one can hold 8 electrons, etc.

When these orbits become filled with the maximum number of electrons that they can hold, they are called **closed shells**, and the next electron is forced to go to next outer orbit. The electrons in the innermost orbits are very strongly bonded to the nucleus, while the outer electrons are weakly bonded.

The electrons of an atom can be separated by externally bombarding it with particles. Observations show that it is more difficult to detach the electrons in closed orbits and inner orbits, as they are more strongly bonded to the nucleus. However, electrons in an outermost and incomplete orbit can be detached much more easily.

When atoms gather to form a rigid body, the state of these weakly-bonded electrons in the outermost orbit determines the electrical conductivity of that matter. The one and two electrons in the outermost orbit of metals get released easily without any bombardment and circulate freely in the solid medium. In insulators, they are firmly bonded to the atoms, as the outermost orbits are closed-shell orbits.

The source of electrical conductivity is these **free electrons**. Metals are conductive because free electrons can easily move in a solid medium. Insulators do not transfer charge because they have no free electrons.

14.2 COULOMB'S LAW

French scientist Charles Coulomb (1736–1806) was the first to examine the nature of the force between electric charges. He had to invent new ways to measure such forces. After charging a metal sphere, he put it in contact with an identical second metal sphere and assumed that the charge was distributed evenly among two spheres. He thus obtained half, quarter and one eighth charged, etc., spheres. He used a torsion balance to measure the very small forces among these spheres.

And he established the law bearing his name:

Coulomb's Law

The force of attraction or repulsion between two electric charges is directly proportional to the product of the two charges and inversely proportional to the square of the distance between them:

$$F = k \frac{q_1 q_2}{r^2} \quad (14.2)$$

Like charges repel, unlike charges attract each other.

Let us emphasize the important points of Coulomb's law:

- Coulomb's law is valid for stationary charges. It is therefore also known as the **electrostatic force**.
- The proportionality constant indicated with k in Coulomb's law is called the **Coulomb's constant**:

$$k = 8.99 \times 10^9 \approx 9 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2 \quad (14.3)$$

The approximate value will be used in the problems.

- Later we shall see that it is more convenient to define a new constant ϵ_0 instead of k in some of the expressions we will be developing:

$$k = \frac{1}{4\pi\epsilon_0} \quad \leftrightarrow \quad \epsilon_0 = \frac{1}{4\pi k} = 8.85 \times 10^{-12} \text{ C}^2/(\text{N}\cdot\text{m}^2) \quad (14.4)$$

The constant ϵ_0 is called the **electric permittivity of free space**.

Coulomb Force for a System of Charges

If there are more than two charges in an environment, the net force on any charge is the *vector sum* of the forces that the other charges exert upon it. For example, consider four charges as q_1, q_2, q_3 and q_4 . The net force exerted upon one of these, for example, upon q_1 , is written as follows:

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} \quad (14.5)$$

We take the vector sum of the forces F_{ij} after finding their magnitudes using the Coulomb's law.

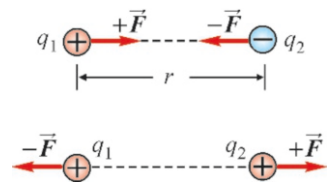


Figure 14.10: Coulomb's law. Unlike charges attract, like charges repel.

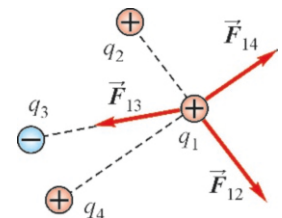
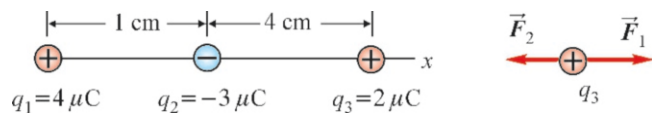


Figure 14.11: System of charges.

Example 14.1



Calculate the total force exerted upon the charge $q_3 = 2 \mu\text{C}$ by the charges $q_1 = 4 \mu\text{C}$ and $q_2 = -3 \mu\text{C}$ shown in the figure.

Answer

Charges with the same sign repel and charges with opposite signs attract each other. Accordingly, the directions of the force F_1 exerted by the charge q_1 and the force F_2 exerted by the charge q_2 are shown on the right-hand side of the figure.

We calculate the magnitudes of both forces with the Coulomb's law. The signs of the charges are ignored when calculating the magnitudes:

$$F_1 = k \frac{q_1 q_3}{r_1^2} = 9 \times 10^9 \times \frac{(4 \times 10^{-6}) \times (2 \times 10^{-6})}{(0.01 + 0.04)^2}$$

$$F_1 = 29 \text{ N}$$

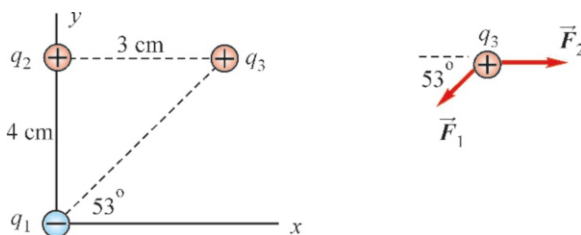
$$F_2 = k \frac{q_2 q_3}{r_2^2} = 9 \times 10^9 \times \frac{(3 \times 10^{-6}) \times (2 \times 10^{-6})}{0.04^2}$$

$$F_2 = 34 \text{ N}$$

We find the total force by taking the positive x -direction towards the right:

$$F = F_1 - F_2 = 29 - 34 = -5 \text{ N.}$$

Example 14.2



For the charges $q_1 = -3 \mu\text{C}$ and $q_2 = +2 \mu\text{C}$ in the figure, (a) Calculate the components of the total force acting on the

charge $q_3 = +4 \mu\text{C}$.

(b) Find the magnitude and direction of the total force.

Answer

(a) The figure shows the forces \vec{F}_1 and \vec{F}_2 exerted by the charges q_1 and q_2 . We first calculate the magnitudes of these forces:

$$F_1 = k \frac{q_1 q_3}{r_1^2} = 9 \times 10^9 \times \frac{(3 \times 10^{-6}) \times (4 \times 10^{-6})}{0.05^2}$$

$$F_1 = 43 \text{ N}$$

$$F_2 = k \frac{q_2 q_3}{r_2^2} = 9 \times 10^9 \times \frac{(2 \times 10^{-6}) \times (4 \times 10^{-6})}{0.03^2}$$

$$F_2 = 80 \text{ N}$$

The total force is $\vec{F} = \vec{F}_1 + \vec{F}_2$, and we calculate its components as follows:

$$F_x = F_2 - F_1 \cos 53^\circ = 80 - 43 \times 0.6 = 54 \text{ N}$$

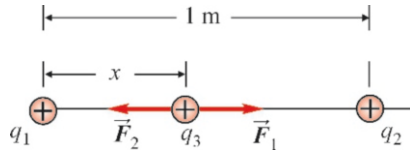
$$F_y = -F_1 \sin 53^\circ = -43 \times 0.8 = -35 \text{ N}$$

(b) We calculate the magnitude and angle of the force with the known components:

$$F = \sqrt{F_x^2 + F_y^2} = 64 \text{ N}$$

$$\tan \theta = \frac{F_y}{F_x} = \frac{-35}{54} = -0.65 \rightarrow \theta = -33^\circ$$

Example 14.3



The distance between the charges $q_1 = +1 \text{ mC}$ and $q_2 = +9 \text{ mC}$ shown in the figure is 1 m. Where should a third charge q_3 be placed such that the net force exerted on it is zero?

Answer

As the charges q_1 and q_2 have the same sign, they exert force in the same direction upon the charges on the outside.

Only the forces they exert at the points in between will be in the opposite directions. Let x be the coordinate of the point upon which the forces cancel each other out. Accordingly, we write the total force and set it as equal to zero:

$$F_1 = F_2 \rightarrow k \frac{q_1 q_3}{x^2} = k \frac{q_2 q_3}{(1-x)^2}$$

Simplifying, we get a quadratic equation:

$$\frac{q_1}{x^2} = \frac{q_2}{(1-x)^2} \rightarrow (q_2 - q_1)x^2 + 2q_1x - q_1 = 0$$

$$8x^2 + 2x - 1 = 0$$

The roots of this equation are -0.5 and $+0.25$. We will not consider the negative root, as it will be outside of the charges.

Therefore, the solution is the positive root:

$$x = 0.25 \text{ m.}$$

14.3 ELECTRIC FIELD

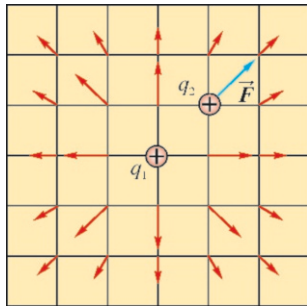


Figure 14.12: First the charge q_1 produces the electric field. The force $\vec{F} = q_2 \vec{E}$ acts on charge q_2 in this field.

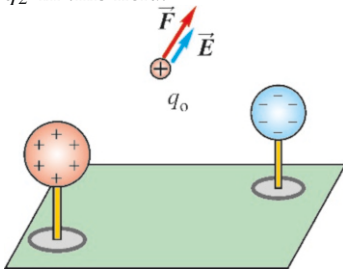


Figure 14.13: If a force \vec{F} acts on test charge q_0 there is an electric field $\vec{E} = \vec{F}/q_0$ at that point.

The Coulomb law is a force exerted between two charges at a distance. Newton’s gravitational force is likewise exerted between two masses a distance apart from each other. This property of “action at a distance” was disturbing for many scientists, including Newton. Just think about it: A charge q_1 examines its surroundings, detects the presence of another charge q_2 , determines its distance and charge, and then exerts Coulomb’s force accordingly. Is this believable?

The English scientist Michael Faraday (1791–1867) suggested the concept of an electric field to resolve this problem. According to Faraday, when a charge q_1 is placed anywhere, it affects every point of the surrounding space and produces an electric field (Figure 14.12). This electric field always remains there, even if no other electrical charge exists. A second charge q_2 that is placed later interacts with this electric field through Coulomb’s law. In other words, the first charge does not check the position and value of the second charge. The concept of field was later taken much further in modern physics and was understood to be a correct approach.

Since a charge interacts with the surrounding electric field, let us take a very small positive q_0 **test charge** and place it at a point in space. If the electrostatic force acting on this charge q_0 is \vec{F} , then the **electric field** at that point in space is defined as:

$$\vec{E} = \frac{\vec{F}}{q_0} \quad (\text{The Electric Field}) \quad (14.6)$$

The inverse is also true: The force acting on a charge q in the presence of an electric field \vec{E} is,

$$\vec{F} = q \vec{E} \quad (14.7)$$

Let us emphasize the main properties of an electric field:

- An electric field can also be considered as the “force acting on a unit charge.”
- The unit of electric field is newton/coulomb (N/C).
- The positive test charge q_0 used in the definition is selected as very small, in order not to disturb the distribution of the electric field in the environment. In a more correct definition, it would be necessary to take the limit $q_0 \rightarrow 0$, but we will not worry about this detail.
- According to this definition, the electric field \vec{E} and the force vector \vec{F} on a charge q will have the same direction if the charge q is positive, and the opposite direction if negative. An easy-to-remember rule is derived from this:

Positive charges always try to go along with and negative charges always try to go opposite to the electric field.

Electric Field of a Point Charge

As the simplest case, let us calculate the electric field produced by a point charge q located at the origin, at a point with position vector \vec{r} . According to the definition, we place a positive test charge q_0 at position \vec{r} and examine the force exerted upon it (Figure 14.14). If we write Coulomb’s law for the charges q and q_0 , the magnitude of the force is

$$F = k \frac{qq_0}{r^2}$$

From here, according to the definition of the electric field, we find that

$$E = \frac{F}{q_0} = \frac{kq}{r^2}$$

The direction of this electric field depends on the sign of charge q (Figure 14.15): If q is positive, it will be away from the origin, in other words, in the same direction as \vec{r} , as it shall repel the other positive test charge q_0 . If q is negative, it will attract the test charge, q_0 , in other words, the force will be towards the origin and in the direction of $-\vec{r}$.

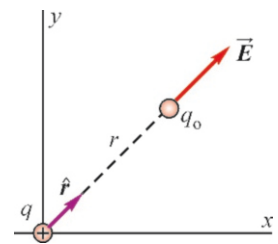
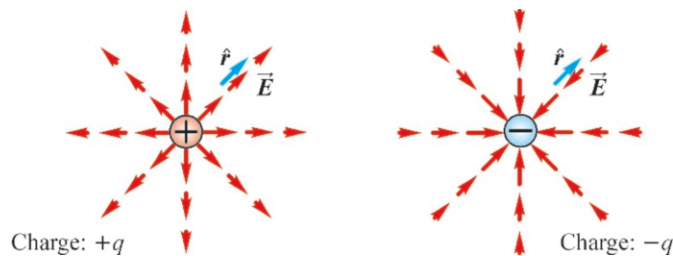


Figure 14.14: Electric field of a point charge.

Figure 14.15: Electric field vectors are outwards from the + charge (along unit vector \hat{r}), inwards towards the – charge (opposite to \hat{r}).

It is possible to show both cases in a single expression. Remember the concept of unit vector that we defined in Chapter 1: We defined the vector \hat{a} as the unit vector in the same direction as the vector \vec{a} . Here, if we use \hat{r} to indicate the unit vector along the position vector \vec{r} , the electric field vector of a point charge can be expressed as follows:

$$\vec{E} = \frac{kq}{r^2} \hat{r} \quad (\text{Electric field of a point charge}) \quad (14.8)$$

This expression gives the correct direction for both a positive and a negative charge q . If q is positive, \vec{E} and \vec{r} are in the same direction. If q is negative, \vec{E} is in opposite direction to \vec{r} .

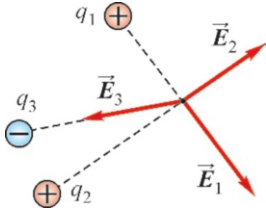


Figure 14.16: The electric field of multiple point charges.

Electric Field of a System of Charges

Just as the Coulomb force of several charges could be written as the sum of vectors, the same is also true for the electric field. The net electric field at any point P due to N charges such as q_1, q_2, \dots, q_N will be the vector sum of the electric field exerted by each one:

$$\vec{E}_P = \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_N = \sum_i \vec{E}_i$$

If $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$ are the position vectors of point P as measured from each charge, and if we use the unit vectors along them, we get

$$\vec{E}_P = \frac{kq_1}{r_1^2} \hat{r}_1 + \frac{kq_2}{r_2^2} \hat{r}_2 + \dots + \frac{kq_N}{r_N^2} \hat{r}_N = k \sum_i \frac{q_i}{r_i^2} \hat{r}_i \quad (14.9)$$

Electric Field Lines

Unfortunately, the electric field is not visible, and it is difficult to imagine certain cases. Michael Faraday, who suggested the concept of the electric field, developed the technique of **electric field lines** to make it easier to visualize them.

We rewrite the electric field \vec{E} produced by a positive charge q at any point \vec{r} :

$$\vec{E} = \frac{kq}{r^2} \hat{r}$$

We would have to draw an infinite number of small arrows if we wanted to indicate this electric field as a separate vector at each point in space. This would be both difficult and impractical. Instead, let us consider the following method:

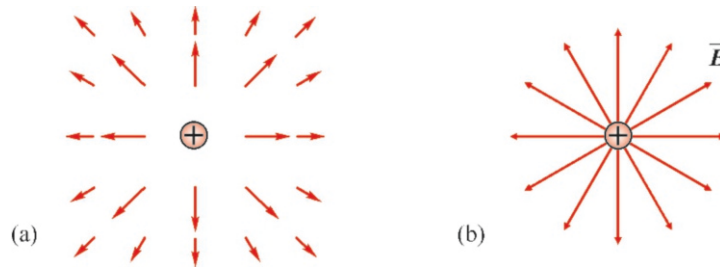


Figure 14.17: Rather than drawing lots of arrows, it is more meaningful to draw rays starting from the charge.

Let us draw the electric field of a point charge at a few points by going further away from the charge (Figure 14.17a). The lengths of these \vec{E} vectors will gradually get smaller. Now, we draw a single line connecting successive vectors in a given direction, extending from the origin to infinity (Figure 14.17b). We do the same thing in another direction. We thus obtain a bundle that spreads out from the origin like light beams. We put an arrow indicating the direction of the electric field on each of these lines.

This simple example shows how to draw and interpret the field lines in the most general case:

- *At any given point, the electric field vector \vec{E} is tangent to the field lines.* The arrow on the field lines determines the direction towards which we will draw the tangent.
- *The magnitude of the electric field at any point is proportional to the density of the field lines around that point.* For example, the electric field of a point

charge decreases with distance from the origin; therefore, the field lines also diverge from each other. In Figure 14.18, the electric field at point A is greater than that at point B.

- The electric field will be towards the origin at each point if the point charge at the origin is a negative. Therefore, *electric field lines start at positive charges and end at negative charges or at infinity.*
- *Electric field lines never intersect.* If the opposite was true, it would be as if there could be two tangents, in other words, two electric fields at that point.



Figure 14.18: At which point is the magnitude of the electric field greater?

Example 14.4

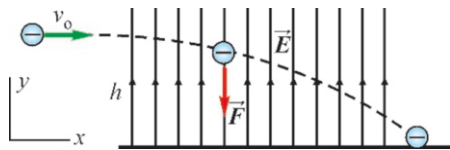
- (a) The magnitude of the electric field is measured as 10 000 N/C at a distance of 1.5 m from a point charge. What is the charge?
- (b) What is the force exerted on an electron placed at 1 mm distance from such a charge? (Electron charge: $q_e = -e = -1.6 \times 10^{-19}$ C).

Answer

(a) We write the expression for the electric field at distance r from a point charge q and solve for q :

$$E = \frac{kq}{r^2} \rightarrow q = \frac{r^2 E}{k}$$

Example 14.5



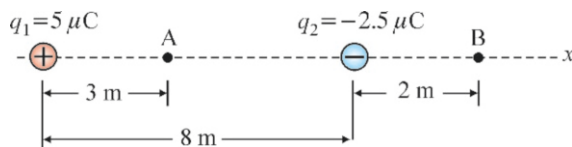
There is a vertically upward constant electric field $E=10$ N/C in a region. An electron is thrown with a horizontal velocity of $v_0=10^6$ m/s at a height of $h=2$ m from the ground. (For the electron, $m_e=9.1 \times 10^{-31}$ kg and $q_e = -e = -1.6 \times 10^{-19}$ C.) Ignoring gravitational force,

- (a) What is the acceleration of the electron?
- (b) Calculate the horizontal range of the electron.

Answer

The force acting on a charge q in a field \vec{E} is $\vec{F} = q\vec{E}$. As the charge of the electron is $q = -e$, the exerted force shall be

Example 14.6



Calculate the total electric field at points A and B due to charges q_1 and q_2 shown in the figure.

Answer

The electric field of a point charge is outward from positive

We substitute the values and calculate q :

$$q = \frac{1.5^2 \times 10000}{9 \times 10^9} = 2.5 \times 10^{-6} \text{ C} = 2.5 \mu\text{C}$$

(b) The force acting on the charge q_e in the field \vec{E} is $\vec{F} = q_e \vec{E}$. If we write the expression for the electric field E of a point charge, the magnitude of the force is as follows:

$$F = |q_e|E = e \frac{kq}{r^2}$$

Substituting the value q found in item (a), the electron charge e and the distance $r = 0.001$ m, we calculate the force as follows:

$$F = 1.6 \times 10^{-19} \times \frac{9 \times 10^9 \times 2.5 \times 10^{-6}}{0.001^2} = 3.6 \times 10^{-9} \text{ N.}$$

opposite to the field \vec{E} , in other words, downward. We write Newton's second law:

$$F = ma \rightarrow qE = ma \rightarrow a = \frac{qE}{m}$$

This acceleration will be downward. We substitute the values and calculate the acceleration:

$$a = \frac{1.6 \times 10^{-19} \times 10}{9.1 \times 10^{-31}} = 1.8 \times 10^{12} \text{ m/s}^2$$

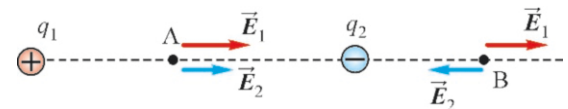
(b) This is a projectile motion problem with the acceleration g replaced by acceleration a . We find the range R by finding the time t from the vertical component of the motion and using it in the horizontal motion:

$$h = \frac{1}{2}at^2 \rightarrow t = \sqrt{2h/a}$$

$$R = v_0 t = v_0 \sqrt{2h/a}$$

$$R = 10^6 \times \sqrt{\frac{2 \times 2}{1.8 \times 10^{12}}} = 1.5 \text{ m}$$

charges and inwards towards negative charges. Accordingly, the fields \vec{E}_1 and \vec{E}_2 at points A and B are shown below:



Therefore, we first calculate the field magnitudes E_1 and E_2 at these points and take their vector sum.

At point A:

$$E_1 = \frac{kq_1}{r_1^2} = \frac{(9 \times 10^9) \times (5 \times 10^{-6})}{3^2} = 5000 \text{ N/C}$$

$$E_2 = \frac{kq_2}{r_2^2} = \frac{(9 \times 10^9) \times (2.5 \times 10^{-6})}{5^2} = 900 \text{ N/C}$$

We add the parallel electric fields at point A:

$$E_A = E_1 + E_2 = 5900 \text{ N/C} \quad (\text{towards right}).$$

At point B:

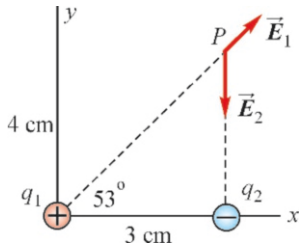
$$E_1 = \frac{kq_1}{r_1^2} = \frac{(9 \times 10^9) \times (5 \times 10^{-6})}{(8 + 2)^2} = 450 \text{ N/C}$$

$$E_2 = \frac{kq_2}{r_2^2} = \frac{(9 \times 10^9) \times (2.5 \times 10^{-6})}{2^2} = 5600 \text{ N/C}$$

We subtract the opposite electric fields at point B:

$$E_B = E_1 - E_2 = -5150 \text{ N/C} \quad (\text{towards left}).$$

Example 14.7



Calculate the components of the total electric field due to charges $q_1 = 5 \mu\text{C}$ and $q_2 = -3 \mu\text{C}$ at point P.

Answer

The figure shows the electric fields \vec{E}_1 and \vec{E}_2 produced by

the charges q_1 and q_2 at point P.

We first calculate the magnitudes of these fields separately:

$$E_1 = \frac{kq_1}{r_1^2} = \frac{9 \times 10^9 \times 5 \times 10^{-6}}{5^2} = 1800 \text{ N/C}$$

$$E_2 = \frac{kq_2}{r_2^2} = \frac{9 \times 10^9 \times 3 \times 10^{-6}}{4^2} = 1690 \text{ N/C}$$

We then calculate the components of the vector $\vec{E} = \vec{E}_1 + \vec{E}_2$:

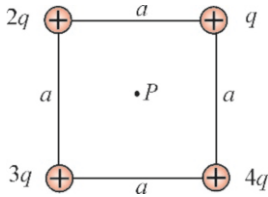
$$E_x = E_1 \cos 53^\circ = 1800 \times 0.6 = 1080 \text{ N/C}$$

$$E_y = E_1 \sin 53^\circ - E_2 = 1800 \times 0.8 - 1690 = -250 \text{ N/C}.$$

According to this result, as the x -component is positive and the y -component is negative, the vector \vec{E} is in the 4th quadrant of the plane.

Example 14.8

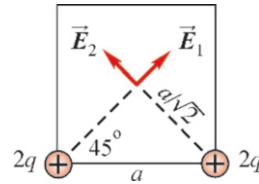
The figure shows the charges $q, 2q, 3q$ and $4q$ respectively placed on the corners of a square with side length a . Calculate the electric field at point P in the center of the square.



Answer

This problem is a good example of the use of symmetry. It is not necessary to calculate the electric field of four separate charges, because there is mutual symmetry. The electric field is zero at the center of two diagonally opposite equal charges, because they are equal and in opposite directions. Therefore,

we can mutually decrease the charges at diagonally opposite sides of point P. We get the following in the end:

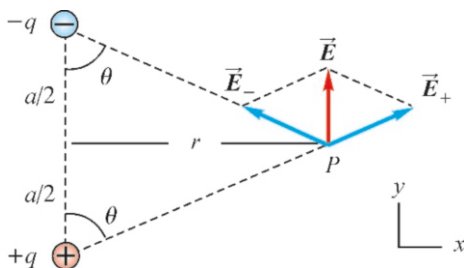


Therefore, we can solve this simpler problem. The x -component of the total electric field of equal $2q$ charges at point P is zero. For the y -component, it is sufficient to find one of the y -components and multiply it by 2:

$$E = E_y = 2E_{1y} = 2E_1 \sin 45^\circ = 2 \frac{k2q}{(a/\sqrt{2})^2} \cdot \frac{1}{\sqrt{2}}$$

$$E = \frac{4\sqrt{2}kq}{a^2}$$

Example 14.9



Electric dipole. The system consisting of two equal and opposite $\pm q$ charges with a small distance a in between is called an

electric dipole. The properties of this neutral system is important in many applications.

The electric field of the dipole plays an important role in the structure of matter. The attraction between neutral molecules, the operating principle of radio and TV antennas, the behavior of dielectric materials, etc., all result from the electric properties of dipoles.

- (a) Find the expression for the total electric field at point P located at distance r from the perpendicular bisector of the charges.
- (b) Find the limit of the electric field very far away from the dipole, when $r \gg a$.

Answer

(a) If we use \vec{E}_+ and E_- to indicate the electric fields of the charges $\pm q$, the total electric field at point P will be the vector sum of these two:

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

The magnitude of the electric fields of point charges $\pm q$ is equal at distance $\sqrt{r^2 + (a/2)^2}$:

$$E_+ = E_- = \frac{kq}{r^2 + a^2/4}$$

If we use the components of these two vectors to calculate the components of the total vector \vec{E} , the x -components are equal and in opposite directions and cancel each other out due to symmetry. Therefore, we only calculate and add the y -components:

$$E_x = E_{+x} + E_{-x} = 0$$

$$E_y = E_{+y} + E_{-y} = 2E_+ \cos \theta$$

$$E = E_y$$

We find the cosine of the angle in the figure and substitute it as follows:

$$\cos \theta = \frac{a/2}{\sqrt{r^2 + a^2/4}}$$

$$E = \frac{kqa}{(r^2 + a^2/4)^{3/2}}$$

We define a new quantity called the *dipole moment* here:

$$p = qa \quad (\text{dipole moment}) \quad (14.10)$$

Accordingly, the expression for the electric field of a dipole is as follows:

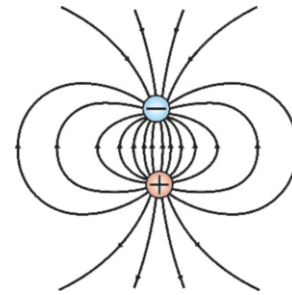
$$E = \frac{kp}{(r^2 + a^2/4)^{3/2}}$$

(b) At very long distances, in other words, when $r \gg a$, the term a^2 in the denominator can be neglected with respect to the other term. The limit field of the electric dipole is thus as follows:

$$E \approx \frac{kp}{r^3} \quad (\text{for } r \gg a)$$

The essential property of the dipole is that the dipole field decreases as $1/r^3$, whereas the field of a point charge decreases as $1/r^2$.

The electric field of a dipole has the following distribution in space:

**Electric Field of Continuous Charge Distributions**

Although an electric charge consists of point charges at the atomic scale, the small distances between them are indistinguishable at the macroscopic scale and appear as continuously distributed. The integration technique must be used to calculate the electric field produced by continuous charge distributions.

Let us divide a charge continuously distributed over a region into small elements with charges $\Delta q_1, \Delta q_2, \dots$, each with a very small dimension (Figure 14.19). We can use Eq. (14.8), which we found for a point charge, to express the contribution of any one of these items to the electric field at point P. For example, the contribution of element i is

$$\Delta \vec{E}_i = \frac{k \Delta q_i}{r_i^2} \hat{r}_i$$

The electric field of the whole charge distribution will be approximately the vector sum of these small contributions:

$$\vec{E} \approx \sum_i \Delta \vec{E}_i = \sum_i \frac{k \Delta q_i}{r_i^2} \hat{r}_i$$

Then, the approximate expression becomes exactly correct at the limit $\Delta q_i \rightarrow 0$ and the sum turns into an integral:

$$\vec{E} = \lim_{\Delta q_i \rightarrow 0} \sum_i \frac{k \Delta q_i}{r_i^2} \hat{r}_i$$

$$\vec{E} = k \int \frac{dq}{r^2} \hat{r} \quad (\text{continuous charge distribution}) \quad (14.11)$$

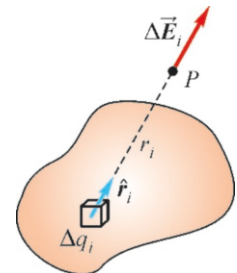


Figure 14.19: The $\Delta \vec{E}$ contribution in the electric field by a small item Δq .

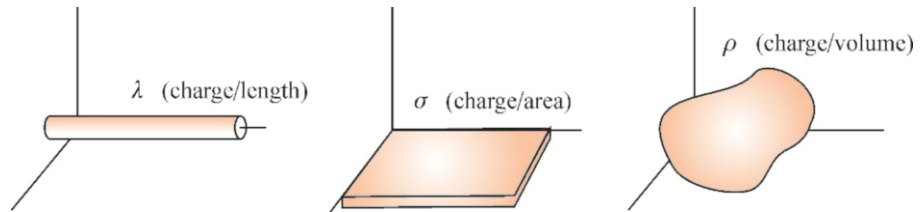
This vector integral should not be confused with the one-dimensional integral with which we are familiar. The contribution of each charge item is in a different direction and a separate integral is required for each of their components. We shall see how this is performed in the worked examples below.

Charge Densities

Distribution of a continuous charge over a region can be expressed more clearly using the concept of charge density. This will make it easier to incorporate the charge element dq into the integral expression above.

Let us review possible charge distributions and their corresponding charge densities (Figure 14.20):

Figure 14.20: Linear charge density λ , surface charge density σ and volume charge density ρ .



- **Linear charge density (λ):** If the total charge Q is evenly distributed over a rod with length L , the linear charge density is

$$\lambda = \frac{Q}{L} \quad (\text{unit: C/m}) \quad (14.12)$$

- **Surface charge density (σ):** If the total charge Q is evenly distributed over a surface A , the surface charge density is

$$\sigma = \frac{Q}{A} \quad (\text{unit: C/m}^2) \quad (14.13)$$

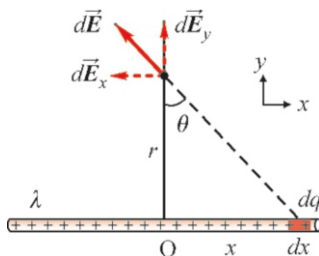
- **Volume charge density (ρ):** If the total charge Q is evenly distributed in a volume V , the volume charge density is

$$\rho = \frac{Q}{V} \quad (\text{unit: C/m}^3) \quad (14.14)$$

With the above definitions of charge densities, we can express the charge element dq in terms of either line, surface or volume elements (dL , dA , dV), depending on the given distribution, as will be seen in the worked examples below:

$$dq = \lambda dL \quad dq = \sigma dA \quad dq = \rho dV \quad (14.15)$$

Example 14.10



Infinite line of charge. An infinite linear wire has uniform linear charge density λ . Calculate the electric field at a distance r from the wire.

Answer

Let us take the wire as the x -axis, as shown in the figure. We write the contribution of a small charge element with thickness dx located at a distance x , to the electric field at

point P . If the unit length has charge λ , then dx will have charge $dq = \lambda dx$:

$$dE = \frac{k dq}{x^2 + r^2} = \frac{k \lambda dx}{x^2 + r^2}$$

The components of these small $d\vec{E}$ contributions will be the components of the total field. However, the E_x components will add to zero due to symmetry, because the mutual contributions of points on both sides of the origin are opposite along the x direction:

$$E_x = \int dE_x = 0$$

Therefore, the field E is the integral of only the y -components:

$$E = E_y = \int dE_y = \int dE \cos \theta$$

$$E = k\lambda \int \frac{\cos \theta dx}{x^2 + r^2}$$

Substituting the cosine term as $\cos \theta = r / \sqrt{x^2 + r^2}$ and taking into consideration that the charges are distributed over the range $[-\infty, +\infty]$, we get

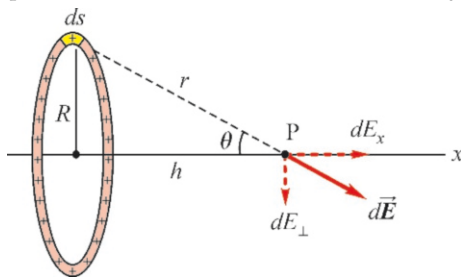
$$E = k\lambda r \int_{-\infty}^{+\infty} \frac{dx}{(x^2 + r^2)^{3/2}} = \frac{2}{r^2}$$

We find the result of this integral to be $2/r^2$ from the integral tables. Therefore, the electric field of an infinite line of charge is as follows:

$$E = \frac{2k\lambda}{r}$$

Example 14.11

Ring of charge. A total charge Q is evenly distributed over a thin wire bent into a ring with radius R . Calculate the electric field at a point P located at distance h on the axis of the ring.



Answer

According to the coordinate system in the figure, let us consider a piece of arc with length ds on the circle. We write the contribution of the small charge dq on this arc to the electric field at point P :

$$dE = \frac{k dq}{r^2} = \frac{k dq}{h^2 + R^2}$$

If the circle with circumference $2\pi R$ has total charge Q , the amount of charge on the arc ds is calculated by proportioning and then substituted:

$$dq = \frac{Q}{2\pi R} ds$$

$$dE = \frac{kQ}{2\pi R} \frac{ds}{h^2 + R^2}$$

As the piece of arc ds rotates around the circle, the perpendicular components dE_{\perp} of these small contributions dE will cancel each other out due to symmetry. However, the dE_x components shall always be added in the same direction. Therefore, the total electric field will be the integral of these dE_x components:

$$E_{\perp} = \int dE_{\perp} = 0$$

$$E = \int dE_x = \int dE \cos \theta = \frac{kQ}{2\pi R} \int \frac{ds \cos \theta}{h^2 + R^2}$$

If we substitute the cosine of the angle as $\cos \theta = h / \sqrt{h^2 + R^2}$ and take all of the constants out of the integral, we get

$$E = \frac{kQ}{2\pi R} \cdot \frac{h}{(h^2 + R^2)^{3/2}} \int ds$$

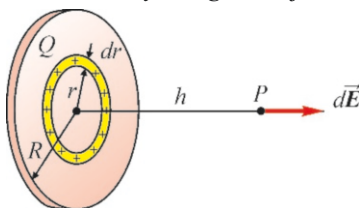
The integral, in other words, the sum of the arcs ds around the circle, will be the circumference $2\pi R$ of the circle. We simplify and find the electric field of the ring:

$$E = \frac{kQh}{(h^2 + R^2)^{3/2}}$$

The electric field will be perpendicular to the ring.

Example 14.12

Charged disk. Calculate the electric field at a point P located at distance h along the axis of a disk with radius R and with charge Q distributed evenly along its surface.



Answer

We can solve this problem using the result for the charged

ring in the previous example. Consider a ring at a radius r with thickness dr in the small interval $[r, r + dr]$.

If we use dq to show the small amount of charge on this ring, according to the previous example, its contribution to the electric charge at point P will be as follows:

$$dE = \frac{kh dq}{(h^2 + r^2)^{3/2}}$$

In order to find the amount of charge dq , let us calculate the surface area of the piece with thickness dr : If we cut this ring out and spread, we get a long thin rectangle with width dr and length $2\pi r$. Its surface area will be approximately $2\pi r dr$. If a disk with surface area πR^2 has a total charge of Q , we find the amount of charge on this small surface and substitute it as follows:

$$dq = \frac{Q}{\pi R^2} 2\pi r dr$$

$$dE = \frac{kQh}{R^2} \frac{2r dr}{(h^2 + r^2)^{3/2}} \quad (\text{along the axis})$$

As the contributions of these small rings are always in the same direction, we can directly integrate to find the total electric field, without any need to separate it into its components:

$$E = \frac{kQh}{R^2} \int_0^R \frac{2r dr}{(h^2 + r^2)^{3/2}}$$

This can be calculated using the change of variable method.

We define the variable $u = r^2 + h^2$ and calculate du :

$$u = r^2 + h^2 \rightarrow du = 2r dr$$

The integral simplifies with this variable u :

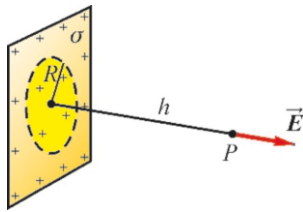
$$E = \frac{kQh}{R^2} \int u^{-3/2} du = \frac{kQh}{R^2} \left(-\frac{2}{\sqrt{u}} \right) = \frac{kQh}{R^2} \left[-\frac{2}{\sqrt{r^2 + h^2}} \right]_0^R$$

Substituting the limit values and simplifying, we find the electric field of the charged disk:

$$E = \frac{2kQ}{R^2} \left[1 - \frac{h}{\sqrt{h^2 + R^2}} \right]$$

Example 14.13

Infinite plane of charge. Calculate the electric field at distance h from an infinite plane that has a constant surface charge density σ .



Answer

We can solve this problem using the result of the charged disk in the previous example. Let us write the electric field expression that we found above for a charged disk with radius R :

$$E = \frac{2kQ}{R^2} \left[1 - \frac{h}{\sqrt{h^2 + R^2}} \right]$$

First, let us write this electric field expression in terms of

charge density σ . If the disk's total charge is Q and its surface area is πR^2 , its surface charge density will be

$$\sigma = \frac{Q}{\pi R^2}$$

We solve this expression for Q and substitute. Also, let us express the constant k as $k = 1/4\pi\epsilon_0$:

$$E = \frac{\sigma}{2\epsilon_0} \left[1 - \frac{h}{\sqrt{h^2 + R^2}} \right]$$

Now, what will happen if the radius R of the disk goes to infinity with the surface charge density σ remaining constant? No difference remains between the infinite disk and the infinite plane. Therefore, this expression gives the electric field of the infinite plane at the limit $R \rightarrow \infty$. The second term inside of the brackets becomes zero when the limit $R \rightarrow \infty$ is taken. We thus find the electric field of the infinite plane of charge:

$$E = \frac{\sigma}{2\epsilon_0}$$

Note that the electric field is independent of h and has the constant value $\sigma/2\epsilon_0$ everywhere.

Multiple-choice Questions

- Which of the following are correct for the electric charge?
 - The smallest charge is 1 Coulomb.
 - There is no smallest charge, it continuously increases or decreases.
 - The smallest charge is the electron.
 - The charge varies as multiples of the electron charge.

(a) I & II (b) II & III (c) III & IV (d) I & IV
- Which of the following carries current in conductors?

(a) Electron (b) Proton (c) Ion (d) Neutron
- What is the direction of the force acting on an electron placed at a point at which the electric field is towards the right?

(a) Right (b) Left (c) Up (d) Down
- Which of the following is true if charge $q_A = +3\text{ C}$ and charge $q_B = +2\text{ C}$ interact?
 - The force on charge A is greater.
 - The force on charge B is greater.
 - The forces are equal.
 - It is impossible to tell.
- Which of the following are correct about the relation between the structure of the atom and conductivity?
 - The electrons at the outermost orbit are weakly bonded in conductors.
 - The electrons at the outermost orbit are strongly bonded in insulators.
 - The electrons at the innermost orbit are weakly bonded in conductors.
 - The electrons at the innermost orbit are weakly bonded in insulators.

(a) I & II (b) I & III (c) II & IV (d) I & IV
- Which of the following are correct?
 - A neutral object has no electrical charge.

II. A neutral object has equal number of + and - charges.

III. There are excess electrons in a positively charged object.

IV. There is a shortfall of electrons in a positively charged object.

- (a) I & II (b) I & III (c) II & III (d) II & IV

7. Why do insulators not conduct electricity well?

- (a) They have no electric charges.
 (b) Their electric charges are strongly bonded to the nucleus of the atoms.
 (c) Their atoms are neutral.
 (d) Their electric charges are weakly bonded to the nucleus of the atoms.

8. If two charges are attracting each other with a force of 100 N, what will the force of attraction be if the distance is doubled?

- (a) 10 N (b) 25 N (c) 50 N (d) 75 N

9. Two equal Q charges are repelling each other with force F . What will the force be if both charges are increased to $3Q$?

- (a) $3F$ (b) $6F$ (c) $9F$ (d) $12F$

10. Which of the following is the definition of the electric field?

- (a) The force acting on a 1 C charge.
 (b) The force acting on one electron.
 (c) The energy of a unit charge.
 (d) The momentum of one electron.

11. If the magnitude of the electric field is E_0 at distance R from a point charge, at what distance will it have the value $E_0/4$?

- (a) $2R$ (b) $4R$ (c) $R/2$ (d) $R/4$

12. What is the force exerted upon a 5 C charge placed at a point at which the electric field is 3 N/C?

- (a) 3 N (b) 5 N (c) 8 N (d) 15 N

13. How does the electric field of an infinite line of charge vary with the distance r from the wire?

- (a) It is proportional to r .
 (b) It is inversely proportional to r .
 (c) It is inversely proportional to r^2 .
 (d) It is inversely proportional to r^3 .

14. How does the electric field of an infinite plane of charge vary with distance?

- (a) It is proportional to r .
 (b) It is inversely proportional to r .
 (c) It is inversely proportional to r^2 .
 (d) It is independent of distance.

15. How does the electric field of an electric dipole change with distance?

- (a) It is proportional to r .
 (b) It is inversely proportional to r .
 (c) It is inversely proportional to r^2 .
 (d) It is inversely proportional to r^3 .

16. A charge Q placed at a corner of a square produces an electric field E at the center of the square. What will the electric field at the center be if equal charges Q are placed at four corners of the square?

- (a) 0 (b) $E/4$ (c) $4E$ (d) $16E$

17. An upward force is exerted upon an electron placed at a point in space. In which direction is the electric field at that point?

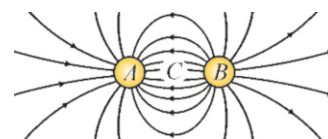
- (a) Up (b) Down (c) Right (d) Left

18. A metal sphere with charge $+Q$ is connected to another neutral identical metal sphere by a conducting wire. What will the charge of the second sphere be?

- (a) $-Q/2$ (b) $-Q$ (c) $+Q/2$ (d) $+Q$

19. Where is the positive (+) charge in the following figure?

- (a) A (b) B (c) C (d) Nowhere



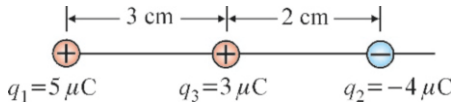
20. Where is the negative (-) charge in the above figure?

- (a) A (b) B (c) C (d) Nowhere

Problems

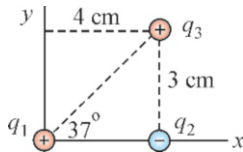
14.2 Coulomb's Law

14.1 Two small conducting spheres are placed with 30 cm of distance in between. One has a $+12\ \mu\text{C}$ charge and the other $-6\ \mu\text{C}$. (a) What is force on one of the spheres? (b) The spheres are connected with a conducting wire. What will the new force be? [A: (a) 7.2 N, (b) 0.9 N.]



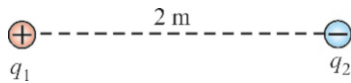
Problem 14.2

14.2 Calculate the total force exerted upon the charge $q_3 = 3\ \mu\text{C}$ by the charges $q_1 = 5\ \mu\text{C}$ and $q_2 = -4\ \mu\text{C}$ shown in the figure. [A: 420 N.]



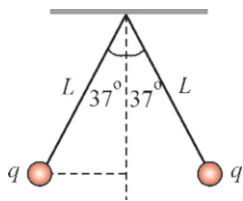
Problem 14.3

14.3 (a) Calculate the components of the total force exerted upon the charge $q_3 = +3\ \mu\text{C}$ by the charges $q_1 = +5\ \mu\text{C}$ and $q_2 = -2\ \mu\text{C}$ in the figure. (b) Find its magnitude and direction. [A: (a) $F_x = 43$, $F_y = -27$ N, (b) $F = 51$ N and -33°]



Problem 14.4

14.4 The distance between the charges $q_1 = +9\ \text{mC}$ and $q_2 = -4\ \text{mC}$ shown in the figure is 2 m. Where should a third charge q_3 be placed such that the net force exerted upon it is zero? [A: 4 m to the right of q_2 .]

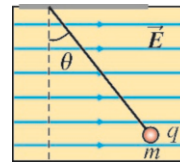


Problem 14.5

14.5 The two identical spheres shown in the figure with mass $m = 30\ \text{g}$ and charge q are suspended from the ceiling with two ropes of length $L = 1\ \text{m}$. What is the charge of the spheres if the ropes each have an angle of 37° with the vertical in equilibrium? [A: $|q| = 6\ \mu\text{C}$.]

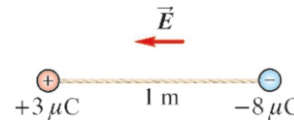
14.3 Electric Field

14.6 A $3.2 \times 10^{-15}\ \text{N}$ force acts on an electron placed at a point in space. What is the magnitude of the electric field at that point? (Electron charge: $e = -1.6 \times 10^{-19}\ \text{C}$.) [A: 20 000 N/C.]



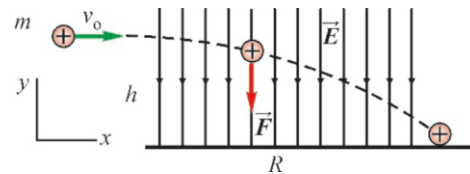
Problem 14.7

14.7 A ball with a mass of 3 g and a charge of $4\ \mu\text{C}$ is suspended from the ceiling with a rope as shown in the figure, in a region with a uniform horizontal electric field of $E = 10\ 000\ \text{N/C}$. Calculate the angle of the rope with respect to the vertical. [A: 53° .]



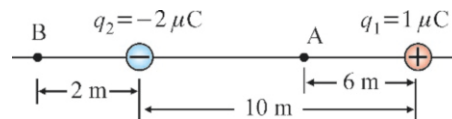
Problem 14.8

14.8 The charges $q_1 = 3\ \mu\text{C}$ and $q_2 = -8\ \mu\text{C}$ in the figure are tied to each other with a rope of length 1 m. These two charges are placed in a region with a uniform electric field of $E = 5 \times 10^6\ \text{N/C}$ and the charge q_1 is nailed and fixed to its location. Calculate the tension in the rope. [A: $T = 40\ \text{N}$.]



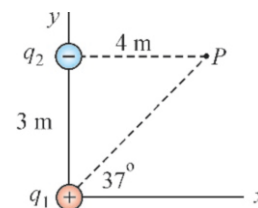
Problem 14.9

14.9 There is a vertically downward constant electric field $E = 1000\ \text{N/C}$ in a region. A proton is thrown with a horizontal velocity of $v_0 = 10^6\ \text{m/s}$ at a height of $h = 1\ \text{m}$ from the ground. (a) What is the acceleration of the proton? (b) Calculate the horizontal range of the proton. ($m = 1.7 \times 10^{-27}\ \text{kg}$ and $q = e = 1.6 \times 10^{-19}\ \text{C}$ for the proton and ignore gravity.) [A: (a) $9.4 \times 10^{10}\ \text{m/s}^2$, (b) $R = 4.6\ \text{m}$.]



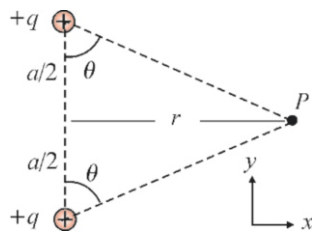
Problem 14.10

14.10 Calculate the total electric field of the charges q_1 and q_2 shown in the figure at points A and B. [A: (a) $-1380\ \text{N/C}$, (b) $+4430\ \text{N/C}$.]



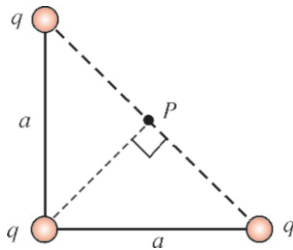
Problem 14.11

14.11 Calculate the components of the total electric fields at point P due to charges $q_1 = 5\ \mu\text{C}$ and $q_2 = -2\ \mu\text{C}$. [A: $E_x = 315$, $E_y = 1080\ \text{N/C}$.]



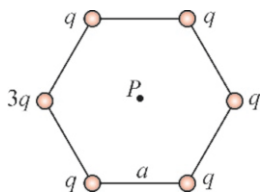
Problem 14.12

14.12 Two equal charges $+q$, with a small distance a in between are shown in the figure. (a) Find the expression for the electric field at distance of r from the perpendicular bisector of the charges. (b) What will the limit of the electric field be when $r \gg a$? [A: (a) $2kqr/(r^2 + a^2/4)^{3/2}$, (b) $2kq/r^2$.]



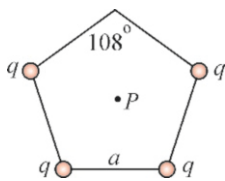
Problem 14.13

14.13 What is the electric field at point P of the isosceles triangle shown in the figure? [A: $2kq/a^2$.]



Problem 14.14

14.14 The figure shows the charges at the corners of a regular hexagon with side length a . Calculate the electric field at point P in the center of the hexagon. [A: $E = 2kq/a^2$ to the right.]



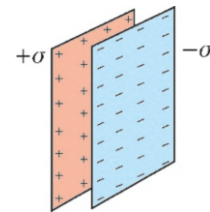
Problem 14.15

14.15 The regular pentagon shown in the figure with side length a has equal charges q on all but one corner. Calculate the electric field at point P in the center of the pentagon. (Hint: Use symmetry.) [A: $E = 1.38kq/a^2$ upwards.]

14.16 In example 14.11, the expression for the electric field of a charged ring with radius R , at distance h along its axis was found to be

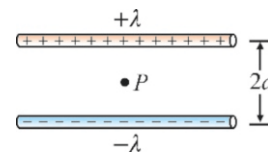
$$E = \frac{kQh}{(h^2 + R^2)^{3/2}}$$

As $E=0$ for $h=0$ and for $h \rightarrow \infty$, the electric field must be at a maximum at a point in between. At what distance h will the electric field be at a maximum? (Hint: The first derivative is zero at the maximum.) [A: $h = R/\sqrt{2}$.]



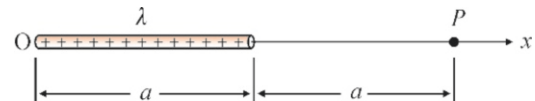
Problem 14.17

14.17 Find the total electric field in the region between the two parallel planes shown in the figure above with surface density $\pm\sigma$. (Hint: Use the result from Example 14.13.) [A: σ/ϵ_0 .]



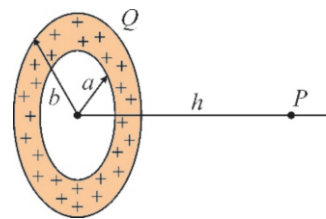
Problem 14.18

14.18 The figure shows two infinite wires placed in parallel at a distance of $2a$ and with linear charge densities $\pm\lambda$. Find the total electric field at the point P in the center. (Hint: Use the result from Example 14.10.) [A: $4k\lambda/a$.]



Problem 14.19

14.19 The figure shows a finite line of charge with length a and uniform linear charge density λ . Use integration to calculate the electric field at point P located along the extension of the wire at distance a . [A: $E = k\lambda/2a$.]



Problem 14.20

14.20 The inner radius of the hollow disk shown in the figure is a and its outer radius is b . The total charge Q is evenly distributed over the surface of the disk. Calculate the electric field at distance h along the disk axis. (Hint: Use the method in example 14.12.)

$$[A: E = \frac{2kQ}{b^2 - a^2} \left[\frac{h}{\sqrt{a^2 + h^2}} - \frac{h}{\sqrt{b^2 + h^2}} \right].]$$

15

GAUSS'S LAW



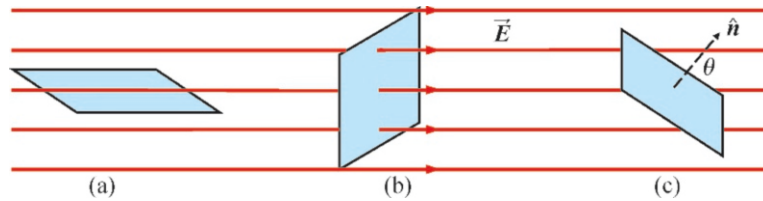
This natural phenomenon in the north and south polar regions, known as the *northern lights* or *aurora borealis*, occurs as a result of the collision of charged particles incoming from the sun with the atoms in the high atmosphere.

In the previous chapter, we learned how to calculate electric fields produced by systems of point charges and by the continuous distribution of charges. These involved vector sums and integrals. These integrals can sometimes be very complex and difficult. However, there is a very powerful and elegant technique for calculating electric field. Known as Gauss's law, this method can be used in symmetrical charge distributions and it yields results very quickly. Although it may seem a bit abstract, learning this method will develop your ability to think mathematically.

15.1 ELECTRIC FLUX (Φ)

Let us place a surface A on the path of the electric field lines in a region. If the field \vec{E} is parallel to the surface (Figure 15.1a), then no line will pass through this surface. And, if we wish to have the maximum amount of field lines pass through, we should orient the surface perpendicular to the field lines (Figure 15.1b).

Figure 15.1: (a) Parallel surface, (b) perpendicular surface and (c) tilted surface to the electrical field lines.



The number of these field lines that pass through the surface A will be proportional to the electric field, the surface area and the angle all at once. Accordingly, the **electric flux** indicated with the symbol Φ is defined as follows (Figure 15.2):

$$\Phi = EA \cos \theta = E_{\perp} A \tag{15.1}$$

Here, θ is the angle between the vector \vec{E} and the **normal vector** \hat{n} . The normal vector is the unit vector perpendicular to the surface at that point.

This definition effectively shows that: (a) When $\theta=0$ (or, the field lines are perpendicular to the surface), we have maximum flux $\Phi = EA$, and (b) When $\theta=90^\circ$ (or, the field lines are parallel to the surface), we have $\Phi=0$. The coefficient $\cos \theta$ in this definition also shows that, if the flux of a vector passing through the surface in one direction is positive, the flux of a vector passing in the opposite direction ($\theta=180^\circ$) will be negative. In general, flux flowing outward from a closed surface is considered to be positive and that flowing inward to be negative.

The definition above is for a constant field \vec{E} . The flux of a variable electric field passing through a surface of any shape is defined with an integral. In order to set it up, we divide the surface A into infinitely small ΔA_i elements and use the flux definition above for each element. The total flux is the limit sum of these fluxes as $\Delta A_i \rightarrow 0$, in other words, the integral:

$$\Phi = \lim_{\Delta A_i \rightarrow 0} \sum_i E_i \Delta A_i \cos \theta_i = \int_{\text{surface}} E dA \cos \theta$$

This is usually a double integral, as it must be taken over the entire surface. If the surface A is *closed*, a circle is added to the integral sign to indicate it:

$$\Phi = \oint_{\text{surface}} E dA \cos \theta \quad (\text{electric flux}) \tag{15.2}$$

The field E and the angle θ inside of the integral are usually variable, and therefore not taken outside of the integral.

15.2 GAUSS'S LAW

In order to understand the technique of Gauss's law, let us consider a positive point charge q at the origin (Figure 15.3). Let us draw electric field lines that spread out from this charge. Let us calculate the total flux of these electric field lines on an imaginary spherical surface that we draw with radius r . (This imaginary surface is called a **Gaussian surface**.)

On this Gaussian surface, the field \vec{E} has the same value everywhere and is perpendicular to it ($\theta = 0$). Therefore, we can calculate the total flux without having to integrate:

$$\Phi = EA \cos 0^\circ = EA$$

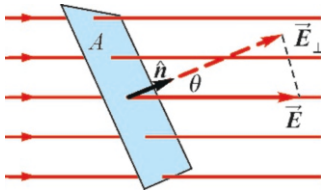


Figure 15.2: Electric flux.

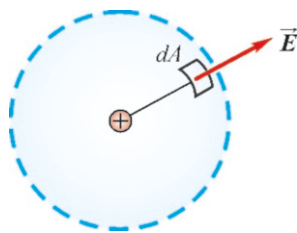


Figure 15.3: A closed Gaussian surface around a point charge.

If we substitute $E = kq/r^2$ for the electric field of a point charge q and $A = 4\pi r^2$ for the surface area of a sphere, we get

$$\Phi = EA = \frac{kq}{r^2} 4\pi r^2$$

$$EA = 4\pi k q = \frac{q}{\epsilon_0}$$

In this last expression, we replaced the constant $k = 1/4\pi\epsilon_0$ with the permittivity of free space ϵ_0 .

This result is rather remarkable. While both the electric field and the surface area on the left-hand side are dependent on the distance r , on the right-hand side, we get a result that is independent of the distance r and only proportional to the charge q inside! Of course, we can see how this happens: As the field E is inversely proportional to r^2 and the surface area A is directly proportional to r^2 , they cancel each other out in the product.

We can argue the case that this result will be valid for all closed surfaces and for all charge distributions as follows:

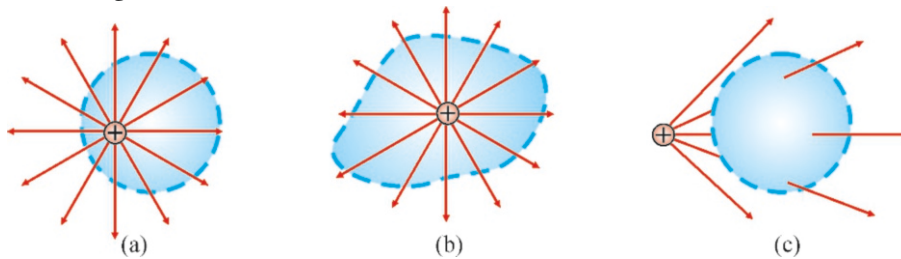


Figure 15.4: (a) The charge is not in the center of the Gaussian surface, (b) the Gaussian surface is not spherical, (c) the charge is outside the Gaussian surface.

- The result would be the same if the charge q was not located at the center of the sphere (Figure 15.4a). Although the field lines that pass through the surface are more frequent in one place and sparser in another, the number of lines that cross the surface is still the same.
- The result would be the same if there was any other closed surface around q instead of a sphere (Figure 15.4b). The total number of lines crossing this surface will still be equal to that crossing the sphere.
- The situation is different if the charge q is outside of the Gaussian surface (Figure 15.4c). In this case, each field line entering the spherical surface will come out somewhere else. Therefore, the sum of positive and negative fluxes will be zero.

$$\Phi = \int E dA \cos \theta = 0 \quad (\text{Charge outside Gaussian surface})$$

In the light of this analysis, we can express Gauss's law as follows without proof:

Gauss's Law

The net electric flux over any closed surface is proportional to the net charge inside of the surface:

$$\oint_{\text{surface}} E dA \cos \theta = \frac{q_{\text{inside}}}{\epsilon_0} \quad (15.3)$$

The charge q_{inside} on the right-hand side of the equation is the algebraic sum of the charges inside the Gaussian surface.

Let us emphasize the important points of Gauss's law:

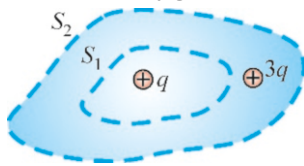
- Although the law is expressed as an integral, we will actually avoid it through symmetry considerations. If the given charges are symmetrically distributed, we choose such a Gaussian surface that the field E remains the same on this surface, and can thus be taken outside of the integral. The remaining integral will usually be much easier to calculate.
- Regardless of how much charge is present outside of the Gaussian surface, in the end, only the net charge inside of the surface is included in the formula. It may seem surprising that the left-hand side of the formula has the total electric field and the right-hand side has the sum of the charges inside, but it is correct.
- Selection of the Gaussian surface is arbitrary and any surface can be chosen. However, the surface must be *closed* for the law to be valid. Otherwise, the flux passing through the open part will not be counted.

15.3 APPLICATIONS OF THE GAUSS'S LAW

We now show the power of Gauss's law in electric field calculations of various charge distributions. In the following examples, we shall find formulas rather than numerical answers. Students will thus gain the ability to make abstract calculations.

Example 15.1

Calculate the total electric flux passing through the closed surfaces S_1 and S_2 shown in the figure, with $q = 1 \mu\text{C}$.



Answer

According to Gauss's law, the electric flux passing through a closed surface is proportional only to the charge inside of

the surface:

$$\Phi = \oint_{\text{surface}} E dA \cos \theta = \frac{q_{\text{inside}}}{\epsilon_0}$$

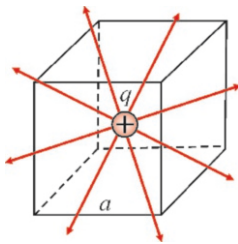
Therefore, it is sufficient to calculate flux from the right-hand side of this equation. The surface S_1 contains only the charge q :

$$\Phi_1 = \frac{q}{\epsilon_0} = \frac{1 \times 10^{-6}}{8.85 \times 10^{-12}} = 1.1 \times 10^5 \text{ N}\cdot\text{m}^2/\text{C}$$

The surface S_2 contains both charges:

$$\Phi_2 = \frac{q + 3q}{\epsilon_0} = 4.4 \times 10^5 \text{ N}\cdot\text{m}^2/\text{C}$$

Example 15.2



The cube with side length $a = 1 \text{ m}$ shown in the figure contains the charge $q = 3 \mu\text{C}$. What is the electric flux passing through one face of the cube?

Answer

It would be difficult to calculate the electric flux on one face

of the cube by integration, because the electric field varies everywhere on this face. However, the problem becomes easy if we consider symmetry. All six faces of the cube are equivalent with respect to charge q . Therefore, the flux passing through one face will be $1/6$ of the total flux:

$$\Phi_1 = \frac{\Phi_{\text{total}}}{6}$$

Now, as the total surface is closed, we can find the flux using Gauss's law:

$$\Phi_{\text{total}} = \frac{q_{\text{inside}}}{\epsilon_0} = \frac{q}{\epsilon_0}$$

From here, we calculate the flux passing through one face:

$$\Phi_1 = \frac{\Phi_{\text{top}}}{6} = \frac{q_{\text{inside}}}{6\epsilon_0} = \frac{10^{-6}}{6 \times 8.85 \times 10^{-12}}$$

$$\Phi_1 = 1.8 \times 10^4 \text{ N}\cdot\text{m}^2/\text{C}$$

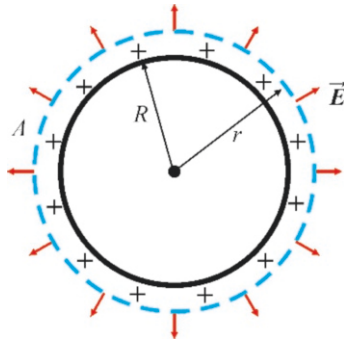
Example 15.3

Charged spherical shell. A total of charge Q is evenly distributed over a spherical surface with radius R .

- (a) Find the electric field at distance r outside of the sphere.
 (b) Find the electric field at distance r inside of the sphere.

Answer

(a) Let us draw an imaginary Gaussian surface with radius r outside of the sphere ($r > R$) (the blue surface in the figure below):



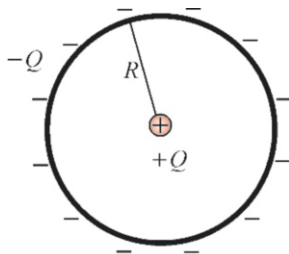
The electric field \vec{E} at each point on this surface should be perpendicular to the surface due to spherical symmetry. Likewise, the electric field at each point on the surface should have the same value because, again, according to spherical symmetry, we go from one point to another when we rotate the sphere.

Consequently, the electric field \vec{E} should be equal and perpendicular to the surface at every point on the Gaussian surface. Let us write Eq. (15.3) for Gauss's law:

$$\oint_{\text{surface}} E dA \cos \theta = \frac{q_{\text{inside}}}{\epsilon_0}$$

The field E inside of the integral can be taken outside, as it is constant over the surface. Also, as the angle with the normal vector is $\theta = 0^\circ$, we get $\cos 0^\circ = 1$:

$$E \oint_{\text{surface}} dA = \frac{q_{\text{inside}}}{\epsilon_0}$$

Example 15.4

A point charge $+Q$ is placed at the center of a spherical shell with radius R , containing evenly distributed charge $-Q$. Find the electric field inside and outside of the sphere.

Answer Inside of the sphere: Let us draw a spherical Gaus-

sian surface with radius r such that $r < R$. Due to symmetry, the electric field will be the same and perpendicular to the surface at each point on this sphere.

$$EA = q_{\text{inside}}/\epsilon_0$$

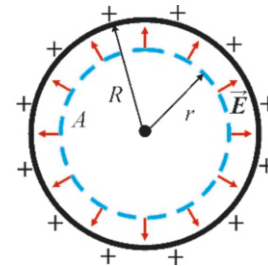
The surface area of a sphere with radius r is $A=4\pi r^2$. The q_{inside} is the total charge Q . We substitute these values:

$$E 4\pi r^2 = Q/\epsilon_0$$

We find the solution by solving this expression for E :

$$E = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{kQ}{r^2} \quad (\text{for } r > R)$$

(b) This time, we choose the Gaussian surface with radius r inside of the sphere ($r < R$) (the blue surface in the figure below):



We again use the same thinking as in item (a): According to symmetry, the field \vec{E} should be perpendicular to the surface and have the same magnitude at every point of the surface. Accordingly, we can directly write Gauss's law as the product EA instead of the integral:

$$EA = q_{\text{inside}}/\epsilon_0$$

Now, what is the charge q_{inside} inside of the Gaussian surface? As all of the charges are left outside of the Gaussian surface,

$$q_{\text{inside}} = 0 \quad \rightarrow \quad EA = 0$$

Since the surface A is not equal to zero, we find that

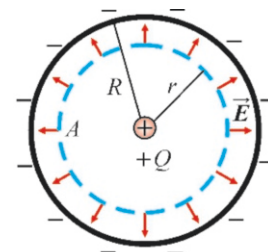
$$E = 0 \quad (\text{for } r < R)$$

Let us summarize the result:

$$E = \begin{cases} kQ/r^2 & (\text{for } r > R) \\ 0 & (\text{for } r < R) \end{cases}$$

This result will be used later for conductors.

sian surface with radius r such that $r < R$. Due to symmetry, the electric field will be the same and perpendicular to the surface at each point on this sphere.



We write the expression for Gauss's law: $EA = q_{\text{inside}}/\epsilon_0$

Only the point charge $+Q$ is located inside of the Gaussian

sphere; the charge $-Q$ on the sphere with radius R is not taken into account, as it is outside of the surface:

$$q_{\text{inside}} = +Q$$

$$E 4\pi r^2 = +Q/\epsilon_0$$

From here, we find the electric field inside of the sphere:

$$E = kQ/r^2 \quad (\text{for } r < R)$$

Outside of the sphere: We draw a Gaussian surface with radius $r > R$. Again, due to symmetry, the field \vec{E} on this

surface is the same and perpendicular to the surface at each point. (We do not need to draw the figure again.) We write the expression for Gauss's law:

$$EA = q_{\text{inside}}/\epsilon_0$$

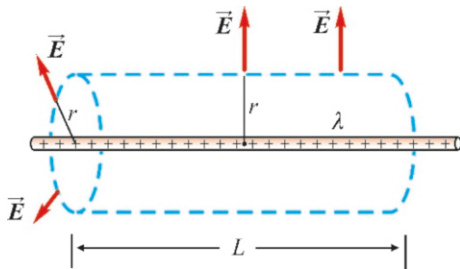
This time, both the point charge $+Q$ and the charge $-Q$ on the sphere with radius R are located inside of the Gaussian surface. Therefore, we get

$$q_{\text{inside}} = +Q - Q = 0$$

$$E 4\pi r^2 = 0 \quad \rightarrow \quad E = 0 \quad (\text{for } r > R)$$

Example 15.5

Infinite line of charge. Calculate the electric field of an infinite linear wire carrying uniform linear charge density λ .



Answer

We solved this problem in Example 14.10 with a lengthy integration. Now let us see how easily it can be solved by Gauss's law.

As the Gaussian surface, let us choose a coaxial cylinder around

the wire with radius r and length L . The field \vec{E} on the lateral surface of this cylinder must be the same everywhere and perpendicular to the surface due to symmetry.

On the base surfaces, the field \vec{E} will be parallel to the surface, in other words, do not cross it. Therefore, the flux is zero on the base surfaces and not taken into consideration. Let us write the expression for Gauss's law:

$$EA = q_{\text{inside}}/\epsilon_0$$

A is the lateral surface here and its area is

$$A = 2\pi r L$$

The charge inside of the cylinder with length L is

$$q_{\text{inside}} = \lambda L$$

We substitute these values:

$$E (2\pi r L) = \frac{\lambda L}{\epsilon_0}$$

We simplify and find the electric field as follows:

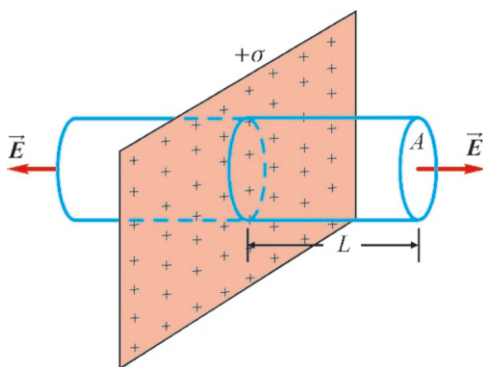
$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{2k\lambda}{r}$$

Example 15.6

Infinite plane of charge. Calculate the electric field of an infinite plane carrying a uniform surface charge density σ .

Answer

Again, we previously solved this problem in Example 14.13 using integration. Now, let us see how easily it can be solved using Gauss's law.



As the Gaussian surface, let us choose a cylinder extending equally to both sides of the surface by L and having a base area of A . The field \vec{E} should be perpendicular to the plane

everywhere, due to symmetry, because a certain angle in any direction would violate symmetry. Also, it will have the same value everywhere on the base surface, because, as the plane is infinite, the charge distribution does not change if any point at the base of the cylinder is shifted onto another point.

\vec{E} will be parallel to the lateral surface of the cylinder, hence the flux is zero on the lateral surface and only the flux on the base surfaces are taken into consideration. The flux is positive, as E is outwards in both bases.

Accordingly, we write the expression for Gauss's law for both bases as follows:

$$EA + EA = q_{\text{inside}}/\epsilon_0$$

The amount of charge inside of the cylinder is found by using the surface charge density σ :

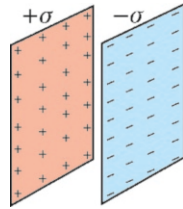
$$q_{\text{inside}} = \sigma A$$

We substitute these values and solve for the electric field:

$$2EA = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0} \quad (\text{Outward from the plane})$$

\vec{E} would have been towards the plane if its charge had been negative.

Example 15.7

Two infinite plates with charge densities $+\sigma$ and $-\sigma$ are placed in parallel. Calculate the electric field in the three regions.

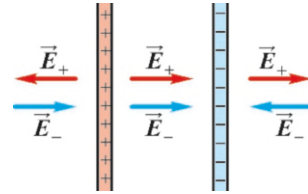
Answer

After finding the electric field of a single plate in the previous example, the electric field of two plates can easily be found as their vector sum. If we use \vec{E}_+ and E_- to indicate the electric fields of the positive and negative plates, the electric field magnitude of both plates will be

$$E_{\pm} = \frac{\sigma}{2\epsilon_0}$$

and will have the same value. However, as the field is outward from the positively charged plate and inward towards the

negatively charged plate, the electric fields in between and outside of the plates will be as follows:



As shown in the figure, we find the sum and difference of two vectors with equal magnitudes:

$$E = \begin{cases} \frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = 0 & \text{(Outside of the plates)} \\ \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} & \text{(In between the plates)} \end{cases}$$

The electric field between two equal and opposite charged plates is σ/ϵ_0 and no electric field flows over outside of the plates. This set up is frequently used in technology (televvisions, capacitors, etc.) to produce a region with uniform electric field.

Example 15.8

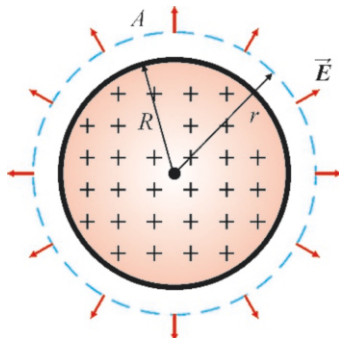
Charged solid sphere. A total charge Q is uniformly distributed in a spherical volume with radius R . Find the electric field outside and inside of the sphere.

Answer

Outside of the sphere: Let us take a spherical Gaussian surface with radius r such that $r > R$. By symmetry, the field \vec{E} on this sphere will be the same everywhere and perpendicular to the surface. We write the expression for Gauss's law:

$$EA = q_{\text{inside}}/\epsilon_0$$

The charge left inside of the Gaussian surface is the whole charge Q :



$$q_{\text{inside}} = Q$$

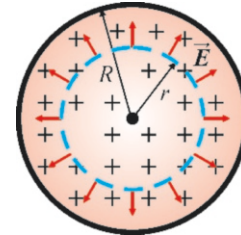
The surface area of a sphere with radius r is $4\pi r^2$. We substitute these expressions and solve for E :

$$E 4\pi r^2 = Q/\epsilon_0$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{kQ}{r^2} \quad (r > R)$$

This result shows that, outside of the sphere, the electric field behaves as if all of the charges were located at the center.

Inside of the sphere: Again, let us take a spherical Gaussian surface with radius r such that $r < R$. The field \vec{E} on this sphere will be the same and perpendicular to the surface:



We write the expression for Gauss's law:

$$EA = q_{\text{inside}}/\epsilon_0$$

This time, some of the charges remain outside of the Gaussian surface, and are thus not taken into consideration. The charge inside the Gaussian surface is found using proportion. If a sphere with radius R contains Q , a sphere with radius r will contain

$$q_{\text{inside}} = \frac{Q}{4\pi R^3/3} \frac{4\pi r^3}{3} = \frac{Qr^3}{R^3}$$

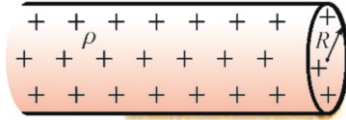
We substitute these expressions and solve for E :

$$E 4\pi r^2 = \frac{Qr^3}{\epsilon_0 R^3}$$

$$E = \frac{Qr}{4\pi\epsilon_0 R^3} = \frac{kQ}{R^3} r \quad (r < R)$$

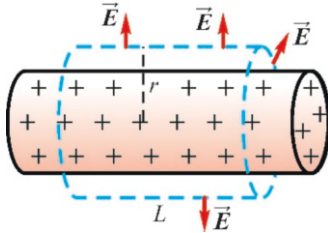
Example 15.9

Infinite cylinder. A cylinder with infinite length and radius R has uniform volume charge density ρ in its volume. Find the electric field in the regions outside and inside of the cylinder.



Answer

Outside of the cylinder ($r > R$), the problem is the same as the infinite wire problem in Example 15.5. We take a cylinder with length L and radius r as the Gaussian surface.



We write the expression for Gauss's law:

$$EA = q_{\text{inside}}/\epsilon_0$$

Here, $A = 2\pi rL$ is the area of the lateral surface. The amount of charge inside of the cylinder is the charge inside of a volume with length L and base radius R :

$$q_{\text{inside}} = \rho V = \rho(\pi R^2 L)$$

We substitute these values:

$$E(2\pi rL) = \frac{\rho(\pi R^2 L)}{\epsilon_0}$$

We simplify and find the electric field as follows:

$$E = \frac{\rho R^2}{2\epsilon_0 r} \quad (r > R)$$

Inside of the cylinder ($r < R$), we again take a cylindrical Gaussian surface with radius r :

$$EA = q_{\text{inside}}/\epsilon_0$$

This time, some charge is left outside of this surface and is not taken into consideration. We calculate the charge inside of the cylinder with length L and radius r :

$$q_{\text{inside}} = \rho V' = \rho(\pi r^2 L)$$

We find the electric field using this expression for charge:

$$E(2\pi rL) = \rho\pi r^2 L/\epsilon_0$$

$$E = \frac{\rho r}{2\epsilon_0} \quad (r < R)$$

15.4 ELECTRIC FIELD IN CONDUCTORS

Electric fields inside of and around conductors have different properties due to the free electrons in the conductors' structures. Gauss's law helps us understand these properties.

Now let us review these properties:

1. Electric field is zero everywhere inside of a conductor in equilibrium.

Remember the atomic structure of conductors: A neutral piece of copper contains an equal number of positive and negative charges. The positively charged ions are stationary, but some of the electrons can freely move inside of the conductor. Now, suppose that there exists a nonzero field inside of the conductor ($\vec{E} \neq 0$). In such a case, a force $\vec{F} = q\vec{E}$ would be exerted upon the electrons. The free electrons would thus start moving and would continue moving until the opposite electric field produced by the electrons in their new position makes $\vec{E} = 0$ inside of the conductor.

A conductor placed in an external electric field will again ensure that $\vec{E} = 0$ inside. As shown in Figure 15.5, electrons that are randomly distributed in the conductor start to gather in the direction opposite to the electric field under the action of the force $\vec{F} = q\vec{E}$, and the opposite electric field that they produce will cancel the external electric field inside of the conductor.

2. Any excess charge given to a conductor distributes itself at the surface.

Let us remember Gauss's law:

$$\oint_{\text{surface}} E dA \cos \theta = \frac{q_{\text{inside}}}{\epsilon_0}$$

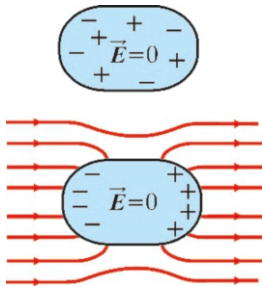


Figure 15.5: The free charges in a conductor placed in an external electric field get repositioned to make $\vec{E} = 0$ inside.

As $\vec{E}=0$ is always true inside of the conductor, the left-hand side of this equation will be zero. Accordingly the inner charge on the right-hand side should also be zero:

$$q_{\text{inside}} = 0$$

We can expand this Gaussian surface and make it encompass the conductor's surface from the inside (Figure 15.6). Even in this case, $q_{\text{inside}}=0$ must be true, as the electric field is zero inside. Therefore, the only place where the excess charge can reside is the surface of the conductor.

But, what if there were a cavity inside of the conductor and a charge $+q$ was placed there (Figure 15.7)? What sort of equilibrium would occur in that case? The conductor cannot move that charge to the surface. In order to fulfill the condition $\vec{E} = 0$ inside, an amount of $-q$ charge will have to move from the outer surface to its inner surface. In this way, we will get $q_{\text{inside}} = 0$ for all Gaussian surfaces drawn inside of the conductor.

3. The electric field just outside of the surface of a conductor is always perpendicular to the surface.

If the electric field was not perpendicular to the surface, it would have a component tangent to the surface. The tangential component would exert a force $\vec{F}=q\vec{E}$ and move the free electrons. As there is static equilibrium, it means that there is no tangential field component acting on the electrons.

A conductor placed inside an external electric field (Figure 15.8) will position its charges such that the electric field lines are perpendicular to the surface.

We will take these properties into consideration when applying Gauss's law to conductors.

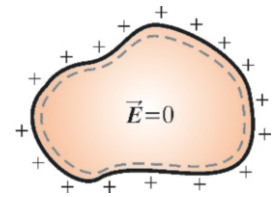


Figure 15.6: The widest Gaussian surface inside of a conductor.

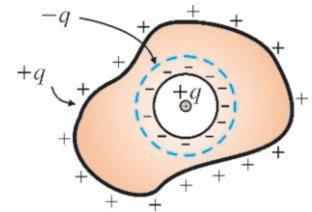


Figure 15.7: $+q$ charge in a cavity inside of a conductor.

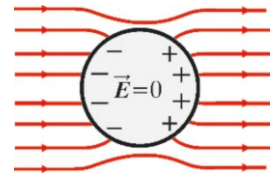
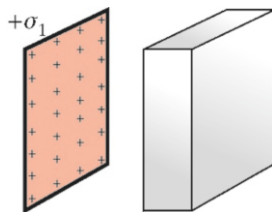


Figure 15.8: Electric field lines will always be perpendicular to the surface.

Example 15.10

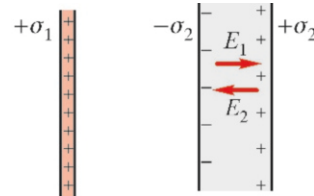
A thick infinite conducting slab is placed in parallel against a thin infinite insulator plate carrying uniform surface charge density σ_1 . The conducting slab is neutral. (a) Calculate the surface charge densities formed on the surfaces of the slab. (b) Calculate the electric field between the plate and the slab.



Answer

(a) Equal and opposite $\pm\sigma_2$ surface densities occur on two surfaces of the conducting slab placed opposite to the charged thin plate. In Examples 15.5 and 15.6, we showed that the electric field of a single plate is $E_1=\sigma/2\epsilon_0$ and the electric field between two inversely charged plates is $E_2=\sigma/\epsilon_0$. Therefore, the charge density σ_2 should be such that there should be a

total of $\vec{E} = 0$ inside of the conductor:



We set the total electric field to zero by taking into consideration the directions of these two vectors inside the conductor:

$$E = E_1 + E_2 = \frac{\sigma_1}{2\epsilon_0} - \frac{\sigma_2}{\epsilon_0}$$

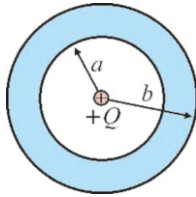
From here, we find the charge σ_2 :

$$\sigma_2 = \frac{\sigma_1}{2}$$

(b) In Example 15.6, we showed that the electric field is zero at the outer region of two equal and opposite charge planes. Therefore, in this problem, we shall only find the electric field of the insulator plate at the region in between the conductor and the insulator:

$$E = \frac{\sigma_1}{2\epsilon_0}$$

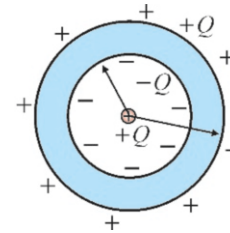
Example 15.11



A point charge $+Q$ is placed at the center of a spherical neutral conducting shell with inner radius a and outer radius b . Calculate the electric field in the three regions.

Answer

We have previously learned that $\vec{E} = 0$ is always true inside of a conductor. Accordingly, it transfers $-Q$ of its free electrons from its outer surface to its inner surface in order to cancel out the effect of the central point charge inside of the conductor and the following distribution occurs:



According to this distribution, we apply Gauss's law in each region:

We take the area of the chosen spherical Gaussian spheres as $A = 4\pi r^2$ and calculate:

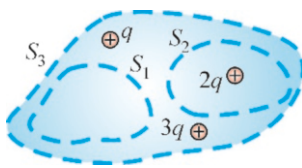
For $r < a$: $E(4\pi r^2) = \frac{+Q}{\epsilon_0} \rightarrow E = \frac{kQ}{r^2}$

For $a < r < b$: $E(4\pi r^2) = \frac{+Q - Q}{\epsilon_0} \rightarrow E = 0$

For $r > b$: $E(4\pi r^2) = \frac{+Q - Q + Q}{\epsilon_0} \rightarrow E = \frac{kQ}{r^2}$

Problems

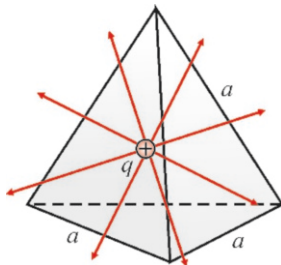
15.3 Applications of Gauss's Law



Problem 15.1

15.1 Calculate the total electric fluxes passing through the closed surfaces S_1 , S_2 and S_3 shown in the figure.

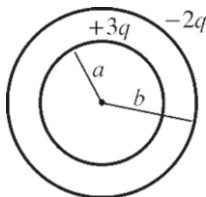
[A: $\Phi_1 = 0$, $\Phi_2 = 2q/\epsilon_0$, $\Phi_3 = 6q/\epsilon_0$.]



Problem 15.2

15.2 The regular tetrahedron shown in the figure has side length a , and a charge $+q$ is placed at its center. What is the electric flux passing through one face of the tetrahedron?

[A: $q/4\epsilon_0$.]

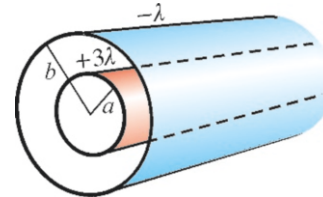


Problem 15.3

15.3 There are two concentric spherical shells, as shown in the figure. A total charge $3q$ is distributed over the one with radius a and a charge $-2q$ is distributed over the one with

radius b . Calculate the electric field in the three regions.

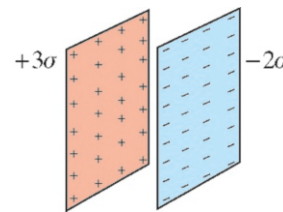
[A: $E=0$ for $r < a$, $3kq/r^2$ for $a > r > b$, kq/r^2 for $r > b$]



Problem 15.4

15.4 There are two coaxial infinite cylindrical shells, as shown in the figure. The one with radius a has a linear charge density 3λ and the one with radius b has a charge density $-\lambda$. Calculate the electric field in the three regions.

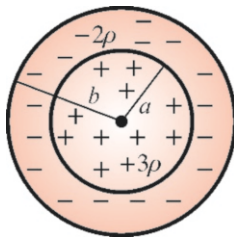
[A: $E = 0$ for $r < a$, $6k\lambda/r$ for $a > r > b$, $4k\lambda/r$ for $r > b$]



Problem 15.5

15.5 Two infinite plates with charge densities $+3\sigma$ and -2σ are placed in parallel. Calculate the electric field in the three regions.

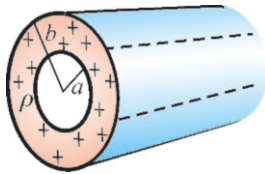
[A: $\sigma/2\epsilon_0$ on the left, $5\sigma/2\epsilon_0$ in the middle, $\sigma/2\epsilon_0$ on the right.]



Problem 15.6

15.6 The spherical volume with radius a in the figure has a uniform volume charge density $+3\rho$ and the spherical shell with inner radius a and outer radius b has a uniform volume charge density -2ρ . Calculate the electric field in the three regions.

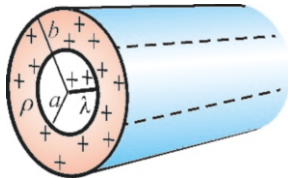
[A: $E = \rho r/\epsilon_0$ for $r < a$, $5\rho a^3/(3\epsilon_0 r^2) - 2\rho r/3\epsilon_0$ for $a < r < b$, $(5a^3 - 2b^3)\rho/(3\epsilon_0 r^2)$ for $r > b$.]



Problem 15.7

15.7 The infinite cylindrical shell with inner radius a and outer radius b shown in the figure has a volume charge density ρ . Calculate the electric field in the three regions.

[A: $E=0$ for $r < a$, $\rho(r^2 - a^2)/2\epsilon_0 r$ for $a < r < b$, $\rho(b^2 - a^2)/2\epsilon_0 r$ for $r > b$.]



Problem 15.8

15.8 An infinite wire carrying a linear charge density λ is surrounded by an infinite conducting cylindrical shell with inner radius a and outer radius b and carrying a volume charge density ρ , as shown in the figure. Calculate the electric field in the three regions.

[A: $2k\lambda/r$ for $r < a$, $2k\lambda/r + \rho(r^2 - a^2)/(2\epsilon_0 r)$ for $a < r < b$, $2k\lambda/r + \rho(b^2 - a^2)/(2\epsilon_0 r)$ for $r > b$.]

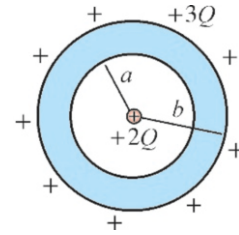
15.4 Electric Field in Conductors



Problem 15.9

15.9 What are the surface charge densities produced on both

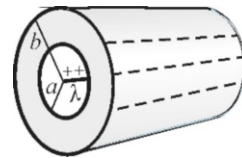
faces of a thick conductor slab placed perpendicularly to a uniform electric field \vec{E}_0 ? [A: $\sigma = \pm\epsilon_0 E_0$.]



Problem 15.10

15.10 A $+3Q$ charge is given to a conducting spherical shell with inner radius a and outer radius b . A point charge $+2Q$ is placed at the center of the cavity inside of the conductor. (a) What is the charge at the inner and outer surfaces of the conductor? (b) Calculate the electric field in the three regions.

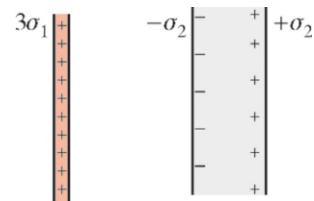
[A: (a) $-2Q$ on the inside, $+5Q$ on the outside, (b) $2kQ/r^2$ for $r < a$, 0 for $a < r < b$, $5kQ/r^2$ for $r > b$.]



Problem 15.11

15.11 An infinite wire carrying a linear charge density λ is surrounded by an infinite cylindrical conducting shell with inner radius a and outer radius b , as shown in the figure. Calculate the electric field in the three regions.

[A: $2k\lambda/r$ for $r < a$, 0 for $a < r < b$, $2k\lambda/r$ for $r > b$.]



Problem 15.12

15.12 The thin planar layer shown in the figure has $3\sigma_1$ in surface charge density. A neutral thick conductor slab is placed in parallel to it. (a) What will the surface charge density σ_2 of the conductor slab be? (b) What is the electric field in the region in between?

[A: (a) $\sigma_2 = \pm 3\sigma_1/2$, (b) $3\sigma_1/2\epsilon_0$.]

16

ELECTRIC POTENTIAL



A “hair raising” experience. This metal sphere, called the **van de Graaff generator**, is used to obtain very high potentials up to 1 million volts with only a small amount of charge. When you touch the sphere, your body will be charged and your hair will stand on end.

What is the relation between electric potential and potential energy? These concepts will help us better understand the nature of electricity.

Up until now we have dealt with Coulomb’s force and electric field, both of which are difficult to work with because they are vector quantities. Now, it is time to adapt the concepts of work and energy that we developed in mechanics into their electrical equivalents. Indeed, like the gravitational force, the electric force is also conservative, hence we can define a potential energy for electric forces as well.

The electric potential that we shall define is a scalar quantity, and thus much easier to deal with and to solve related problems. Also, the electric potential will help us to better understand conductors, capacitors, electric circuits, etc., and other technological applications.

16.1 ELECTRIC POTENTIAL

Electric Potential Energy

In Chapter 5, we defined the work performed by a variable force:

$$W = \int_1^2 \vec{F} \cdot d\vec{r}$$

Let us also remember the force on a charge q placed in an electric field \vec{E} :

$$\vec{F} = q\vec{E}$$

Therefore, the work performed by the electrical force when this charge q is moved from one point to another will be

$$W = q \int_1^2 \vec{E} \cdot d\vec{r}$$

Again, as we discussed in Chapter 5, the work performed *against* conservative forces is equal to the potential energy difference between the two points (Equation 5.12):

$$- \int_1^2 \vec{F}_c \cdot d\vec{r} = U_2 - U_1 \quad (16.1)$$

It can easily be observed that the Coulomb force between two charges will also be conserved: In Chapter 5, we showed that a potential energy could be defined for the gravitational force $F = GmM_E/r^2$. Likewise, the Coulomb force $F = kq_1q_2/r^2$ will also be conservative, as it is also inversely proportional to r^2 .

Therefore, an **electric potential energy** can also be defined for the electrical force as the work performed against the force $\vec{F} = q\vec{E}$:

$$U_2 - U_1 = -q \int_1^2 \vec{E} \cdot d\vec{r} \quad (\text{Electric potential energy}) \quad (16.2)$$

We shall directly proceed to the concept of potential without discussing potential energy, as we already examined it in detail in Mechanics.

Electric Potential

The electric potential energy of the point charge q defined above in Eq. (16.2) is proportional to q . Therefore, there is no use in keeping the charge q always in the formulas. If we know the potential energy of the unit charge, we can easily calculate the potential energy of any charge q at that point by just multiplying them.

Definition: If a charge q has a potential energy U at a point in space, the **electric potential** at that point will be:

$$V = \frac{U}{q} \quad (\text{electric potential}) \quad (16.3)$$

Although the potential looks like the potential energy per unit charge, this simple definition is somehow misleading: The potential at a point exists even when no charge q is present there. Hence, we should stress that the potential is a property of the electric field, and not of the charge q .

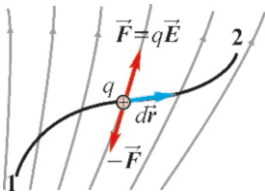


Figure 16.1: The work performed against the electrical force exerted upon a charge q in an electric field.

Using this definition in reverse, the potential energy of a charge q placed at a point with potential V will be

$$U = qV \quad (16.4)$$

Therefore, if we divide Eq. (16.2) for potential energy difference by q , we will obtain the expression for the potential difference between two points:

$$V_2 - V_1 = - \int_1^2 \vec{E} \cdot d\vec{r} \quad (\text{Electric potential difference}) \quad (16.5)$$

More precisely, the work performed against electric forces when moving the unit charge from one point to another, is the potential difference between these two points.

It is useful to remember the following two simple rules according to this definition:

- Potential decreases when moving in the direction of electric field lines.
- Potential increases when approaching positive charges and decreases when approaching negative charges.

For a small displacement Δr , the small potential difference ΔV can be expressed without integration as:

$$\Delta V = -\vec{E} \cdot \Delta\vec{r} \quad (16.6)$$

Unit of Potential

According to the definition above, the unit of potential is joule/coulomb (J/C) and is called the **volt**, indicated with the symbol V:

$$1 \text{ joule/coulomb} = 1 \text{ volt} = 1 \text{ V}$$

The unit of electric field can also be re-expressed in terms of Volts. We had previously defined the unit of electric field as (N/C). However, if we compare the units in the formula $\Delta V = E \Delta r$, the electric field unit will be,

$$1 \frac{\text{newton}}{\text{coulomb}} = 1 \text{ V/m}$$

This V/m unit is used in technology, as it is more practical. The term *voltage* is also used instead of potential difference.

Potential of Constant Electric Field

In Chapter 15, we learned that a constant electric field forms between two oppositely charged parallel plates. Let us place two such plates along the x -axis, as shown in Figure 16.2. Let the electric field be in the $-x$ direction. (The reason that we place them so is that the potential will increase in the opposite direction to the electric field and we want it to increase in the $+x$ direction.) We use Eq. (16.5) to find the potential difference between position x_1 and position x_2 in this electric field. As E is constant and in the $-x$ direction, we get

$$V(x_2) - V(x_1) = - \int_{x_1}^{x_2} (-E) dx = E \int_{x_1}^{x_2} dx = E(x_2 - x_1)$$

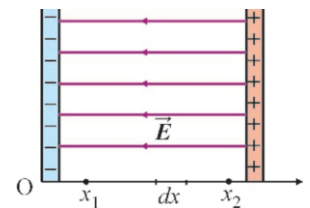


Figure 16.2: Potential difference in a constant electric field.

If we take this integral from the origin $x_1 = 0$ to the point $x_2 = x$, we get

$$V(x) - V(0) = Ex$$

The zero reference point of potential is always arbitrary. If we choose the negative plate at $x = 0$ as the zero potential, the potential change between the plates is found to be as follows:

$$V(x) = Ex \quad (\text{potential of a constant electric field}) \quad (16.7)$$

If the distance between the plates is d , the potential difference between the plates will be $V = Ed$.

Example 16.1

(a) A $2 \mu\text{C}$ charge gains 0.006 J potential energy when placed in an electric field. What is the potential at that point?

(b) What will the velocity of a proton be if you convert all of its potential energy at a point with potential 1000 V into kinetic energy? (The proton's mass is $m_p = 1.7 \times 10^{-27} \text{ kg}$ and the charge is $e = 1.6 \times 10^{-19} \text{ C}$)

Answer

(a) We write the definition of potential in terms of potential

energy U :

$$V = U/q = 0.006/(2 \times 10^{-6}) = 3000 \text{ V}$$

(b) The potential energy of the proton at a point with potential V will be eV . This is fully converted into kinetic energy:

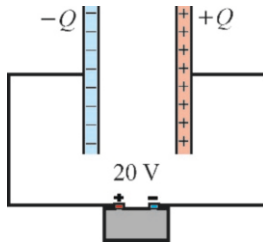
$$\frac{1}{2}m_p v^2 = eV \rightarrow v = \sqrt{2eV/m_p}$$

We substitute the given values and calculate the velocity:

$$v = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 1000}{1.7 \times 10^{-27}}} = 4.3 \times 10^5 \text{ m/s}$$

Example 16.2

Two parallel conducting plates with 5 mm of distance in between are connected to the terminals of a 20 V battery. (a) What is the magnitude of the electric field between the plates? (b) How much charge accumulates on one of the plates if its surface area is 100 cm^2 ?



Answer

(a) We know that a constant electric field E is formed between two oppositely charged plates. Also, we had found Eq. (16.7) for the potential change in a constant electric field:

$$V = Ex$$

We substitute the position $x = 0.005 \text{ m}$ of the positive charged plate and find the electric field:

$$E = V/x = 20/0.005 = 4000 \text{ V/m}$$

(b) We can assume the plates to be infinite if their dimensions are large with respect to the distance in between. Then, the electric field is given by:

$$E = \frac{\sigma}{\epsilon_0}$$

We write the surface charge density of a plate with charge Q and surface area A :

$$\sigma = \frac{Q}{A}$$

$$E = \frac{(Q/A)}{\epsilon_0} \rightarrow Q = \epsilon_0 EA$$

We find the charge of a plate by substituting the numerical data:

$$Q = 8.85 \times 10^{-12} \times 4000 \times 100 \times 10^{-4}$$

$$Q = 35 \times 10^{-11} \text{ C} = 0.35 \text{ nC}$$

16.2 POTENTIAL OF A SYSTEM OF POINT CHARGES

Potential of a Point Charge

Let us rewrite the electric field expression of a positive charge Q located at the origin:

$$E = \frac{kQ}{r^2}$$

The electric field will be outwards, as the charge Q is positive (Figure 16.3). We use Eq. 16.5 to calculate the potential difference between position r_1 and position

r_2 in this electric field. As the field E and the displacement dr are in the same direction here, the scalar product can be written as the product $E dr$:

$$V(r_2) - V(r_1) = - \int_{r_1}^{r_2} E dr$$

$$\begin{aligned} V(r_2) - V(r_1) &= - \int_{r_1}^{r_2} \frac{kQ}{r^2} dr = -kQ \int_{r_1}^{r_2} \frac{dr}{r^2} = -kQ \left[-\frac{1}{r} \right]_{r_1}^{r_2} \\ &= kQ \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \end{aligned}$$

If we take this integral from $r_1 = \infty$ to $r_2 = r$, we get

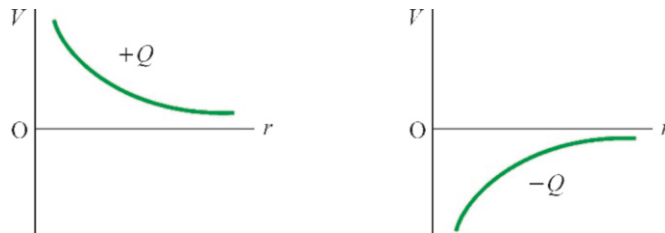
$$V(r) - V(\infty) = \frac{kQ}{r}$$

We choose the zero reference point of the potential at infinity, $V(\infty) = 0$. With this choice, the potential for a point charge is as follows:

$$V(r) = \frac{kQ}{r} \quad \text{(Potential of point charge)}$$

(16.8)

In other words, *the potential of a point charge is the work performed to bring it from infinity to that point.*



According to this result, the potential of a positive charge is positive everywhere and the potential of a negative charge is negative. Potential increases towards a positive charge and decreases towards a negative charge.

Potential of a System of Point Charges

When calculating the electric field of several charges, we have seen that the electric fields of the charges were added as vectors. However, the potential of a system of point charges at a point P will be the *algebraic sum* of the potential of each charge.

We can write the potential produced by charges $q_1, q_2 \dots q_N$ at a point P using Eq. (16.8), which we found above for a point charge:

$$V = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \dots + \frac{kq_N}{r_N} = \sum_i \frac{kq_i}{r_i} \quad (16.9)$$

The distance r_i here is the distance of charge q_i to the point P (Figure 16.5).

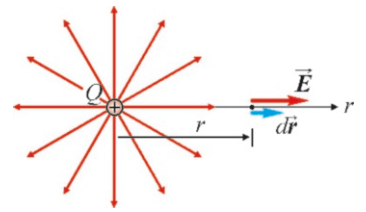


Figure 16.3: Potential difference of a point charge.

Figure 16.4: Potential increases towards a positive charge and decreases towards a negative charge.

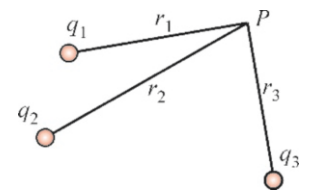


Figure 16.5: Potential of multiple charges at point P .

Potential Energy of a System of Point Charges

Let us consider a system in which many charges are located at certain fixed positions, as shown in Figure 16.6. What is the potential energy of this system? In other words, how much work was done to bring the charges from infinity to these positions?

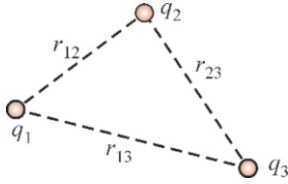


Figure 16.6: Potential energy of three charges.

Let us find the answer to this question step-by-step. Let us first bring the charge q_1 from infinity. No work needs to be done for this, therefore its potential energy is zero:

$$U_1 = 0$$

Then, we bring charge q_2 . It will be subject to the potential of charge q_1 . As the potential is $V = kq_1/r$, if we use the formula $U = qV$ to calculate the potential energy of the charge q_2 placed at distance r_{12} , we get

$$U_2 = q_2 V_1 = q_2 \frac{kq_1}{r_{12}} = k \frac{q_1 q_2}{r_{12}}$$

and this potential energy is the energy of the (q_1, q_2) charge system. It would be incorrect to consider this as belonging to q_2 , as we could have first brought charge q_2 and then charge q_1 .

Now let us bring charge q_3 to its position. This charge will be subject to the potential of the charges (q_1, q_2) . Its potential energy at its position is

$$U_3 = q_3 \left(\frac{kq_1}{r_{13}} + \frac{kq_2}{r_{23}} \right) = k \left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

Now let us stop here and examine the potential energy of the whole system:

$$U = U_1 + U_2 + U_3 = k \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \quad (16.10)$$

This result can easily be generalized for more than three charges: We find the interaction of each charge with the others and add them. The total energy of N charges is expressed as follows:

$$U = k \sum_{i < j} \frac{q_i q_j}{r_{ij}} \quad (16.11)$$

This potential energy is due to the work performed in bringing these charges from infinity to these positions.

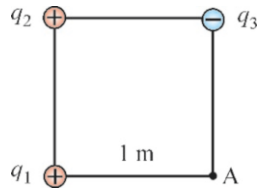
Law of Energy Conservation in Electrostatics

The law of energy conservation that we developed in Chapter 5 is also valid in electrostatics. Since the potential energy of a point charge q is $U = qV$ in a given potential V , Eq. (5.17) for energy conservation can be written as follows:

$$\frac{1}{2} m v_1^2 + qV_1 = \frac{1}{2} m v_2^2 + qV_2 \quad (16.12)$$

However, unlike gravitational potential energy, the charge q can have positive or negative contribution to potential energy depending on its sign, as we shall see in the examples below.

Example 16.3



The charges $q_1=1\ \mu\text{C}$, $q_2=5\ \mu\text{C}$ and $q_3=-2\ \mu\text{C}$ are placed on three corners of a square with side length 1 m.

- (a) What will the potential at point A be?
- (b) How much work needs to be done to bring a new charge $q_4 = +4\ \mu\text{C}$ from infinity to point A?

Answer (a) The potential of point A is the algebraic sum of

potentials of all three charges at that point:

$$V_A = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3}$$

$$V_A = 9 \times 10^9 \left(\frac{1 \times 10^{-6}}{1} + \frac{5 \times 10^{-6}}{\sqrt{2}} + \frac{-2 \times 10^{-6}}{1} \right)$$

$$V_A = 22\ 800\ \text{V}.$$

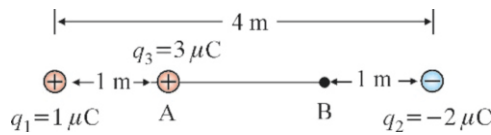
(b) Another definition of potential energy is the work performed to move a charge q from infinity to point A. Therefore, we can write the potential energy of charge q_4 at point A in terms of the potential at that point as follows:

$$U_A = q_4 V_A$$

We substitute the values and calculate as follows:

$$U_A = 4 \times 10^{-6} \times 22800 = 0.09\ \text{J}$$

Example 16.4



The charges $q_1=1\ \mu\text{C}$ and $q_2=-2\ \mu\text{C}$ shown in the figure with 4 m of distance in between are fixed in their current locations. The charge $q_3=3\ \mu\text{C}$ with a mass of 1 g is released from rest between these two charges at point A. What will its speed be when it reaches point B?

Answer

A charge q at a point with potential V has a potential energy $U = qV$. We thus write the conservation of energy for points A and B:

$$\frac{1}{2}mv_A^2 + q_3V_A = \frac{1}{2}mv_B^2 + q_3V_B$$

As charge q_3 is released from rest at point A, we take $v_A = 0$ and solve for v_B :

$$v_B = \sqrt{\frac{2q_3(V_A - V_B)}{m}}$$

Let us calculate the potential differences separately:

$$V_A = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} = 9 \times 10^9 \times \left(\frac{1 \times 10^{-6}}{1} - \frac{2 \times 10^{-6}}{3} \right)$$

$$= 3000\ \text{V}$$

$$V_B = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} = 9 \times 10^9 \times \left(\frac{1 \times 10^{-6}}{3} - \frac{2 \times 10^{-6}}{1} \right)$$

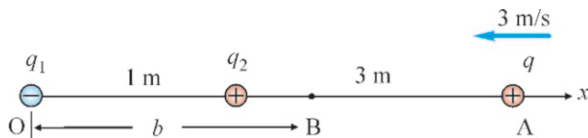
$$= -15\ 000\ \text{V}$$

From here, we calculate the speed v_B :

$$v_B = \sqrt{\frac{2 \times 3 \times 10^{-6} (3000 + 15000)}{0.001}}$$

$$v_B = 10\ \text{m/s}.$$

Example 16.5



Charges $q_1=-5\ \mu\text{C}$ and $q_2=+2\ \mu\text{C}$ are fixed with 1 m of distance in between. A third charge $q=1\ \mu\text{C}$ with a mass of 1 g is thrown with an initial velocity of 3 m/s towards q_2 from point A located along the extension of these charges at a distance of 3 m from q_2 . Calculate the maximum distance that charge q can approach.

Answer

The charge q will have zero speed at some point B that it approaches most: $v_B=0$. Let us use b to show the distance of this point to the origin. We write the conservation of energy between points A and B:

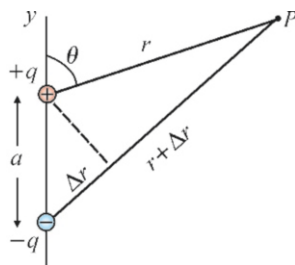
$$\frac{1}{2}mv_A^2 + qV_A = \frac{1}{2}mv_B^2 + qV_B$$

$$\frac{1}{2}mv_A^2 + kq \left(\frac{q_1}{4} + \frac{q_2}{4-1} \right) = 0 + kq \left(\frac{q_1}{b} + \frac{q_2}{b-1} \right)$$

If we substitute the given values and solve for b , we get the following result:

$$b = 1.7\ \text{m}$$

Example 16.6



Electric dipole. Calculate the potential of an electric dipole at any point in space and find its limit value for $r \gg a$.

Answer

We had found the electric field of an electric dipole in Example 14.8. Now, let us calculate its potential.

The charges $\pm q$ in the figure are placed along the y -axis with distance a in between. We can immediately write the electric

potential of this dipole at a point P located at distance r from the charge $+q$ and with angle θ with the y -axis:

$$V = \frac{kq}{r} - \frac{kq}{r + \Delta r} = \frac{kq \Delta r}{r(r + \Delta r)}$$

Here, we indicate the distance of point P to charge $+q$ with r and to charge $-q$ with $r + \Delta r$. The potential can be found by calculating this expression for any given point P .

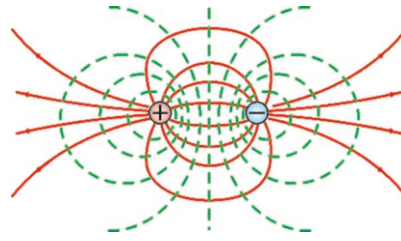
We use the following approximate expressions for $r \gg a$:

$$\Delta r \approx a \cos \theta$$

$$r(r + \Delta r) \approx r^2$$

Also, the product $p = qa$ was defined as the electric dipole moment. Accordingly, the expression for potential at distances very far from the dipole is as follows:

$$V = k \frac{p \cos \theta}{r^2}$$



The equipotential surfaces of the electric dipole on the xy -plane are shown in the figure above. The most important feature of this potential is that it decreases with r^2 . The potential of a point charge decreases with r and the potential of a dipole decreases much faster. The electric field lines of the dipole are shown in the same figure from the positive charge to the negative charge, as perpendicularly intersecting the equipotential surfaces.

16.3 POTENTIAL OF CONTINUOUS CHARGE DISTRIBUTIONS

Integration is used to calculate the potential of charges distributed over a volume, surface or line. As shown in Figure 16.7, if we consider a small charge element dq in the region where charge is distributed, its small contribution dV to the total potential will be like that of a point charge:

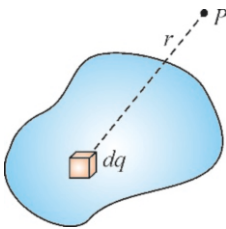


Figure 16.7: The contribution of a small charge dq on the potential at point P .

$$dV = \frac{k dq}{r}$$

The potential of the whole charge distribution will be the sum of these small contributions at the limit $dq \rightarrow 0$, in other words, their integral:

$$V = k \int \frac{dq}{r} \quad (\text{Potential of a continuously distributed charge}) \quad (16.13)$$

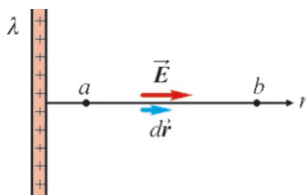
The charge element dq here is to be expressed in terms of charge density, by examining the given charge distribution. It can be one of the following three expressions if distributed over a line, surface or volume:

$$dq = \lambda dL \quad dq = \sigma dA \quad dq = \rho dV \quad (16.14)$$

Also, note that the definition of potential as the integral of electric field \vec{E} (Eq. 16.5) can also be used to calculate V :

$$V_2 - V_1 = - \int_1^2 \vec{E} \cdot d\vec{r} \quad (16.15)$$

Example 16.7



Infinite line of charge. Find the potential difference between two points located at distances a and b from an infinite wire

with linear charge density λ .

Answer

It is easier to use the definition in terms of electric field:

$$V_b - V_a = - \int_a^b \vec{E} \cdot d\vec{r} = - \int_a^b E dr$$

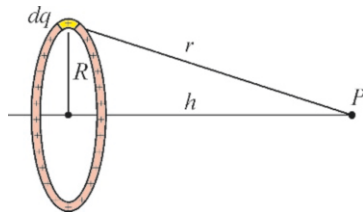
Since \vec{E} and $d\vec{r}$ are parallel, we have $\vec{E} \cdot d\vec{r} = E dr$. The electric field of an infinite line of charge was found to be $E = 2k\lambda/r$ in Example 15.5. We can use that result and integrate:

$$V_b - V_a = - \int_a^b \frac{2k\lambda dr}{r} = -2k\lambda \ln r \Big|_a^b = -2k\lambda \ln(b/a)$$

$$V_b - V_a = -2k\lambda \ln(b/a)$$

As $b > a$, this result shows that the potential decreases when moving away from a positively charged wire.

Example 16.8



Charged ring. Calculate the potential along the axis at a distance h from the center of a ring with radius R carrying a total charge Q .

Answer

Let us choose a small charge dq corresponding to a small arc

ds on the ring. We write its contribution to the potential at point P and add the contributions of ds all over the ring, in other words, take its integral:

$$V = k \int \frac{dq}{r} = k \int \frac{dq}{\sqrt{R^2 + h^2}}$$

The denominator inside of the integral can be taken outside, as it is constant throughout the ring:

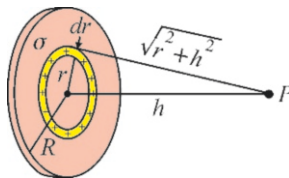
$$V = \frac{k}{\sqrt{R^2 + h^2}} \int dq$$

The remaining integral is the sum of all of the charges Q :

$$V = \frac{kQ}{\sqrt{R^2 + h^2}}$$

Example 16.9

Charged disk. Calculate the potential of a disk with radius R and uniform surface charge density σ , at point P located at distance h on the axis of the disk.



Answer

Consider a ring with thickness dr with a radius in the interval $[r, r + dr]$. If we use dq to indicate the small amount of charge on this ring, according to the previous example, its contribution to the potential at point P will be as follows:

$$dV = \frac{k dq}{\sqrt{r^2 + h^2}}$$

In order to find the charge dq , we have to multiply the area of the small ring with the surface charge density:

$$dq = \sigma dA = \sigma (2\pi r dr)$$

We substitute this value and find the contribution of all the rings by integrating:

$$V = \pi k \sigma \int_0^R \frac{2r dr}{\sqrt{r^2 + h^2}}$$

We make a change of variable by defining a new variable $u = r^2 + h^2$. Then, $du = 2r dr$. The integral is simplified with this variable u :

$$V = \pi k \sigma \int u^{-1/2} du = \pi k (2u^{1/2}) = \pi k \sigma [2\sqrt{r^2 + h^2}]_0^R$$

Substituting the limit values and simplifying, we find the expression for the potential of the disk:

$$V = 2\pi k \sigma [\sqrt{h^2 + R^2} - h]$$

16.4 CONDUCTORS AND EQUIPOTENTIAL SURFACES

Surfaces on which the potential has the same value are called **equipotential surfaces**. This is similar to the isotherms used in meteorology. For example, let us examine the expression for the potential of a charge q at the origin:

$$V = \frac{kq}{r}$$

As shown in Figure 16.8, the equipotential surfaces of a point charge are spherical surfaces centered around the charge q . Likewise, the equipotential surfaces of an infinite line of charge are cylindrical surfaces with the line as their axis.

Let us emphasize the important features of the equipotential surfaces.

- **Conductor surfaces are equipotential surfaces.** In order to show this, let us write Eq. (16.5) for potential difference:

$$V_2 - V_1 = - \int_1^2 \vec{E} \cdot d\vec{r}$$

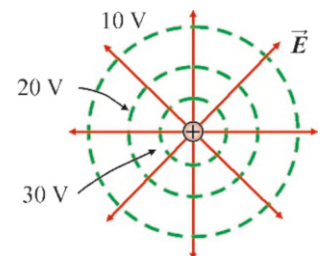


Figure 16.8: The equipotential surfaces around a point charge are spherical.

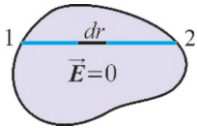


Figure 16.9: Two points on a conductor surface.

Let points 1 and 2 be located on the conductor’s surface (Figure 16.9). As potential difference is independent of the path taken, let us choose the path that connects these two points through the inside of the conductor. However, we had previously seen that $\vec{E} = 0$ is always true inside of a conductor. Therefore, the right hand-side of this integral is zero:

$$\vec{E} = 0 \implies V_2 = V_1$$

In other words, no work is performed to take a charge q from one point to another on a conductor’s surface.

- **Electric field lines are always perpendicular to equipotential surfaces.** For a small displacement $d\vec{r}$ the potential difference was given by Eq. (16.6): $dV = -\vec{E} \cdot d\vec{r}$. As two points very close to each other by $d\vec{r}$ will have the same potential on the equipotential surface (Figure 16.10), $dV=0$ and we get

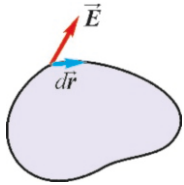


Figure 16.10: A small displacement $d\vec{r}$ on a conductor surface.

$$dV = 0 \implies -\vec{E} \cdot d\vec{r} = 0 \implies \vec{E} \perp d\vec{r}$$

We had previously proven with Gauss’s law that electric field is perpendicular to a conductor surface.

- **Electric field as the gradient of potential.** Let us consider that we take a small step $d\vec{r}$ from an equipotential surface with potential V , in the perpendicular direction (in other words, towards \vec{E}). As the vectors \vec{E} and $d\vec{r}$ are in the same direction, the scalar product turns into a simple product:

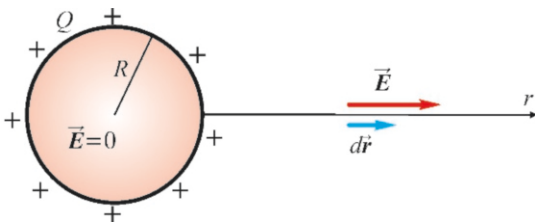
$$dV = -E dr$$

From here we get the relation between electric field and potential:

$$E = -\frac{dV}{dr} \tag{16.16}$$

The rate of potential increase perpendicular to the equipotential surface is called the **potential gradient**. Temperature gradient is mentioned in meteorology, and its meaning is likewise the temperature increase per unit length in the direction perpendicular to the isotherm. Therefore, **electric field is the negative potential gradient**. The negative sign tells us that the potential decreases in the direction of the electric field.

Example 16.10



Conducting sphere. Calculate the potential outside and inside of a conducting sphere with radius R and charge Q .

Answer

It is easier in this problem to calculate potential as the integral

of the electric field. We write the formula (16.5):

$$V_2 - V_1 = - \int_1^2 \vec{E} \cdot d\vec{r}$$

If we take $r_1 = \infty$ in this formula, we get $V_1 = 0$, and the potential at point r is written as $V_2 = V(r)$:

$$V(r) = - \int_{\infty}^r \vec{E} \cdot d\vec{r} = \int_r^{\infty} \vec{E} \cdot d\vec{r} = \int_r^{\infty} E dr$$

(We reversed the limits of the integral and used $\vec{E} \cdot d\vec{r} = E dr$, as \vec{E} and $d\vec{r}$ are in the same direction.)

Outside of the conducting sphere ($r > R$): the electric field is the same as the field $E = kQ/r^2$ of a point charge:

$$V(r) = \int_r^\infty \frac{kQ dr}{r^2} = \left| -\frac{kQ}{r} \right|_r^\infty$$

$$V(r) = \frac{kQ}{r} \quad (r > R)$$

Inside of the conducting sphere ($r < R$): Let us write the integral in two parts for a point r :

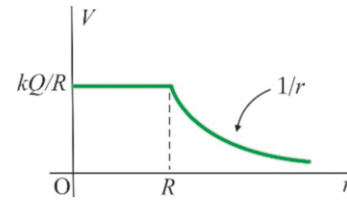
$$V(r) = \int_r^\infty E dr = \int_r^R E dr + \int_R^\infty E dr$$

As $E = 0$ is always true inside of the conductor, the first integral will be zero. The other is the value that the potential outside of the conductor takes at the surface $r = R$:

$$V(r) = \frac{kQ}{R} \quad (r < R)$$

The potential inside of the conductor is constant and equal to its value at the surface.

Let us look at a graphic of the potential of a conducting sphere:



Example 16.11



Two conducting spheres are located far away from each other. The one with a radius of 1 m is charged with $q_1=1\mu\text{C}$ and the other with a radius of 2 m is charged with $q_2=8\mu\text{C}$. These two spheres are connected with a metal wire. What is the new charge of each sphere?

Answer After the spheres are connected with a wire, they all constitute one single conductor, and thus their potential

will be equal.

As the spheres are far away from each other, the potential on the surface of each sphere only results from its own charge. Let us denote q'_1 and q'_2 as the new charges on the spheres. These new charges should fulfill two conditions:

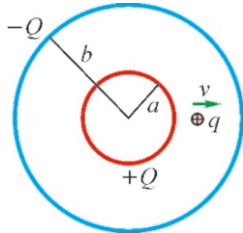
Charge conservation: $q'_1 + q'_2 = q_1 + q_2 = 1 + 8$
 $q'_1 + q'_2 = 9\mu\text{C}$ (1)

Equipotential: $\frac{kq'_1}{r_1} = \frac{kq'_2}{r_2} \rightarrow \frac{q'_1}{1} = \frac{q'_2}{2}$ (2)

Solving (1) and (2), we find the new charges:

$$q'_1 = 3\mu\text{C}, \quad q'_2 = 6\mu\text{C}$$

Example 16.12



Two concentric conducting spheres have radii $a=1\text{ m}$ and $b=2\text{ m}$, with charges $Q=+1\mu\text{C}$ and $-Q$, respectively.

- (a) Calculate the potential difference between the two spheres.
- (b) A $q=2\mu\text{C}$ point charge with a mass of 1 g is released from rest near the positively charged sphere. What will its speed be when it reaches the negatively charged sphere?

Answer (a) The potential due to the outer conducting sphere will be constant inside of it and equal to the value on its surface, and it will not contribute to the potential difference between a and b . Hence, the potential difference between the spheres will be the potential difference due to the inner sphere charged with $+Q$ at distances a and b :

$$V_a - V_b = kQ \left(\frac{1}{a} - \frac{1}{b} \right)$$

$$V_a - V_b = 9 \times 10^9 \times 10^{-6} \left(\frac{1}{1} - \frac{1}{2} \right) = 4500\text{ V}$$

(b) We write the energy conservation law for charge q :

$$\frac{1}{2}mv_A^2 + qV_A = \frac{1}{2}mv_B^2 + qV_B$$

$$v = \sqrt{\frac{2q(V_A - V_B)}{m}} = \sqrt{\frac{2 \times 2 \times 10^{-6} \times 4500}{0.001}}$$

$$v = 4.2\text{ m/s}$$

Multiple-choice Questions

1. If a 2 C charge gains 10 J in potential energy when placed at a point, what is the potential of that point?
 (a) 5 V (b) 10 V (c) 20 V (d) 40 V
2. If the distance between two parallel conducting plates is 2 m and their potential difference is 10 V, what is the electric field between the plates?
 (a) 5 V/m (b) 10 V/m (c) 20 V/m (d) 40 V/m
3. Of three conducting spheres, sphere A has a radius of 1 m and is charged with $1\mu\text{C}$, sphere B has a radius of 2 m and is charged with $3\mu\text{C}$ and sphere C has a radius of 3 m and is charged with $6\mu\text{C}$. Which sphere has greater potential?
 (a) A (b) B (c) C (d) Equal
4. Of two conducting spheres with equal potentials, if the

one with a radius of 1 m has a charge of $4\mu\text{C}$, then what is the charge on the one with a radius of 2 m?

- (a) $1\mu\text{C}$ (b) $2\mu\text{C}$ (c) $4\mu\text{C}$ (d) $8\mu\text{C}$

5. Which of the following are correct?

- I. Electrons are attracted towards higher potential.
 II. Protons are attracted towards higher potential.
 III. Electrons are attracted towards lower potential.
 IV. Protons are attracted towards lower potential.

- (a) I & II (b) I & III (c) II & III (d) I & IV

6. Which of the following is true at the middle point between point charges $+Q$ and $-Q$ with distance r in between?

- (a) $E = 0$ (b) $V = 0$ (c) $E = kQ/r^2$ (d) $V = 2kQ/r$

7. Which of the following complies with the definition of potential?

- I. The potential energy of a unit charge.
 II. The work performed to bring a unit charge from infinity to that point.
 III. The product of electric field and charge.
 IV. The electric field of a unit charge.

- (a) I & II (b) I & III (c) II & III (d) I & IV

8. Which of the following are correct?

- I. The potential is zero inside of a conductor.
 II. The potential is constant inside of a conductor.
 III. The potential inside of a conductor is equal to that on the surface.
 IV. The potential is infinity inside of a conductor.

- (a) I & II (b) I & III (c) II & III (d) I & IV

9. Which of the following is the potential energy of a charge q at a place where potential is V ?

- (a) V/q (b) qV (c) q^2V (d) $qV/2$

10. Which of the following are correct?

- I. Potential increases towards a positive charge.
 II. Potential increases towards a negative charge.
 III. Potential decreases towards a positive charge.
 IV. Potential decreases towards a negative charge.

- (a) I & II (b) I & III (c) II & IV (d) I & IV

11. When a $+1\text{ C}$ charge moves freely from a point at which the potential is 50 V to a point at which the potential is 20 V , how much does its kinetic energy change?

- (a) 0 (b) -30 J (c) $+30\text{ J}$ (d) $+70\text{ J}$

12. Which of the following systems consisting of two charges has a greater potential energy?

- A: $q_1 = q_2 = 1\text{ C}$ and separated by 1 m,
 B: $q_1 = q_2 = 2\text{ C}$ and separated by 4 m,
 C: $q_1 = q_2 = 3\text{ C}$ and separated by 3 m.

- (a) A (b) B (c) C (d) Equal

13. What is the electric field in a region where potential is constant?

- (a) 0
 (b) Constant.
 (c) Increasing linearly.
 (d) Decreasing linearly.

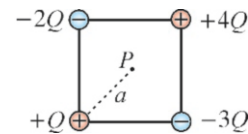
14. What is the potential in a region where electric field is constant?

- (a) 0
 (b) Constant
 (c) Increasing linearly.
 (d) Infinite.

15. If the radius of a conducting sphere is doubled and its charge is increased by a factor of 4, by what factor will its potential increase?

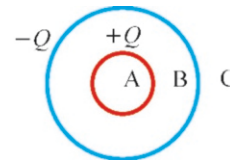
- (a) 2 (b) 4 (c) $1/2$ (d) Equal

16. What is the total potential produced at point P located at the center of the square by the charges shown on the corners of the square in the figure below?



- (a) 0 (b) $3kQ/a$ (c) $5kQ/a$ (d) $10kQ/a$

17. Which regions of two concentric conducting spheres charged with $\pm Q$ have constant potential?



- (a) A & C (b) A & B (c) Only A (d) Only C

18. Which of the following are correct for equipotential surfaces?

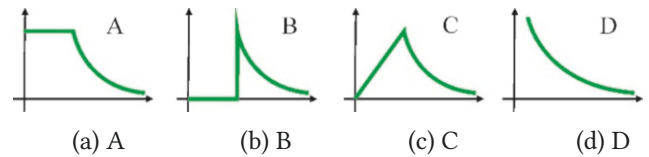
- I. Conductors are equipotential surfaces.
 II. Insulators are equipotential surfaces.
 III. The electric field is perpendicular to the equipotential surface.
 IV. The electric field is tangent to the equipotential surface.

- (a) I & II (b) I & III (c) II & IV (d) I & IV

19. Which is the expression for the potential of a point charge?

- (a) kq/r^2 (b) kq/r (c) kq^2/r^2 (d) kq^2/r

20. Which of the following figures is the potential of a charged conducting sphere?



Problems

16.1 Electric Potential

16.1 (a) How much potential energy will a $3\mu\text{C}$ charge have when placed at a point with a potential of 500 kV ? (b) An electron at rest at infinity is accelerated towards a metal plate and hits it with a speed of $6 \times 10^6\text{ m/s}$. What is the potential of the plate? (The electron's mass is $m_e = 9.1 \times 10^{-31}\text{ kg}$ and the charge is $-e = -1.6 \times 10^{-19}\text{ C}$.)

[A: (a) 1.5 J , (b) 102 V .]

16.2 Two identical conducting plates are placed parallel to each other with 2 mm distance in between and are connected to the terminals of a 24 V battery. (a) What is the magnitude of the electric field between the plates? (b) If the surface area of one plate is 100 cm^2 , how much charge accumulates on each plate?

[A: (a) 12 kV/m , (b) 1.1 nC .]

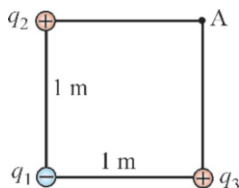
16.2 Potential of a System of Point Charges

16.3 At what distance from a point charge $q=5\text{ nC}$ will the potential be 30 V ?

[A: 1.5 m .]

16.4 The potential is 100 V and the electric field is 80 V/m at a certain distance from a point charge. Find the amount of charge and the distance.

[A: $r = 1.25\text{ m}$, $q = 140\mu\text{C}$.]



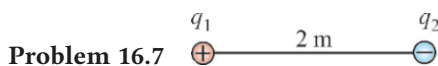
Problem 16.5

16.5 The charges $q_1 = -1\mu\text{C}$, $q_2 = 2\mu\text{C}$ and $q_3 = 3\mu\text{C}$ are placed on three corners of a square with side length of 1 m . (a) What will the potential be at point A ? (b) How much work needs to be done to bring a charge $q_4 = +4\mu\text{C}$ from infinity to point A ?

[A: (a) 39 kV , (b) 0.15 J .]

16.6 Six identical $2\mu\text{C}$ charges are placed in equal intervals around a circle with a radius of 1 m . How much work is performed to bring these charges from infinity to these positions?

[A: 0.52 J .]



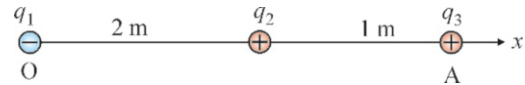
Problem 16.7

16.7 The distance between the charges $q_1 = 4\mu\text{C}$ and $q_2 = -1\mu\text{C}$ shown in the figure is 2 m . At what point will the potential be zero?

[A: 1.6 m to the right from q_1 .]

16.8 Two identical charges with masses $m_1 = m_2 = 1\text{ g}$ and charges $q_1 = q_2 = 5\mu\text{C}$ are released from rest with 2 m of distance in between. What will their speeds be when the distance is doubled?

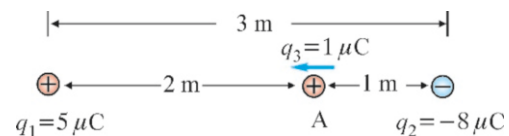
[A: 7.5 m/s .]



Problem 16.9

16.9 The charges $q_1 = -1\mu\text{C}$ and $q_2 = +2\mu\text{C}$ in the figure above are fixed with 2 m of distance in between. A third charge $q = 3\mu\text{C}$ with a mass of 1 g is released from rest at point A located along the extension of these charges at a distance of 1 m from q_2 . What will be its speed at infinity?

[A: 9.5 m/s .]



Problem 16.10

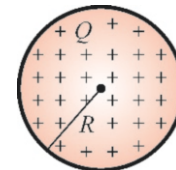
16.10 The charges $q_1 = 5\mu\text{C}$ and $q_2 = -8\mu\text{C}$ shown in the figure with 3 m of distance in between are fixed in their current locations. A third charge $q_3 = 1\mu\text{C}$ with mass $m_3 = 1\text{ g}$ is thrown from point A between these two charges with a speed of 10 m/s towards q_1 . What is the nearest distance at which it can approach q_1 ?

[A: 1.1 m .]

16.3 Potential of Continuous Charge Distributions

16.11 The potential is 400 V on the surface of a raindrop with a radius of 1 mm . (a) What is the charge of the drop? (b) When two such drops merge, what will the potential at the surface of the new drop be?

[A: (a) $4.4 \times 10^{-11}\text{ C}$, (b) 635 V .]



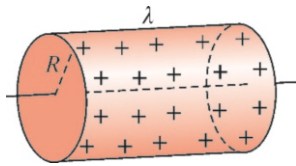
Problem 16.12

16.12 The electric field outside and inside of a sphere with radius R and with a total charge Q evenly distributed over its volume was found as follows using Gauss's law:

$$E = \begin{cases} kQ/r^2 & (r > R) \\ kQr/R^3 & (r < R) \end{cases}$$

Calculate the potential outside and inside of the sphere.

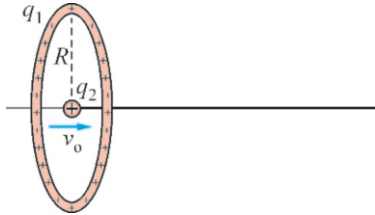
[A: kQ/r for $r > R$, $kQ(3 - r^2/R^2)/(2R)$ for $r < R$.]



Problem 16.13

16.13 An infinite cylindrical conducting shell with radius R has linear charge density λ on its surface. Calculate the potential difference between the surface of the cylinder ($r = R$) and a point located outside at distance r .

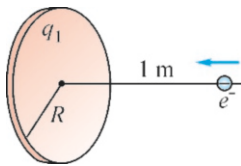
[A: $V_r - V_R = -2k\lambda \ln(r/R)$.]



Problem 16.14

16.14 A $q_2=2\mu\text{C}$ point charge with mass $m_2=1\text{g}$ is thrown with $v_0=10\text{ m/s}$ from the center and along the axis of a ring with radius $R=1\text{ m}$ carrying charge $q_1=1\mu\text{C}$. What will its speed be at infinity? (Hint: Use the result of the Example 16.8.)

[A: $v_\infty = 12\text{ m/s}$.]



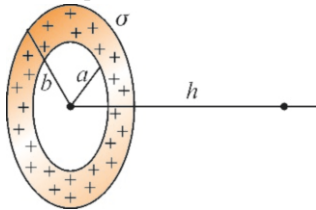
Problem 16.15

16.15 An electron is released from rest at a distance $h=1\text{ m}$ from a point on the axis of a disk with radius $R=1\text{ m}$ and on which a total charge $q_1=1\mu\text{C}$ is distributed uniformly. With what speed will it collide with the disk? (Hint: Use the result of Example 16.9.)

[A: $v = 6.1 \times 10^7\text{ m/s}$.]

16.16 In Example 16.9, the expression for the potential at distance h on the axis of a disk with radius r and carrying σ surface charge was found as follows:

$$V = 2\pi k\sigma \left[\sqrt{h^2 + R^2} - h \right]$$



Problem 16.16

Using this result, calculate the potential at distance h on the axis of a hollow disk with inner radius a and outer radius b and carrying surface charge density σ . (Hint: There is

?

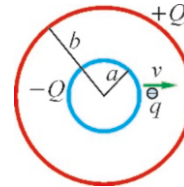
no need to take the integral again. Consider that there is a second disk with surface charge $-\sigma$ at the center of a full disk.)

[A: $V = 2\pi k\sigma \left[\sqrt{b^2 + h^2} - \sqrt{a^2 + h^2} \right]$.]

16.4 Conductors and Equipotential Surfaces

16.17 A conducting sphere with a radius of 3 m is kept at a 100 V potential. At what distance from the center of this sphere will the potential drop to 50 V ?

[A: 6 m .]



Problem 16.18

16.18 Of two concentric conducting spheres, the one with radius $a=1\text{ m}$ is charged with $-4\mu\text{C}$ and the one with radius $b=2\text{ m}$ is charged with $+4\mu\text{C}$. A point charge of $q=-1\mu\text{C}$ with a mass of 1 g is released from rest near the negatively charged sphere. What will its speed be when it reaches the positively charged sphere?

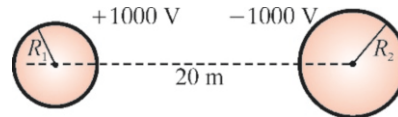
[A: 6 m/s .]



Problem 16.19

16.19 Two identical conducting spheres located far away from each other have a radius of 1 m . These two spheres are connected with a metal wire and are together set to a potential of 90 kV . (a) What is the charge on each sphere? (b) While the spheres are interconnected, the radius of the second sphere is increased to 2 m . What will the new charges of the spheres be? (c) What is the new potential of the spheres?

[A: (a) $q_1 = q_2 = 10\mu\text{C}$, (b) $q_1 = 6.7\mu\text{C}$, $q_2 = 13.3\mu\text{C}$, (c) 60 kV .]



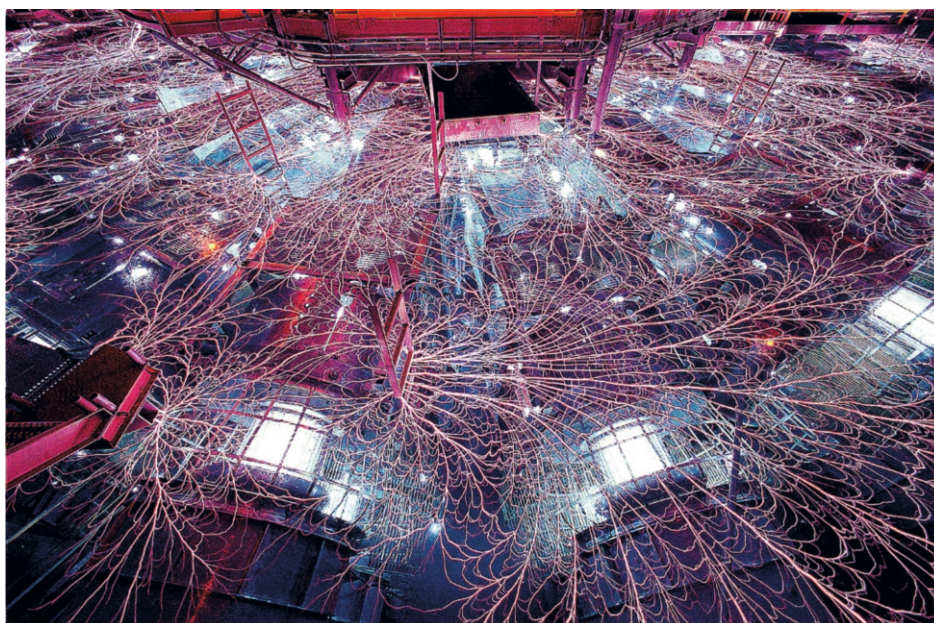
Problem 16.20

16.20 Of two conducting spheres separated by $d=20\text{ m}$ from their centers, one has radius $R_1=1\text{ m}$ and a potential of 1000 V and the second has radius $R_2=2\text{ m}$ and a potential of -1000 V . Calculate the charge on each sphere. (Hint: The total potential of each sphere is the sum of its own potential and the potential of the sphere at distance d . As the spheres are far away, assume that the charges are evenly distributed on their surfaces.)

[A: $q_1 = +0.12\mu\text{C}$, $q_2 = -0.23\mu\text{C}$.]

17

CAPACITORS AND DIELECTRICS



The Z machine at the Sandia Laboratory in New Mexico. The most powerful laboratory radiation source in the world, it works by charging a giant bank of capacitors. When it operates, it generates more power than 2,500 lightning bolts. It is used in fusion energy and other military-related researches. What makes capacitors useful in technology? How can we change their properties using insulating (dielectric) materials?

We discussed the basic properties of electrical forces in the previous chapters. We can now start discussing the technological applications of electricity. We shall first learn about a circuit component called the capacitor, which is used to store charge and electrical energy. Capacitors are commonly used today in electrical technology, in automobile spark plugs, radios and television, in camera flash lights and nanotechnology.

In this chapter, we will also take a closer look at the behavior of materials placed inside of an electric field. We will examine how these properties can change the performance of capacitors in energy and charge storage.

17.1 CAPACITANCE

A system consisting of two conductors carrying equal and opposite charges $\pm Q$ in vacuum or in an insulating medium is called a **capacitor**. The primary function of capacitors is to store charge under a certain potential difference. Regardless of their geometric shapes, the charge $\pm Q$ stored by all capacitors is proportional to the applied potential V :

$$Q = C V$$

The proportionality coefficient C is called **capacitance**. It is the ratio of the charge on the conductor to the applied potential difference:

$$C = \frac{Q}{V} \quad (\text{capacitance}) \quad (17.1)$$

Here, V is the absolute value of the potential difference $V_2 - V_1$ between the two conductors: $V = |V_2 - V_1|$. ΔV should actually have been used to indicate this. However, using V will not lead to any confusion in this chapter.

Capacitance depends on the geometric shape of the conductors and the properties of the intermediate insulator medium. You should think as follows to keep this in mind: *A capacitor with larger capacitance stores higher charge.*

The unit of capacitance is coulomb/volt (C/V) and was named as the **farad** (F) in memory of the great scientist Michael Faraday. The farad is a very large unit and its fractions are used in practice:

$$1 \text{ nanofarad (nF)} = 10^{-9} \text{ F}$$

$$1 \text{ picofarad (pF)} = 10^{-12} \text{ F}$$

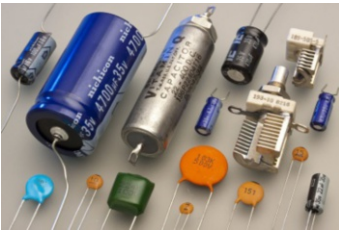


Figure 17.1: Various capacitors.

Figure 17.2: The place of capacitors in technology: Capacitors supplying the flash light of a camera, capacitors under the keys of a keyboard and dynamic memory (DRAM).

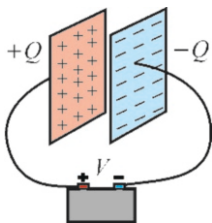
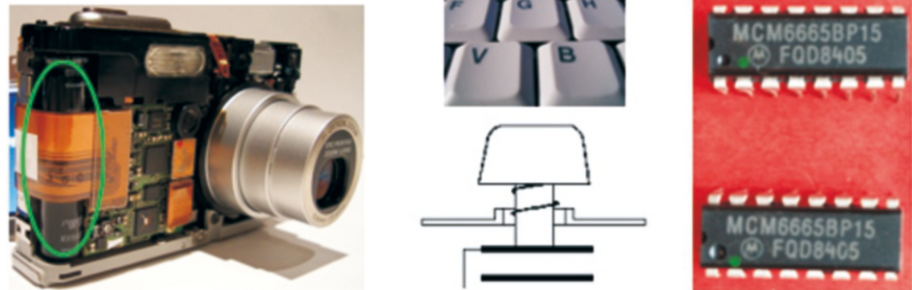


Figure 17.3: Parallel-plate capacitor connected to a battery.

Parallel-plate Capacitor

Let us consider two parallel conducting plates (Figure 17.3). Let each have surface area A and let the distance in between them be d . Two equal and opposite charges $\pm Q$ will accumulate when these plates are connected to the terminals of a battery generating potential difference V .

We can assume these plates to be approximately infinite planes if the dimensions of the plate are very large with respect to the distance d . Accordingly, we had previously found the electric field between two infinite planes carrying $\pm\sigma$ surface charge density:

$$E = \frac{\sigma}{\epsilon_0} = \text{constant}$$

In Chapter 16, we found the potential difference at distance d under a constant electric field:

$$V = V_2 - V_1 = E d = \frac{\sigma d}{\epsilon_0}$$

If the plate carrying charge Q has surface area A , the surface charge density σ will be $\sigma = Q/A$. Accordingly,

$$V = \frac{(Q/A)d}{\epsilon_0}$$

and, if we write this in terms of Q ,

$$Q = \frac{\epsilon_0 A}{d} V$$

Comparing with the definition in Eq. (17.1), we can identify the capacitance of a parallel-plate capacitor as:

$$C = \frac{\epsilon_0 A}{d} \quad (\text{capacitance of a parallel-plate capacitor}) \quad (17.2)$$

As you can see, capacitance is dependent on the geometric dimensions and the permittivity of the intermediate insulator space. Capacitance increases with the surface area of the plates and decreases with the distance between the plates.

Cylindrical Capacitor

Let us consider two coaxial conducting cylinders, both with length L (Figure 17.4). Let the internal cylinder have radius a and the outer b . We wish to calculate the potential difference when these cylinders are charged with equal and opposite charge $\pm Q$.

The cylinders can be assumed to be approximately infinite cylinders if their length L is very large with respect to their radii a, b . In Chapter 15, we used Gauss's law to find the electric field of an infinite cylinder with linear charge density λ :

$$E = \frac{2k\lambda}{r}$$

Let us calculate the potential difference without taking into consideration whether the field E is positive or negative:

$$V = V_b - V_a = \int_a^b E dr = 2k\lambda \int_a^b \frac{dr}{r} = 2k\lambda \ln \frac{b}{a}$$

The linear charge density of a cylinder with length L carrying charge Q will be $\lambda = Q/L$. Also, if we take $k = 1/4\pi\epsilon_0$, we find that

$$V = \frac{Q}{2\pi\epsilon_0 L} \ln \frac{b}{a} \quad \rightarrow \quad Q = \frac{2\pi\epsilon_0 L}{\ln(b/a)} V$$

From here, we can find the capacitance C :

$$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)} \quad (\text{capacitance of a cylindrical capacitor}) \quad (17.3)$$

We likewise observe that capacitance is dependent only on the geometric properties of the conductors.

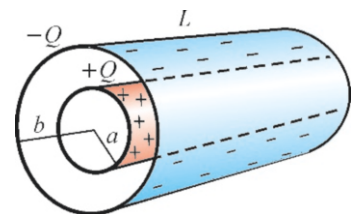


Figure 17.4: Cylindrical capacitor.

Spherical Capacitor

Let us consider two concentric conducting spherical shells with radii a and b with $a < b$. Let the inner sphere with radius a be charged with $+Q$ and the outer with $-Q$. Let us use the definition to calculate the potential difference V between these two spheres:

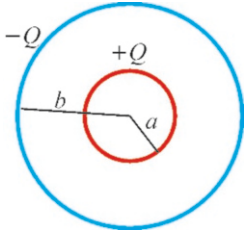


Figure 17.5: Spherical capacitor.

$$V = V_b - V_a = - \int_a^b E dr$$

We had found the electric field in the region between the two spheres ($a < r < b$) as $E = kQ/r^2$, using Gauss's law. Accordingly, the absolute value of the potential difference is

$$V = kQ \int_a^b \frac{dr}{r^2} = kQ \left| -\frac{1}{r} \right|_a^b = kQ \left(-\frac{1}{b} + \frac{1}{a} \right) = \frac{kQ(b-a)}{ab}$$

If we solve this expression for Q and also write the constant k in terms of ϵ_0 ,

$$Q = \frac{4\pi\epsilon_0 ab}{b-a} V$$

The coefficient of V will be the capacitance C of the spherical capacitor:

$$C = \frac{4\pi\epsilon_0 ab}{b-a} \quad (\text{capacitance of a spherical capacitor}) \quad (17.4)$$

Example 17.1

In a parallel-plate capacitor, the plates have a surface area of 30 cm^2 and a distance of 4 mm between them. This capacitor is connected to a 5000 V voltage (potential difference). (a) Find the capacitance of the capacitor. (b) What will the total charge accumulated on the plates be? (c) What is the electric field between the plates?

Answer

(a) We use Eq. (17.2) we found for a parallel-plate capacitor:

$$C = \frac{\epsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times 30 \times 10^{-4}}{4 \times 10^{-3}}$$

$$C = 6.6 \times 10^{-12} \text{ F} = 6.6 \text{ picofarad} = 6.6 \text{ pF}$$

(b) We use the definition of capacitance to find the charge:

$$Q = CV = 6.6 \times 10^{-12} \times 5000 = 33 \times 10^{-9} = 33 \text{ nC}$$

(c) As E is constant between the plates, its relation with potential is $V = Ed$:

$$E = \frac{V}{d} = \frac{5000}{4 \times 10^{-3}} = 1.25 \times 10^6 \text{ V/m}$$

Example 17.2

In a coaxial cable, the inner cylinder has a radius of 2 mm and the outer has a radius of 4 mm . A cylindrical capacitor with a capacitance of 1 pF is to be made from this cable. How long should the cable be?

Answer

We use Eq. (17.3) we found for cylindrical capacitor:

$$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$$

We solve for the length L and substitute the numerical data:

$$L = \frac{C \ln(b/a)}{2\pi\epsilon_0}$$

$$L = \frac{10^{-12} \times \ln(4/2)}{2 \times 3.14 \times 8.85 \times 10^{-12}} = 0.012 \text{ m} = 1.2 \text{ cm}$$

Example 17.3

A parallel-plate capacitor with capacitance $C=3 \text{ pF}$ is connected to a 1000 V battery.

- (a) How much charge is accumulated?
 (b) While the capacitor is connected to the battery, the distance between the plates is doubled. What will the new capacitance and new charge be?

(c) When it is fully charged, the capacitor is disconnected from the battery and then the distance between the plates is doubled. What will the new capacitance and the new potential difference between the plates be?

Answer (a) We calculate from the capacitance formula:

$$Q = CV = 3 \times 10^{-12} \times 1000 = 3 \text{ nC}$$

(b) As the capacitance of a parallel-plate capacitor is inversely

proportional to the distance d in between, C will halve if d is doubled. And V will remain constant, because it is connected to the battery,

$$Q' = C' V = \frac{C}{2} V = \frac{Q}{2} = 1.5 \text{ nC}$$

(c) The charge will not change, as the capacitor is disconnected while it has a charge Q . The potential difference V is again calculated from $Q = CV$, as C is halved:

$$V' = \frac{Q}{C'} = \frac{Q}{(C/2)} = 2V = 2 \times 1000 = 2000 \text{ volt}$$

17.2 COMBINATIONS OF CAPACITORS

Each circuit component used in electrical circuits is indicated with a standard symbol. Capacitors are indicated with the symbol $\text{—}||\text{—}$ in circuits.

A single capacitor that performs the function of multiple capacitors in a circuit is called an **equivalent capacitor**. The way to calculate an equivalent capacitor depends on whether the connections are in series or parallel.

Capacitors in Parallel

Two capacitors are said to be connected **in parallel** if they are connected to the *same potential difference*. Let two capacitors with capacitances C_1 and C_2 be connected to the same battery with potential difference V (Figure 17.6).

Then, the charges transferred to each capacitor can be calculated using the formula $Q = CV$:

$$Q_1 = C_1 V \qquad Q_2 = C_2 V$$

The total charge drawn by the capacitors will be $Q = Q_1 + Q_2$.

Now let us insert a single equivalent capacitor C_{eq} between the same points a and b such that the same charge accumulates under the same potential difference:

$$Q = C_{\text{eq}} V$$

As the charge Q here will be the sum of Q_1 and Q_2 ,

$$\begin{aligned} Q &= Q_1 + Q_2 \\ C_{\text{eq}} V &= C_1 V + C_2 V \end{aligned}$$

Canceling V , we get the equivalent capacitance for the parallel case:

$$C_{\text{eq}} = C_1 + C_2$$

This proof is valid for any number of capacitors connected in parallel:

$$C_{\text{eq}} = C_1 + C_2 + \cdots + C_N \quad (\text{Capacitors in parallel}) \quad (17.5)$$

The equivalent capacitance in parallel is larger than each capacitance.

Capacitors in Series

Two capacitors are said to be connected **in series** if they are connected *end to end*, without separating into another branch between them. When two such capacitors C_1 and C_2 are connected to the terminals of a battery (Figure 17.7), the outer plates will draw charges $+Q$ and $-Q$. The inner plates, although they draw no charge from the battery, cannot remain neutral; the charge $+Q$ on the left plate of C_1 will attract electrons in the wire to the right plate of C_1 , thus leaving the left plate of C_2 with charge $+Q$.

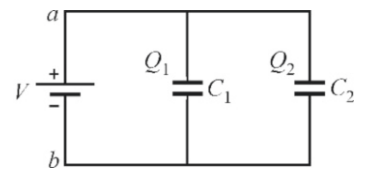


Figure 17.6: Capacitors in parallel.

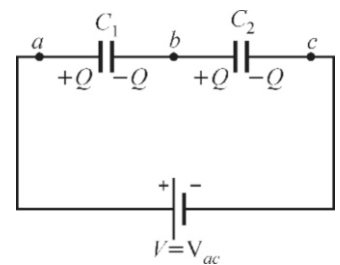


Figure 17.7: Capacitors in series.

We write the potential differences between the points a, b and c in the figure and use the formula $V = Q/C$ for each capacitor:

$$V_{ac} = V_{ab} + V_{bc} = V_1 + V_2$$

$$V = \frac{Q}{C_1} + \frac{Q}{C_2}$$

The equivalent capacitor to be inserted between ac should accumulate the same charge under the same potential difference:

$$\frac{Q}{C_{\text{eq}}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

Canceling Q , we get the equivalent capacitance for the series case:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{or} \quad C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}$$

(Note: The expression on the right is convenient for calculations, because it is a common mistake to forget to take the inverse at the last stage when working with sums of $1/C$.)

This expression is valid for any number of capacitors connected in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} \quad (\text{Capacitors in series}) \quad (17.6)$$

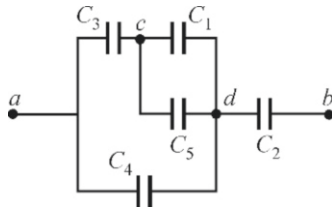
In series connection, the equivalent capacitance is less than each capacitance. Let us remember the following two principles (in addition to the formula $Q = CV$) when solving capacitor circuits:

- (1) Capacitors in parallel have equal voltage.
- (2) Capacitors in series have equal charge.

Example 17.4

In the circuit shown in the figure, we have $C_1 = 1$, $C_2 = 2$, $C_3 = 3$, $C_4 = 4$ and $C_5 = 5 \mu\text{F}$.

- (a) Calculate the equivalent capacitance between the terminals ab .
- (b) The terminals ab are connected to a 12 V potential difference. How much charge accumulates in the equivalent capacitor?
- (c) How much charge accumulates on the capacitor C_4 ?



Answer

(a) We carefully examine the figure to seek components that fit the definition of parallel or series. You should always look

for innermost capacitors.

C_1 and C_5 between points c and d are parallel. We use C' to indicate their equivalent capacitance between cd and calculate:

$$C' = C_1 + C_5 = 1 + 5 = 6 \mu\text{F}$$

Now, if we consider that this capacitance C' is between cd , it will be in series connection with C_3 . We calculate their equivalent capacitance:

$$C'' = \frac{C_3 C'}{C_3 + C'} = \frac{3 \times 6}{3 + 6} = 2 \mu\text{C}$$

Then, C_4 is in parallel with C'' . We calculate their equivalent capacitance:

$$C''' = C_4 + C'' = 4 + 2 = 6 \mu\text{F}$$

Finally, capacitance C_2 is in series with C''' and we find the equivalent capacitance that we are looking for:

$$C_{\text{eq}} = \frac{C_2 C'''}{C_2 + C'''} = \frac{2 \times 6}{2 + 6} = 1.5 \mu\text{F}$$

(b) As the equivalent capacitor is connected to terminals ab , its potential difference is V_{ab} . From here, we calculate the

charge Q :

$$Q = C_{\text{eq}} V_{ab} = 1.5 \times 12 = 18 \mu\text{C}$$

(c) We can solve this problem calculating charges and voltages alternately. First, note that the charge on C_2 is equal to the charge on the equivalent capacitor, because one terminal of the battery directly sees the capacitor C_2 without separating into another branch. We therefore write Q_2 directly as follows:

$$Q_2 = Q = 18 \mu\text{C}$$

From here, we use the formula $Q = CV$ to calculate the

potential difference over C_2 :

$$V_2 = V_{db} = \frac{Q_2}{C_2} = \frac{18}{2} = 9 \text{ V}$$

The potential difference V_{ab} facing the battery can be written in two parts:

$$V_{ab} = V_{ad} + V_{db} \rightarrow 12 = V_{ad} + 9$$

$$V_{ad} = 3 \text{ V}$$

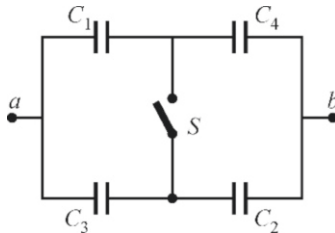
This potential difference is the one faced by C_4 . From here, we calculate the charge Q_4 :

$$Q_4 = C_4 V_4 = Q_4 V_{ad} = 4 \times 3 = 12 \mu\text{C}$$

Example 17.5

In the circuit shown below, we have $C_1=1$, $C_2=2$, $C_3=3$ and $C_4=4 \mu\text{F}$.

- (a) First, the terminals ab are connected to a 24 V battery when the switch S is open. Calculate the equivalent capacitance and the total charge drawn from the battery.
- (b) The circuit is disconnected from the battery after charging and then the switch S is closed. Find the equivalent capacitance and the charge on each capacitor.



Answer

(a) When the switch S is open, C_1 and C_4 are connected in series; likewise, C_2 and C_3 are also connected in series. Then, these two branches are connected in parallel to each other. We calculate the equivalent capacitance accordingly:

$$C_{\text{eq}} = \frac{C_1 C_4}{C_1 + C_4} + \frac{C_2 C_3}{C_2 + C_3}$$

$$C_{\text{eq}} = \frac{1 \times 4}{1 + 4} + \frac{2 \times 3}{2 + 3} = \frac{4}{5} + \frac{6}{5} = 2 \mu\text{F}$$

We calculate the charge of the equivalent capacitance from

the definition of capacitance:

$$Q = C_{\text{eq}} V = 2 \times 24 = 48 \mu\text{C}.$$

(b) The connection between ab changes when the switch S is closed. This time, the C_1 and C_3 are connected in parallel; likewise, the pair C_2 and C_4 are connected in parallel. Then, these two pairs are connected in series. We calculate the equivalent capacitance accordingly:

$$C_{\text{eq}} = \frac{(C_1 + C_3)(C_2 + C_4)}{(C_1 + C_3) + (C_2 + C_4)}$$

$$C_{\text{eq}} = \frac{(1 + 3)(2 + 4)}{1 + 2 + 3 + 4} = 2.4 \mu\text{F}$$

The total charge Q remains the same when the circuit is disconnected from the battery as charged. But, this time, it is differently distributed into two branches. As the capacitors C_1 and C_3 are connected to the common points aS , they have the same voltage:

$$V_{aS} = V_{aS}$$

$$\frac{Q_1}{C_1} = \frac{Q_3}{C_3}$$

Also, the sum of Q_1 and Q_3 should be equal to the initial charge:

$$Q_1 + Q_3 = Q = 48 \mu\text{C}$$

We calculate Q_1 and Q_3 from these two equations:

$$Q_1 = 12 \mu\text{C}, \quad Q_3 = 36 \mu\text{C}$$

The charges on C_2 and C_4 are calculated using the same method applied to the potential difference Sb . The result is:

$$Q_2 = 16 \mu\text{C}, \quad Q_4 = 32 \mu\text{C}$$

Energy of a Capacitor

Capacitors have energy because of the electric charge that they carry. They can discharge this energy in a very short duration to produce high currents or potential differences. For example, although automobile batteries have a 12 Volt potential difference, a capacitor circuit can produce potential differences up to 800–1000 V which are required to ignite spark plugs.

Let a capacitor with capacitance C have a total charge Q . Its energy will be equal to the amount of work required to bring it from zero to the final charge Q .

Consider a parallel-plate capacitor initially without charge. Let us first take a small charge dq_1 from one of the plates and carry it to the other (Figure 17.8). The work is zero, because no potential exists yet:

$$dW_1 = 0$$

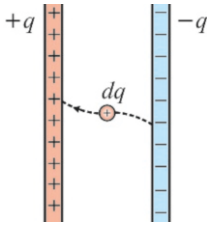


Figure 17.8: Gradual charging of a capacitor.

When we try to carry the next charge dq_2 , there will be a potential difference of dq_1/C between the plates, and therefore the work to be performed will be

$$dW_2 = dq_2 V_1 = dq_2 \frac{dq_1}{C}$$

We carry out this process repeatedly and reach the charge Q in the end. We can calculate the energy of the capacitor by adding the works performed at each stage.

Let q be the charge accumulated in the plates at some stage. The potential difference at that stage will be $V = q/C$. Now, the work to be performed if we wish to carry an additional charge dq will be

$$dW = dq V = dq \frac{q}{C}$$

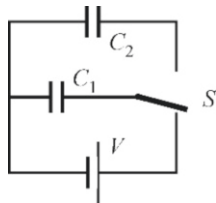
The work to be performed to bring the charge from zero at the start to the value Q , in other words, the increase in the potential energy of the capacitor, will be the integral of the small works dW :

$$W = U_Q - U_0 = \frac{1}{C} \int_0^Q q dq = \frac{1}{C} \left[\frac{q^2}{2} \right]_0^Q = \frac{1}{2} \frac{Q^2}{C}$$

The energy of the neutral capacitor is chosen as zero potential: $U_0=0$. Using the formula $Q = CV$ for the charge, the energy of a capacitor can be written under various forms:

$$U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2 = \frac{1}{2} QV \quad \text{(Energy of a capacitor)} \quad (17.7)$$

Example 17.6



The capacitor shown in the figure with capacitance $C_1=1 \mu\text{C}$ is charged by closing the switch S in the direction of the battery with a potential difference of $V=12 \text{ V}$.

- (a) What are the charge and energy of the capacitor C_1 ?
- (b) The switch S is then closed in the direction of the capacitor $C_2 = 2 \mu\text{C}$. What will the charges and total energies of the two capacitors be?
- (c) How can the energy difference in between be explained?

Answer

(a) C_2 is not taken into consideration when the switch S is closed in the direction of the battery. We calculate the charge as follows:

$$Q = C_1 V = 1 \times 12 = 12 \mu\text{C}$$

We calculate the energy using any one of the formulas (17.7):

$$U = \frac{1}{2} QV = \frac{1}{2} \times 12 \times 10^{-6} \times 12$$

$$U = 72 \times 10^{-6} \text{ J} = 72 \mu\text{J}$$

(b) When the switch S is closed in the direction of C_2 , the charge Q at the start is distributed between C_1 and C_2 . Considering that the potential differences between the terminals of these two capacitors are equal, we get

$$V_1 = V_2 \rightarrow \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

$$Q_1 + Q_2 = Q = 12 \mu\text{C}$$

We calculate Q_1 and Q_2 from these two equations:

$$Q_1 = 4 \mu\text{C} \text{ and } Q_2 = 8 \mu\text{C}$$

We use the formula $U = \frac{1}{2} Q^2/C$ to calculate the energies:

$$U' = \frac{1}{2} \frac{Q_1^2}{C_1} + \frac{1}{2} \frac{Q_2^2}{C_2} = \frac{1}{2} \left(\frac{4^2}{1} + \frac{8^2}{2} \right) \times 10^{-12+6}$$

$$U' = 24 \mu\text{J}$$

(c) In this problem, we have $U' < U$, and it seems as if the energy of the system automatically decreased, in other words, not conserved.

This problem is actually an ideal case that is not found in real life. Circuit wires connecting capacitors usually have a certain resistance. When one capacitor transfers its charge by moving electrons to the other, energy will be lost in the resistance of wires. As a result, the energy difference in be-

tween gets converted into heat in the wires of the circuit. Otherwise, if such losses did not exist, the system would not reach equilibrium and the charge Q would go back and forth from one capacitor to the other.

17.3 DIELECTRICS

In our analysis so far, we have assumed that there was vacuum between the two plates of the capacitor. However, the area between the plates is filled with an insulating material in capacitors used in technology. These materials include paper, glass, plastic, oil, etc. When insulators are placed in an electric field, they try to accommodate their structure with the applied field. They are called **dielectrics** to emphasize this property.

According to experimental observations:

- A capacitor accumulates more charge when a dielectric material is inserted between the plates while it is connected to a battery. According to this observation, the *capacitance of the capacitor increases* due to the formula $C = Q/V$.
- When a dielectric material is inserted between the plates after a charged capacitor is disconnected, the potential difference between the plates decreases. According to this observation, for example, if we recall the formula $V = E d$ for parallel-plate capacitors, *the electric field between the plates decreases*.

How is the material between the plates able to change the properties of the capacitor despite the fact that it is an insulator? In order to understand this, we need to examine the microscopic structure of dielectric materials.

Remember the concept of **electric dipole** from Chapter 13: We discussed that a system consisting of two equal and opposite charges ($\pm q$) separated by distance a is called an electric dipole. This system could produce an electric field in space and interact with other charges even though it is neutral. This system has an electric **dipole moment** defined as $p = qa$.

In some molecules (like H_2O , NO_2 , HCl), the positions of positive and negative charges do not coincide and they form a electric dipole. These are called **polar molecules** (Figure 17.9a). They have a permanent dipole moment, even when there is no electric field. When they are in an external electric field, they try to align their dipole moments with it.

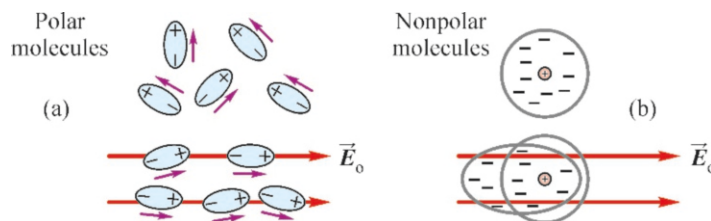
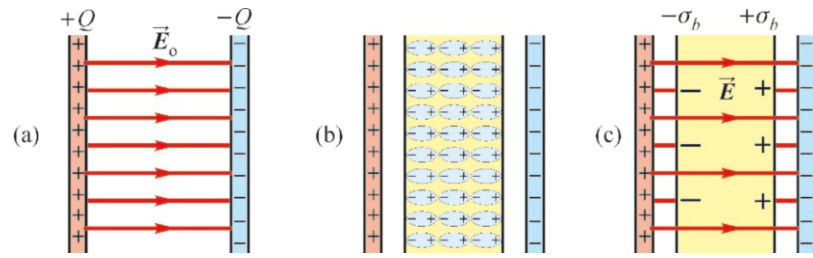


Figure 17.9: Polar and nonpolar molecules placed in an external electric field gain a dipole moment in the direction of the field.

In some other molecules (like O_2 , CO_2 , CH_4), the positions of positive and negative charges coincide and they have no permanent dipole moment. However, when placed in an external electric field, the geometric location of the electrons changes under the forces exerted in the opposite direction to the $+$ and $-$ charges and they gain a dipole moment. These are called **nonpolar molecules** (Figure 17.9b).

Figure 17.10: The molecules of dielectric materials placed in between the plates of a capacitor form surface charges in the opposite direction and cause the field \vec{E} to decrease inside.



Regardless of their structure, dielectric molecules placed in an electric field between the plates of a charged capacitor show a reaction (Figure 17.10). Polar molecules rotate and nonpolar molecules deform to try to align their dipole moments in the direction of the electric field (Figure 17.10b). Although they cannot fully balance the electric field, they partially reduce its effect in the medium. **Induced surface charges** are thus produced on dielectric surfaces facing the plates (Figure 17.10c). This phenomenon is called **polarization**.

If we express the electric field produced by the charges $\pm Q$ on the plates in terms of the surface charge density σ , we get

$$E_0 = \frac{\sigma}{\epsilon_0}$$

If we use σ_b to show the surface charge density induced in the opposite direction by the *bound* charges of the dielectric material, the new electric field will result from the net surface charge density $\sigma - \sigma_b$:

$$E = \frac{\sigma - \sigma_b}{\epsilon_0} = E_0 - \frac{\sigma_b}{\epsilon_0} \quad (17.8)$$

If the external electric field is not too strong, the amount of the induced charge σ_b will be proportional to the electric field E that they *feel* in the medium. Hence, E and E_0 will be proportional, with E being smaller:

$$E = \frac{E_0}{\kappa} \quad (17.9)$$

The denominator of this expression is defined as the **dielectric constant** of the medium. The electric field always decreases in a dielectric medium because $\kappa > 1$.

The decrease in the electric field will also decrease the potential difference through the formula $V = E d$. If V_0 is the potential difference in vacuum,

$$\begin{aligned} V &= E d = \frac{E_0}{\kappa} d = \frac{E_0 d}{\kappa} \\ V &= \frac{V_0}{\kappa} \end{aligned} \quad (17.10)$$

According to the definition $C = Q/V$, we find the capacitance in the presence of a dielectric medium:

$$C = \kappa C_0 \quad (\text{Capacitance of a capacitor with dielectric}) \quad (17.11)$$

All of these results can be expressed simply by defining the permittivity of the dielectric medium as:

$$\epsilon = \kappa \epsilon_0 \quad (17.12)$$

Here, ϵ is the **electric permittivity** of the medium. Therefore, all formulas become valid in capacitors with dielectrics by using ϵ instead of ϵ_0 .

Some dielectric constants are given in the table:

Dielectric constants and dielectric strengths of some insulators

	dielectric constant κ	dielectric strength E_{\max} (10^6 V/m)
Vacuum	1	—
Air	1.0006	3
Paraffin	2.2	10
Paper	3.7	15
Glass	5	14
Porcelain	6	12



Figure 17.11: Use of dielectrics in technology: Coaxial cables, pin insulators, pure silicon layers known as *wafers* in the manufacture of microcircuits.

Dielectric Strength

When the magnitude of the electric field applied on a dielectric is too strong, the positive and negative charges of the molecules are subject to extreme forces in the opposite directions and their bond is broken. Molecules become ionized and the dielectric turns into a conducting medium in which an electric discharge is observed. The high current passing through the dielectric turns into heat, damaging the material (Figure 17.12).

The maximum electric field that a dielectric medium can endure without *dielectric breakdown* is called its **dielectric strength**. The dielectric strengths of some materials are given in the table above.

As can be seen from the table, air cannot withstand high electric fields. But if we insert a dielectric between the plates of a capacitor, according to Eq. 17.9, $E = E_0/\kappa$, it will have a weaker electric field for the same accumulated charge. And the capacitor can operate in higher voltages without getting damaged.

van de Graaff Generator

How large an electric potential can be generated on a conductor? We can now answer this question using what we have learned up to now about conductors and dielectrics. Let us consider a hollow conducting spherical shell with radius R as the simplest case. Let us remember the expressions for the potential and surface electric field of this sphere:

$$V = \frac{kQ}{R}, \quad E = \frac{kQ}{R^2} \quad \text{and} \quad V = RE$$

In principle, the more we can increase Q , the more the potential of the conductor will increase, until infinity. However, this actually has a limit. The electric field around the conductor increases with its potential.

The dielectric strength of the air in which the sphere is located is $E_{\max} = 3 \times 10^6$ V/m, in other words, at higher electric fields, the air molecules get ionized and dielectric breakdown occurs, with charges accumulated on the conducting



Figure 17.12: Cracks formed on a plexiglass plate as a result of dielectric breakdown.

sphere flowing to the ground. Therefore, the maximum potential that the sphere can carry should fulfill the following equation:

$$V_{\max} = R E_{\max}$$

For a sphere with radius $R = 1 \text{ m}$, we find that the maximum charge that it can carry is $Q_{\max} = 1/3 \text{ mC}$ and the maximum potential produced is $V_{\max} = 3 \times 10^6 \text{ V}$.

But how do we charge this sphere? If we bring the charges directly, as the electric field around the sphere will increase gradually, the opposing force will also increase and it will become more difficult to bring additional charges. However, the following property of conductors will make it easier: *Charges placed in the internal surface of a conductor will accumulate in the outer surface*. This is because the electric field inside of the conductor must always be zero.

A simple diagram of the **van de Graaff generator** that uses this principle is shown in Figure 17.13. In this mechanism, an insulating rubber belt moves by turning a pulley on the lower end. A sharp metal brush connected to the positive terminal of a battery stands very close to the belt near the bottom pulley. This brush ionizes the air around it and transfers some of the positive charges to the belt. A second metal brush near the top pulley has zero electric field around it, as it is connected to the conducting sphere and easily collects the charges on the belt and then transfers them to the external surface of the sphere.

The van de Graaff generator is used in nuclear physics researches to accelerate charged particles and operate X-ray tubes. High-voltage shows are also performed in physics laboratories and science fairs. If your shoes are insulated from the ground, when you touch the metal sphere, your hairs will repel each other and stand up due to the high voltage formed on your body. (The van de Graaff generator is not a dangerous device, despite this high potential. Considering that the charge required for 3 million volts is on the order of merely millicoulombs, the current that can flow from your feet to the ground is very low.)

Lightning

In stormy weather, we hear a peal of thunder like a big explosion after a flash of light in the sky and understand that a lightning bolt has struck somewhere. Lightning is the electric discharge that occurs after air becomes conductor when the maximum value of electric field that the dielectric medium (air) can withstand is exceeded, as explained above. There can be different types of lightning with different structures. The most common type of lightning is explained as follows:

Water molecules that evaporate on the surface of the Earth start to rise, as they are lighter than air. They start to condense into water droplets when they reach the colder upper layers of the atmosphere. This mixture of microscopic particles of water and ice collide with other rising water vapor molecules, and ionize them by freeing their electrons. Negative electrons accumulate in the bottom surface of the cloud and positive ions in the top surface.

Consequently, an electric field arises between the negative bottom surfaces of the clouds and the Earth's surface. As the magnitude of the electric field increases, the molecules of the air in between generate a conducting path to ensure a discharge. Hence, before the lightning strike, zigzagged fringes called **leaders** consisting of ionized air molecules start to extend downward from the clouds and upwards from the roofs on the Earth's surface (Figure 17.15).

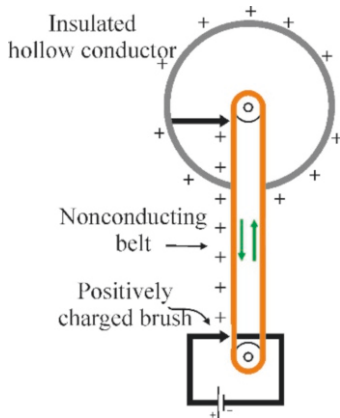


Figure 17.13: Diagram of the van de Graaff generator.



Figure 17.14: A tree struck by lightning.

Once the conducting path produced by these ionized air molecules is completed, the electric discharge occurs instantly and a very large electron current reaches the Earth's surfaces. The heat generated by the current causes the surrounding air molecules to heat and expand. As this expansion occurs instantly, it produces an explosive effect and a shockwave.

As the speed of light is greater than the speed of sound, the light is observed first and the sound later. As the speed of sound in air is 340 m/s, you can guess the distance of the place where the lightning struck by counting the seconds after it strikes (1 km per 3 seconds).

Leaders that prepare the path of lightning are usually produced on high buildings and trees. Therefore, it can be dangerous to stay in open air or to take shelter under a tree. The best thing to do is to crouch, but without touching the ground with your hands, in order to not increase the conducting path.

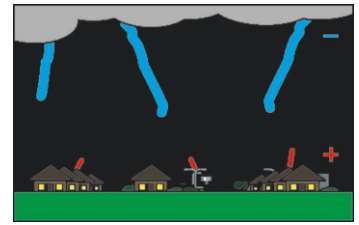


Figure 17.15: Leaders that prepare the path of lightning.

Example 17.7

There is a 2 mm separation between the plates of a parallel-plate capacitor. Its capacitance is $C_0 = 3 \mu\text{F}$ when there is vacuum between the plates.

- The capacitor is connected to a battery with $V_0 = 12 \text{ V}$. What will the charge Q_0 and the electric field E_0 between the plates be?
- The capacitor is disconnected as charged and a glass layer with dielectric constant $\kappa = 5$ is inserted between the plates. Find the capacitance C , charge Q , potential V and electric field E of the capacitor.
- A glass layer is inserted between the plates when the capacitor is still connected to the battery. Find the capacitance C , charge Q , potential V and electric field E of the capacitor.

Answer

(a) We calculate Q_0 from the definition:

$$Q_0 = C_0 V_0 = 3 \times 12 = 36 \mu\text{C}$$

We find the electric field using the formula $V = Ed$ in constant electric field:

$$E_0 = \frac{V_0}{d} = \frac{12}{0.002} = 6000 \text{ V/m} = 6 \text{ kV/m}$$

(b) As the capacitor is disconnected as charged, its charge will remain constant afterwards:

$$Q = Q_0 = 36 \mu\text{C}$$

To calculate the new capacitance when a dielectric is inserted in between, it is sufficient to replace the coefficient ϵ_0 with

$\kappa\epsilon_0$ in the capacitance formula (17.2) of the parallel-plate capacitor:

$$C = \frac{\kappa\epsilon_0 A}{d} = \kappa C_0 = 5 \times 3 = 15 \mu\text{F}$$

We calculate the potential V using this capacitance C and charge Q :

$$V = \frac{Q}{C} = \frac{Q}{\kappa C_0} = \frac{V_0}{\kappa} = \frac{12}{5} = 2.4 \text{ V}$$

The electric field is calculated with this potential difference V :

$$E = \frac{V}{d} = \frac{(V_0/\kappa)}{d} = \frac{E_0}{\kappa} = \frac{6000}{5} = 1.2 \text{ kV/m}$$

(c) The potential difference remains constant if the capacitor is kept connected to the battery:

$$V = V_0 = 12 \text{ V}$$

The change in capacitance is the same when a dielectric is likewise inserted in between:

$$C = \kappa C_0 = 15 \mu\text{F}$$

The charge is calculated using this capacitance and potential difference:

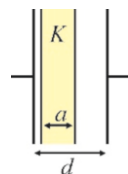
$$Q = CV = (\kappa C_0) V_0 = 15 \times 12 = 180 \mu\text{C}$$

The electric field is calculated using the same method:

$$E = \frac{V}{d} = \frac{V_0}{d} = E_0 = 6 \text{ kV/m}$$

As you may see, a capacitor connected to the same battery accumulates higher charge when there is a dielectric but the electric field between the plates still remains the same.

Example 17.8



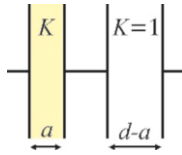
A parallel-plate capacitor has capacitance C_0 when the area between the plates is empty. A dielectric layer with constant κ that fills a distance a of the total distance d between the plates

is inserted. What will the new capacitance be?

Answer

We write the capacitance C_0 of the parallel-plate capacitor before the dielectric layer is inserted as $C_0 = \epsilon_0 A/d$.

We can consider the system as two capacitors connected in series when the dielectric is inserted. The first capacitor has thickness a and dielectric constant κ_1 and the second capacitor has thickness $d - a$ and dielectric constant $\kappa_2 = 1$.



Accordingly, we write the equivalent capacitance formula of the capacitors in series as follows:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{a}{\kappa\epsilon_0 A} + \frac{d-a}{\epsilon_0 A}$$

We factor out C_0 in this expression:

$$\frac{1}{C} = \frac{d}{\epsilon_0 A} \left(\frac{a}{\kappa d} + \frac{d-a}{d} \right) = \frac{1}{C_0} \frac{\kappa d - (\kappa - 1)a}{\kappa d}$$

From here, we find the capacitance C in terms of C_0 and dimensions a and d :

$$C = \frac{\kappa d}{\kappa d - (\kappa - 1)a} C_0$$

In order to check the validity of this expression, we can see that $C = C_0$ at the limit $a = 0$. Likewise, it correctly gives $C = \kappa C_0$ within the limit $a = d$.

Example 17.9

The 4 mm wide space between the plates of a parallel-plate capacitor is filled with air.

- (a) What is the maximum operating voltage of the capacitor?
 (b) The space between the plates is filled with a porcelain layer. What is the maximum operating voltage this time? (Dielectric strengths: Air: $E_{\max} = 3 \times 10^6$ V/m, porcelain: 12×10^6 V/m)

Example 17.10

A parallel-plate capacitor is to be manufactured with a material with dielectric constant $\kappa=30$ and dielectric strength 15×10^6 V/m such that its capacitance shall be 1 pF and be operating up to 30 kV voltage. What should the surface area and distance between the plates of this capacitor be?

Answer

The electric field between the plates by the maximum voltage should be equal to the dielectric strength:

$$V = E_{\max} d$$

Answer

(a) The fact that air's dielectric strength is 3×10^6 V/m means that air will lose its dielectric property and turn into a conductor if the electric field exceeds this value. Therefore, the maximum potential difference between the plates of the parallel-plate capacitor is calculated using this E_{\max} value:

$$V = E_{\max} d = 3 \times 10^6 \times 0.004 = 12\,000 \text{ V} = 12 \text{ kV}$$

(b) A similar method is used for porcelain:

$$V = E_{\max} d = 12 \times 10^6 \times 0.004 = 48\,000 \text{ V} = 48 \text{ kV}$$

From here, we calculate the distance d :

$$d = \frac{V}{E_{\max}} = \frac{30\,000}{15 \times 10^6} = 0.002 \text{ m} = 2 \text{ mm}$$

We write the capacitance formula of the parallel-plate capacitor with dielectric constant κ :

$$C = \frac{\kappa\epsilon_0 A}{d}$$

From here, we calculate the surface area A :

$$A = \frac{Cd}{\kappa\epsilon_0} = \frac{10^{-9} \times 0.002}{30 \times 8.85 \times 10^{-12}}$$

$$A = 0.0075 \text{ m}^2 = 75 \text{ cm}^2$$

Multiple-choice Questions

- What is the unit of capacitance?
 (a) joule (b) volt (c) faraday (d) farad
- In a parallel-plate capacitor, the surface area of the plates and the distance in between are both doubled. By what factor will capacitance increase?
 (a) No change (b) 2 (c) 1/2 (d) 4
- The capacitance of a capacitor and the applied voltage are both doubled. By what factor will the accumulated charge increase?
 (a) No change (b) 2 (c) 1/2 (d) 4
- A parallel-plate capacitor is disconnected after being charged and the distance between its plates is doubled. Which of the following are correct?
 I. The charge remains the same.
 II. The potential doubles.
- The electric field doubles.
 IV. The capacitance doubles.
 (a) I & II (b) I & III (c) I & IV (d) II & IV
- The distance between the plates of a parallel-plate capacitor is doubled while it is connected to a battery. Which of the following are correct?
 I. The potential difference remains the same.
 II. The capacitance halves.
 III. The charge halves.
 IV. The electric field remains the same.
 (a) I & II (b) I & III (c) I & IV (d) I, II & III
- When two capacitors with capacitance $1 \mu\text{F}$ and $2 \mu\text{F}$ are connected in parallel, how many μF will the equivalent capacitance be?
 (a) 1 (b) 3 (c) 2/3 (d) 3/2

7. When two capacitors with capacitance $1\ \mu\text{F}$ and $2\ \mu\text{F}$ are connected in series, how many μF will the equivalent capacitance be?

- (a) 1 (b) 3 (c) $2/3$ (d) $3/2$

8. A dielectric with $\kappa = 3$ is inserted between the plates of a capacitor. By what factor will the capacitance increase?

- (a) No change (b) $1/3$ (c) 3 (d) 9

9. A dielectric with $\kappa = 3$ is inserted between the plates of a capacitor charged with Q . By what factor will the potential difference V increase?

- (a) No change (b) $1/3$ (c) 3 (d) 9

10. A dielectric with $\kappa = 3$ is inserted between the plates of a capacitor charged with Q . By what factor will the electric field increase?

- (a) No change (b) $1/3$ (c) 3 (d) 9

11. By what factor will the energy of a capacitor charged with Q increase when its capacitance is doubled?

- (a) $1/2$ (b) 2 (c) 4 (d) $1/4$

12. By what factor will the energy of a capacitor charged with Q increase when the distance between its plates is doubled?

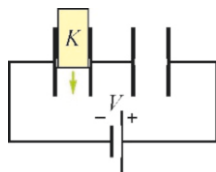
- (a) $1/2$ (b) 2 (c) 4 (d) $1/4$

13. By what factor will the energy of a capacitor increase when the potential difference V applied to it doubles?

- (a) $1/2$ (b) 2 (c) 4 (d) $1/4$

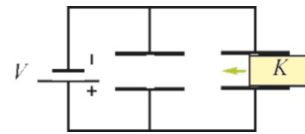
14. By what factor will the ratio Q/V increase for a capacitor whose capacitance is doubled?

- (a) No change (b) 2 (c) 4 (d) $1/2$



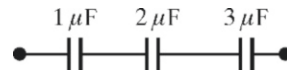
15. While the capacitors are connected to the circuit, a dielectric material is inserted between the plates, as shown in the figure above. What happens to the total charge drawn from the battery?

- (a) It increases.
 (b) It decreases.
 (c) It remains the same.
 (d) It is impossible to tell.



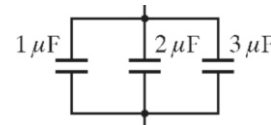
16. While the capacitors are connected to the circuit, a dielectric material is inserted between the plates, as shown in the figure above. What happens to the total charge drawn from the battery?

- (a) It increases.
 (b) It decreases.
 (c) It remains the same.
 (d) It is impossible to tell.



17. What is the equivalent capacitance of the circuit in the figure above?

- (a) $6\ \mu\text{F}$ (b) $6/11\ \mu\text{F}$ (c) $1/6\ \mu\text{F}$ (d) $4\ \mu\text{F}$



18. What is the equivalent capacitance of the circuit in the figure above?

- (a) $6\ \mu\text{F}$ (b) $6/11\ \mu\text{F}$ (c) $1/6\ \mu\text{F}$ (d) $4\ \mu\text{F}$

19. Of two identical capacitors, one has double the charge and half the potential with respect to the other. By what factor is the energy of the second capacitor higher than that of the first one?

- (a) Equal (b) 2 (c) $1/2$ (d) 4

20. Which of the following can explain the decrease in electric field when a dielectric is placed between the plates of a capacitor?

I. An opposite electric field produced by molecular dipoles.

II. The surface charges induced on the dielectric surfaces.

III. Electrons detached from molecules.

IV. The ionization of molecules.

- (a) I & II (b) I & III (c) II & IV (d) I & IV

Problems

17.1 Capacitance

17.1 A capacitor with capacitance 5 pF is to be manufactured using two conducting plates with surface area 80 cm^2 . What should the distance between the plates be? [A: 14 mm .]

17.2 The distance between the plates of a parallel-plate capacitor is 2 mm and each plate has an area of 40 cm^2 . Charges $\pm 5 \text{ nC}$ accumulate in its plates when this capacitor is connected to a battery. (a) Find the capacitance of the capacitor. (b) Find the potential difference of the plates. (c) Find the electric field between the plates.

[A: (a) 18 pF , (b) 280 V , (c) 140 kV/m .]

17.3 A parallel-plate capacitor with capacitance $C = 5 \text{ pF}$ is connected to a 2000 V battery. (a) How much charge is accumulated? (b) The capacitor is disconnected as charged and then the distance between the plates is halved. What will the new capacitance and the new potential difference between the plates be? (c) The distance between the plates is halved while the capacitor is connected to the battery. What will the new capacitance and new charge be?

[A: (a) 10 nC , (b) 10 pF , 1000 V , (c) 10 pF , 20 nC .]

17.4 The capacitance of a spherical capacitor was found in Eq. (17.4) as:

$$C = \frac{4\pi\epsilon_0 ab}{b - a}$$

(a) Show that the capacitance is $C = 4\pi\epsilon_0 a$ when the radius of the external conductor goes to infinity ($b \rightarrow \infty$). (b) Assume that the Earth is a conductor and calculate its capacitance. (Earth's radius: 6400 km .)

[A: $710 \mu\text{F}$.]

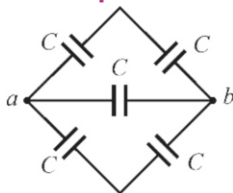
17.5 A spherical capacitor with 10 pF in capacitance consists of two conducting spheres for which the radius of one is double that of the other. Calculate the radius of each sphere.

[A: 4.5 and 9 cm .]

17.6 A cylindrical capacitor with 1 nF in capacitance is to be manufactured using a coaxial cable for which the radius of the inner cylinder is 1 mm and the outer is 3 mm . How long should the cable be?

[A: 20 m .]

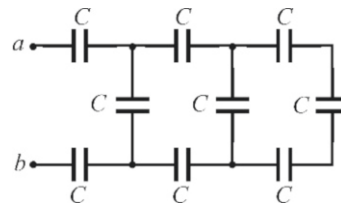
17.2 Connection of Capacitors



Problem 17.7

17.7 Each capacitor in the circuit shown in the figure above has capacitance $C = 1 \mu\text{F}$. What is the equivalent capacitance between ab ?

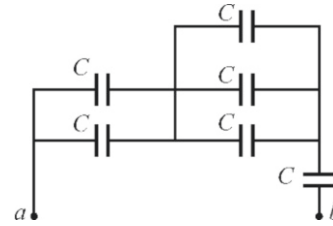
[A: $2 \mu\text{F}$.]



Problem 17.8

17.8 Each capacitor in the circuit shown in the figure above has capacitance $C = 1 \mu\text{F}$. What is the equivalent capacitance between ab ?

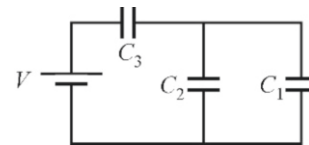
[A: $15/41 \mu\text{F}$.]



Problem 17.9

17.9 Each capacitor in the circuit shown in the figure above has capacitance $C = 1 \mu\text{F}$. What is the equivalent capacitance between ab ?

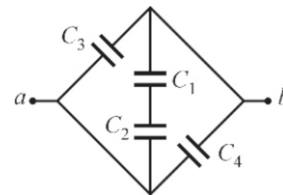
[A: $6/11 \mu\text{F}$.]



Problem 17.10

17.10 In the circuit shown in the figure we have $C_1 = 1$, $C_2 = 2$, $C_3 = 3 \mu\text{F}$ and $V = 12 \text{ V}$. (a) Calculate the equivalent capacitance. (b) Find the charge of the equivalent capacitor. (c) Find the charge on each capacitor.

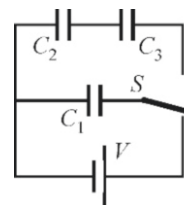
[A: (a) $1.5 \mu\text{F}$, (b) $18 \mu\text{C}$, (c) $q_1 = 6$, $q_2 = 12$, $q_3 = 18 \mu\text{C}$.]



Problem 17.11

17.11 In the circuit shown in the figure we have $C_1 = 1$, $C_2 = 2$, $C_3 = 3$ and $C_4 = 4 \mu\text{F}$. (a) Calculate the equivalent capacitance between the terminals ab . (b) The terminals ab are connected to a 12 V potential difference. How much charge gets accumulated in the equivalent capacitor? (c) How much charge gets accumulated on the capacitor C_4 ?

[A: (a) $23/3 \mu\text{F}$, (b) $92 \mu\text{C}$, (c) $48 \mu\text{C}$.]

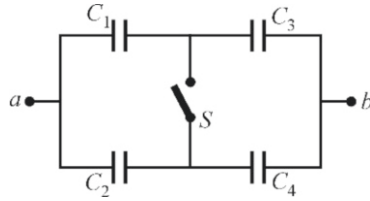


Problem 17.12

17.12 The capacitor shown in the figure above with capac-

itance $C_1=4\mu\text{C}$ is charged by closing the switch S in the direction of the battery $V=12\text{ V}$. (a) What is the charge of the capacitor C_1 ? (b) The switch S is then closed in the direction of the capacitors $C_2=3\mu\text{C}$ and $C_3=6\mu\text{C}$. What is the charge on each capacitor?

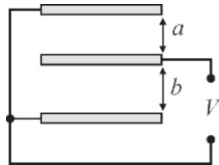
[A: (a) $q_1 = 48\mu\text{C}$, (b) $q_1 = 32$, $q_2 = q_3 = 16\mu\text{C}$.]



Problem 17.13

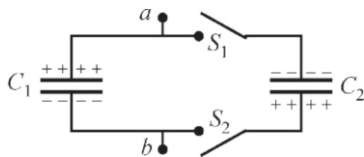
17.13 In the circuit shown in the figure above, we have $C_1=1$, $C_2=2$, $C_3=3$ and $C_4=4$. (a) In this circuit, the terminals ab are connected to a 12 V battery when the switch S is open. Calculate the equivalent capacitance and the total charge drawn from the battery. (b) The circuit is disconnected from the battery when charged and then the switch S is closed. Find the equivalent capacitance and the charge on each capacitor.

[A: (a) $25/12\mu\text{F}$ and $25\mu\text{C}$, (b) $2.1\mu\text{F}$ and $q_1 = 8.3$, $q_2 = 16.7$, $q_3 = 10.7$, $q_4 = 14.3\mu\text{C}$.]



Problem 17.14

17.14 Each of the 3 parallel plates shown in the figure above have surface area A and the distances in between them are a and b . These plates are connected to a potential difference V , as shown in the figure. Prove that the expression for equivalent capacitance is $C = \epsilon_0 A(1/a + 1/b)$. (Hint: The conductor plate in the center is the common plate of two capacitors. Determine whether these two capacitors are connected in parallel or series.)



Problem 17.15

17.15 After being charged under the same potential difference of 20 V , the capacitors with capacitance $C_1=1\mu\text{F}$ and $C_2=4\mu\text{F}$ are connected in reverse, as shown in the figure above, in other words, the negative charged plate of one is connected to the positive charged plate of the other. (a) What are the charges and total energies of the capacitors at the start? (b) What are the final charges and total energies of the

capacitors?

[A: (a) $q_1 = 20$, $q_2 = 80\mu\text{C}$, 1 mJ , (b) $q'_1 = 12$, $q'_2 = 48\mu\text{C}$, 0.36 mJ .]

17.3 Dielectrics

17.16 Paraffin is used to fill the volume of a parallel-plate capacitor whose plates have 5 cm^2 in surface area and 2 mm in distance between them. (a) What is the capacitance of the capacitor? (b) What is the maximum operating voltage? (Refer to the table on Page 293 for the dielectric constant and strength of paraffin.) [A: (a) 4.9 pF , (b) 20 kV .]

17.17 There is a distance of 3 mm between the plates of a parallel-plate capacitor. Its capacitance is $C_0 = 5\mu\text{F}$ when there is vacuum between the plates.

(a) The capacitor is connected to a battery with $V_0 = 24\text{ V}$. What will the charge Q_0 and the electric field E_0 between the plates be?

(b) The capacitor is disconnected as charged and a dielectric layer with dielectric constant $\kappa = 8$ is inserted between the plates. Find the capacitance C , potential V and electric field E of the capacitor.

(c) The same dielectric layer is inserted between the plates while the capacitor is connected to the battery. Find the charge Q , potential V and electric field E of the capacitor. [A: (a) $120\mu\text{C}$ and 8 kV/m , (b) $40\mu\text{F}$, 3 V and 1 kV/m , (c) $960\mu\text{C}$, V and E do not change.]



Problem 17.18

17.18 The parallel-plate capacitor shown in the figure above has capacitance C_0 when the space between the plates is empty. A dielectric with constant κ is inserted that fills one half of the vacuum between the plates. What will the new capacitance be? [A: $C = (1 + \kappa)C_0/2$.]

17.19 The 5 mm wide space between the plates of a parallel-plate capacitor is filled with air. (a) What is the maximum operating voltage? (b) A paper layer is inserted between the plates of the capacitor. What is the maximum operating voltage this time? (Dielectric strengths: Air: $E_{\text{max}} = 3 \times 10^6\text{ V/m}$, paper: $15 \times 10^6\text{ V/m}$) [A: (a) 15 kV , (b) 75 kV .]

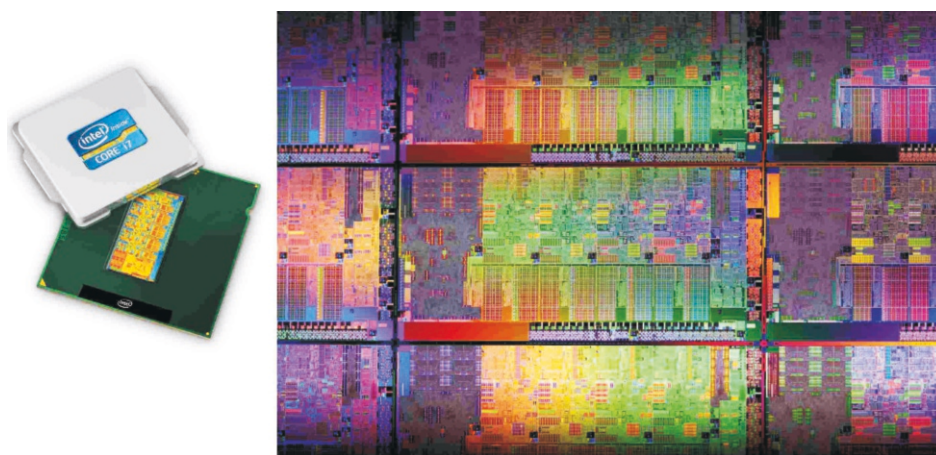
17.20 A parallel-plate capacitor is to be manufactured with a material with a dielectric constant of 12 and a dielectric strength of $5 \times 10^6\text{ V/m}$ such that its capacitance shall be 10 pF and it will be able to operate up to 20 kV of potential difference. What should the surface area and distance between the plates of this capacitor be?

[A: 3.8 cm^2 and 4 mm .]

?

18

CURRENT, RESISTANCE AND CIRCUITS



The Intel Core *i7* chip and the underlying Sandy Bridge architecture. Launched in 2011, this chip contains 2.6 billion transistor circuits.

No other type of power more efficient and practical than electric power has been discovered in modern technology and industry. Electricity can easily be transferred and used in all places and under all operating conditions. It can also be used in sending signals and in operating and controlling machinery.

The electric current lies at the basis of all of these activities. From this chapter onwards, we shall leave the subject of electrostatics and examine the currents produced by moving electric charges and their behavior in matter. We will learn of electrical effects that are much richer and more complex compared to static charges.

18.1 ELECTRIC CURRENT

The amount of charge flowing through a conductor in a given time is called the **current**. More explicitly, if a net charge dq flows through the cross-section of a conductor in a certain direction during a small time dt , then the current, indicated with the symbol I , is

$$I = \frac{dq}{dt} \quad (\text{Current}) \quad (18.1)$$

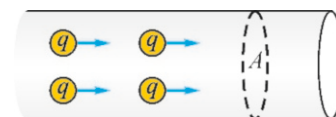


Figure 18.1: Charges flowing through a cross-section.

The unit of current is the **ampere** and is indicated with A:

$$1 \text{ A} = 1 \text{ C/s}$$

However, as the ampere is the basic unit in the SI system, it is not necessary to express ampere in terms of coulombs. On the contrary, the coulomb unit is ampere \times seconds.

Note the expression “net charge flowing through in a certain direction” here. The free electrons in a conductor are continually in motion. When observed in any cross-section, the average amount of charge flowing through in one direction is equal to the average amount of charge flowing in the other direction, and therefore the net charge flowing through is zero. A net charge flow is observed in a certain direction when this conductor is connected to a potential difference.

Direction of Current

Before the microscopic structure of matter was known, the direction of electric current was accepted as the direction of motion of positive charges. However, it was later understood that only the negative electrons were moving. Positively charged ions do not move; they only vibrate, as they are strongly bonded to each other and are very heavy.

Despite this, the direction of current was not changed. According to this assumption, *the direction of current is taken as the direction of motion of positive charges*. In actual fact, in a solid medium these are negatively charged electrons moving in the opposite direction.

This creates no problem macroscopically. Consider this: A $+q$ charge traveling to the right increases the charge on the right by $+q$. However, a $-q$ charge traveling to the left also increases the charge on the right by $+q$. They both lead to the same result. In this chapter, we will examine many cases as if $+q$ charges are moving, but the conclusions that we shall reach will not change.

Drift Speed

We can calculate the current in a conducting wire in terms of the speed of the moving charges inside of the wire. A net electric field E will form inside of a wire when a potential difference V is applied between the two ends of the wire. The charges will start to accelerate as a force $F=qE$ will be exerted upon a charge q in the electric field.

(Let us make one correction here: We had previously stated that $E=0$ was always true in a conductor in electrostatics. However, the conditions of electrostatics are not valid any more, as we are examining moving charges and the electric field may be different from zero.)

Charges under the force $F = qE$ would be expected to accelerate according to Newton’s law. However, this is not possible in a conducting medium (Figure 18.3). This is because moving q charges collide with the ions in the medium and lose energy or change direction. They then accelerate again and repeatedly collide and slow down. You may consider this like an automobile trying to drive along a road with successive red traffic lights. The automobile accelerates, but halts at a red traffic light and waits; it then starts to accelerate again after a green traffic light and is then forced to stop again at the next red traffic light.

Electrons thus travel at an average speed inside of a conductor. This is called the **drift speed** and is indicated with v_d . Now, let us show the relation between drift speed and current.

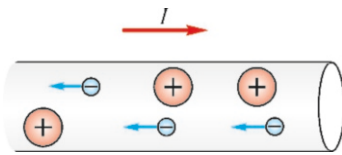


Figure 18.2: Current is formed by $(-)$ charged electrons, and $(+)$ charged ions do not move.

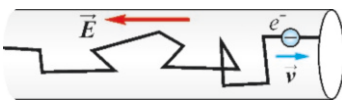


Figure 18.3: Electrons lose energy by colliding with ions when accelerating in a conductor.

Let Δq be the amount of charge flowing through the cross-section A of a conductor during the time interval Δt (Figure 18.4). The individual charges e flowing through each with drift speed v_d during this Δt time interval will be those charges inside of a cylinder with base area A and length $L=v_d \Delta t$. Let the number of free charges inside of a unit volume of the conductor medium be $n=N/V$. Accordingly, if the number of e charges inside of the mentioned cylinder is ΔN , then

$$\Delta N = n \times \text{volume} = n A v_d \Delta t$$

and the amount of charge flowing through the cross-section is

$$\Delta q = e \Delta N = q n A v_d \Delta t$$

If we now use the definition of current, we find the following result:

$$I = \frac{\Delta q}{\Delta t} = e n A v_d \quad (18.2)$$

If current I is known, we can write this expression for speed v_d as follows:

$$v_d = \frac{I}{e n A} \quad (\text{drift speed}) \quad (18.3)$$

Let us see the order of magnitude of drift speed using a simple calculation: A current $I = 20 \text{ A}$ is flowing through a copper wire with cross-section 1 mm^2 . The copper contains approximately 10^{29} free electrons per cubic meters. If we take $q=e=1.6 \times 10^{-19} \text{ C}$ as the charge, we find that

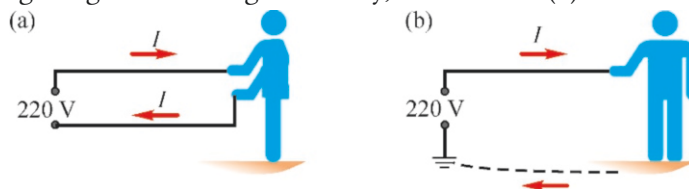
$$v_d = \frac{I}{e n A} = \frac{20}{1.6 \times 10^{-19} \times 10^{29} \times 10^{-6}} \approx 0.001 \text{ m/s}$$

This result may seem surprising. Electrons are moving as slow as 1 mm per second inside the conductor. However, we see a lamp turn on immediately after we switch it on. The reason for this is that there are free electrons everywhere in the conductor. The neighboring charges of the lamp start to move as soon as you switch it on and do not have to travel a long distance.

Biological Effects of Electric Current

The human body has limited resistance to electric voltage and current. The effect called **electric shock** occurs at currents and potential differences above certain limits and may lead to serious injuries that may result in death. Let us emphasize the important points to keep in mind for protection from electricity.

- Electric current can flow through the body in two ways: It may flow through by completing the circuit over the body, as shown in Figure 18.5a. Or by reaching the ground through the body, as shown in (b).



- The important factor in electric shock is not the voltage but current. You may come into contact with a very high voltage, but it will not harm the body if a

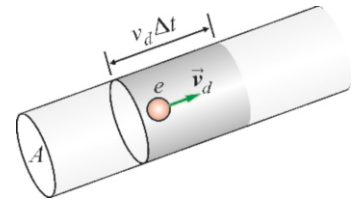


Figure 18.4: Charges flowing through the cross-section A during time interval Δt , are located inside of a cylinder with length $v_d \Delta t$.

Figure 18.5: Current can flow through the body in two ways: (a) By completing the circuit, (b) By reaching the ground through the body.

current does not flow as a result. However, if the high voltage completes the circuit at another point of the body (for example, if two wires are held with two hands or a conducting contact is made with the ground), then the circuit is completed and current starts to flow. (This is why birds are able to land on electric cables without getting electrocuted.)

- Direct currents higher than 0.3 – 0.5 A and alternating currents higher than 60 milliamperes are harmful.
- Electric current can harm the body in two ways:

Burns on the skin and internal organs. The heat energy released and the ionization that occurs along the current path may damage tissues and lead to death.

Disruption of the heart and the nervous system. The nervous system of our body consists of nerve cells called axons that operate with voltages as small as 0.1 V. An external electric current disrupts this system and leads to paralysis in muscles. The heart beat becomes irregular, especially in the event of a paralysis in the heart ventricles. This is called **fibrillation**. The heart may stop as a result of being unable to pump sufficient blood.

80 % of the deaths caused by electric shock are due to burns and the rest due to heart and nervous system paralysis.



Figure 18.6: The defibrillator device used in cardiac arrest.

- The current flowing through the body depends on the electric resistance of the body (see Ohm's law below). This resistance may vary depending on skin temperature or whether it is moist or dry. The resistance of a dry skin is around 5 000 – 10 000 Ω but it may drop down to around 1 000 Ω for moist skin.
- Considering these current and resistance limits, even voltages as low as 10 V may be harmful.
- **Protection.** The first rule to be observed in the event of an electric shock is to cut the contact with the voltage source. You must make sure to isolate yourself while doing this.

Using three-prong plugs with ground connection will provide the necessary protection for electric appliances with current leaks. One of the three wires in these plugs carries the voltage and the other two are neutral. The third wire is connected both to the ground and to the conductor frame of the electrical appliance. Any electrical leak thus goes to the ground through this third path.

Another protection method is to prevent currents flowing to the ground through the body by wearing shoes with insulating soles (rubber, cork).

18.2 OHM'S LAW AND RESISTANCE

The current flowing through a unit cross-section of a conductor is called the **current density** and is indicated with J . If a current I flows through a cross-sectional area A , the current density will be

$$J = \frac{I}{A} \quad (\text{Current density}) \quad (18.4)$$

Its unit is ampere/meter² = A/m².

There is a linear relation between the current density J and the electric field E in the medium in metals and many other conductors. Discovered for the first time by the German scientist Ohm, this relation is known as **Ohm's law** and is expressed as follows:

$$E = \rho J \quad (\text{microscopic Ohm's law}) \quad (18.5)$$

The ratio ρ is called the **resistivity**.

Resistivity depends on the type of material and temperature. The following table lists the resistivities of some materials:

Resistivities of some materials at room temperature (20 °C)		
Material	Resistivity ρ ($\Omega\cdot\text{m}$)	Temperature coefficient α ($1/^\circ\text{C}$)
Silver	1.5×10^{-8}	0.004
Copper	1.7×10^{-8}	0.004
Gold	2.4×10^{-8}	0.003
Aluminum	2.8×10^{-8}	0.004
Tungsten	5.3×10^{-8}	0.005
Carbon	3.5×10^{-5}	-0.001

Eq. (18.5) is an expression of Ohm's law at the microscopic level. In order to turn this into a form that is usable in practical applications, we use the potential difference V applied on both ends of the conductor (Figure 18.7). The electric field inside of a conducting wire with cross-section A and length L is

$$E = \frac{V}{L}$$

and, using current density $J = I/A$, the law (18.5) becomes as follows:

$$\frac{V}{L} = \rho \frac{I}{A} \quad \rightarrow \quad V = \left(\rho \frac{L}{A} \right) I$$

From here, we obtain the macroscopic expression of Ohm's law:

$$V = RI \quad (\text{macroscopic Ohm's law}) \quad (18.6)$$



The coefficient R in this law is called the **resistance**:

$$R = \rho \frac{L}{A} \quad (\text{resistance}) \quad (18.7)$$

The resistance of a conductor depends on its type, dimensions and temperature. The resistance of a conductor increases with its length and decreases with its cross-sectional area.

The unit of resistance is the **ohm** and is indicated with the Ω (omega) symbol:

$$1 \Omega = 1 \frac{\text{V}}{\text{A}}$$

When the resistance is used as a circuit element, it is called a **resistor** and is indicated with the symbol  or .

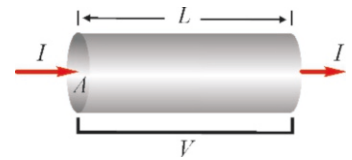


Figure 18.7: Ohm's law.

Variation of Resistivity with Temperature

The resistivity of a conductor increases with temperature and makes current flow more difficult. Microscopically, the vibration of positive ions usually increases with temperature; the ions then collide more frequently with the free electrons in the medium, and thus reduce their drift speed. This, in turn, causes an increase in the resistance of the medium.

Experimental observations show that the variation of resistivity with temperature can be expressed as follows:

$$\rho = \rho_0 [1 + \alpha (T - T_0)] \quad (18.8)$$

The value ρ_0 here is the resistivity at a reference temperature T_0 (usually 0°C or 20°C). The coefficient α is called the **temperature coefficient** of resistivity. The values of α for some materials are given in the table on page 305.

The relation of resistance R to temperature is similar:

$$R = R_0 [1 + \alpha (T - T_0)] \quad (18.9)$$

Example 18.1

A $5\ \Omega$ resistor is to be made using a copper wire with a $0.1\ \text{mm}^2$ cross-section.

- What should the length of the wire be?
- How many amperes of current will flow when this resistor is connected to a potential difference of $116\ \text{V}$?
- What will the magnitude of the electric field inside of the wire be?

Answer

(a) We use Eq. (18.7), which defines the resistance:

$$R = \rho \frac{L}{A} \rightarrow L = \frac{RA}{\rho}$$

We find the resistivity of copper from the table on Page 305 and calculate the required length:

$$L = \frac{5 \times 0.1 \times 10^{-6}}{1.7 \times 10^{-8}} = 29\ \text{m}$$

(b) Eq. (18.6) is solved for the current:

$$I = \frac{V}{R} = \frac{116}{5} = 23\ \text{A}$$

(c) Assuming that there is uniform electric field inside of the wire, we write the potential difference in terms of electric field:

$$V = EL \rightarrow E = \frac{V}{L} = \frac{116}{29} = 4\ \text{V/m}$$

Example 18.2

An electric field with a magnitude of $100\ \text{V/m}$ is formed when a $3\ \text{A}$ current flows through a wire with cross-section $2\ \text{mm}^2$. What is the resistivity of the wire?

Answer

We write Eq. (18.5), the microscopic expression of Ohm's law:

$$E = \rho J$$

$J = I/A$ is the current flowing through the unit cross-section. From here we calculate the resistivity:

$$E = \rho \frac{I}{A} \rightarrow \rho = \frac{EA}{I}$$

We substitute the data and calculate the resistivity:

$$\rho = 100 \times 2 \times 10^{-6} / 3 = 6.7 \times 10^{-5}\ \Omega\cdot\text{m}$$

Example 18.3

A tungsten wire lamp has $10\ \Omega$ in resistance at 20°C .

- What will its resistance be at 600°C ?
- The lamp has $80\ \Omega$ in resistance when it is glowing. Determine the temperature of the lamp.

Answer

(a) Eq. (18.9) is used:

$$R = R_0 [1 + \alpha (T - T_0)]$$

We find the temperature coefficient of tungsten from the table on Page 305:

$$R = 10 [1 + 0.005 \times (600 - 20)] = R = 39\ \Omega$$

(b) The same formula is used to find temperature T :

$$80 = 10 [1 + 0.005 \times (T - 20)]$$

$$T = 1420^\circ\text{C}$$

18.3 DIRECT CURRENT CIRCUITS

Electromotive Force (EMF)

There must be a potential difference between the two points of a circuit for a current to flow through it. A potential difference obtained from other sources (chemical, mechanical, magnetic, optical, etc.) is called the **electromotive force**. It is called **emf** for short and is indicated with the \mathcal{E} symbol.

Emf is not actually a force, but a potential difference, however this name has stuck for historical reasons. The emf potential difference is actually indicated with \mathcal{E} rather than V , to indicate that it is a source.



Figure 18.8: Various sources of emf: bicycle dynamos, dry batteries, solar batteries.

As a circuit component, emf is indicated with the symbol $\begin{array}{c} + \\ | \\ - \end{array}$.

The most common source of emf is the battery that converts chemical energy into electrical energy. For example, in a galvanic battery, a potential difference is produced between two different metal electrodes (zinc and copper) immersed in an electrolyte liquid. Lead and lead-oxide electrodes are used in car batteries.

In all types of battery, one of the metal electrodes will have higher potential (Figure 18.9). This positive terminal is called the *cathode* and the negative terminal the *anode*. When this battery is connected to a circuit, the current flows from the cathode to the anode, in other words, from the positive terminal to the negative terminal. The current inside of the battery flows in the reverse direction to complete the cycle. (As we mentioned earlier, electrons actually flow in the opposite direction.)

Likewise, solar batteries convert solar energy directly into a potential difference, just as a generator converts mechanical energy into electrical energy.

The emf value, in other words, the potential difference \mathcal{E} of the battery, is constant and does not vary, even if you increase the dimensions of the battery. The emf value can only be increased by connecting several batteries in series. For example, emf is 1.1 V in a galvanic battery, 2.1 V in a lead battery, and 12 V in an automobile battery with 6 cells.

Internal Resistance and Terminal Voltage

When an emf source is connected to a circuit, while current flows from the (+) terminal to the (-) terminal on the outside, there must be a current from the (-) terminal to the (+) terminal inside of the electrolyte liquid. An internal resistance appears, as the mobility of the electrons inside of the liquid will be limited. Internal resistance is indicated with r .

Now, let us consider a very simple circuit by connecting the emf source \mathcal{E} to a resistor R . In the circuit, it is necessary to add an internal resistance r for the battery in addition to the resistor R . A current I starting from the (+) terminal (point b) with high potential will circulate throughout the whole circuit (Figure 18.10).

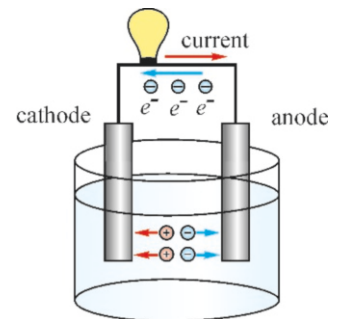
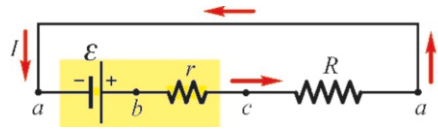


Figure 18.9: In a battery, current flows from cathode (+) to anode (-), but electrons actually flow in the reverse direction.

Figure 18.10: A simple circuit consisting of a resistance R connected to an emf source \mathcal{E} with internal resistance r .



To calculate this circuit, let us start from the (-) terminal of the battery indicated with a , go along the circuit in the direction of the current and return to point a . The potential should return to the value V_a again at the end of this path.

Let the potential at point a be V_a . Since we have to go through the (+) terminal of the emf source to reach point b , the potential will increase and the potential of point b will be higher by $+\mathcal{E}$:

$$V_b = V_a + \mathcal{E}$$

When we reach the internal resistance r , according to Ohm's law, the potential will drop by rI when going along the resistor in the direction of the current. Therefore, the potential of point c is

$$V_c = V_b - rI = V_a + \mathcal{E} - rI$$

Likewise, the potential will drop by RI when going from point c through the resistor R in the direction of the current. Therefore, the potential of point a on the right will be

$$V_a = V_c - RI = V_a + \mathcal{E} - rI - RI$$

We thus return to point a . The graph in Figure 18.11 shows the potential value at each point along the circuit.

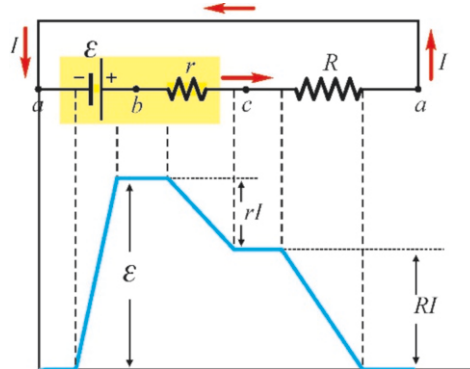


Figure 18.11: The potential differences on a circuit as current flows through the emf and the resistors.

By simplifying, we get

$$\mathcal{E} - rI - RI = 0$$

We thus find the current I in a circuit with resistor R by an emf source \mathcal{E} with internal resistance r :

$$I = \frac{\mathcal{E}}{R + r}$$

As this circuit operates with the current I flowing, the resistor R will be subject to the potential difference between the ac terminals of the battery. And this potential difference will be,

$$V_{ac} = \mathcal{E} - rI \quad (\text{Terminal voltage}) \quad (18.10)$$

This potential difference V_{ac} is called the **terminal voltage** of the battery. Terminal voltage will always be less than the value of \mathcal{E} , because the potential drop across the internal resistor r should also be taken into account.

Power and Energy in Electric Circuits

Electric circuit components (resistors, capacitors, motors...) convert electrical energy into other types of energy, such as heat, light and motion, when a current flows through them. In the most general case, if a current I flows through a circuit component connected between any two points, there will be a potential difference V between its terminals. The potential difference shows the change in the potential energy, i.e., the work performed per unit charge.

Let us suppose that charge dq flows during time interval dt through a circuit component with potential difference V between its terminals. The change in the potential energy of this charge will be $dU = dqV$ according to the definition of potential energy. In other words, the potential energy lost when the charge dq goes from a high potential to a low potential point on this circuit element, performs $dW = dU = dqV$ amount of work on this circuit component:

$$dW = dU = V dq$$

The work performed per unit time gives us the power:

$$P = \frac{dW}{dt} = \frac{V dq}{dt} = V \frac{dq}{dt}$$

As the amount of charge flowing during the unit time is $I = dq/dt$, the expression for power will be as follows:

$$P = VI \quad (\text{Power}) \quad (18.11)$$

This formula is the most general expression of power, and it gives us the power consumed when the potential difference is V and the current flowing through is I on any component of the circuit, including resistors, capacitors, emf, motors, etc.

To find the energy spent during a certain time interval, we write the definition of power for dW :

$$dW = P dt$$

As a particular case, let us find the expression for the power consumed in a resistor: As $V = RI$ according to Ohm's law, we get

$$P = VI = RI^2 \quad (\text{Power converted into heat in a resistor}) \quad (18.12)$$

or, alternately, the amount of energy consumed during time interval dt will be

$$dW = P dt = RI^2 dt \quad (18.13)$$

Note that this energy converted into heat is proportional to the square of the current.



Figure 18.12: Electric energy is converted to heat in an electric heater.

Example 18.4

A 15 V battery is connected to a $7\ \Omega$ resistor. The internal resistance of the battery is $0.5\ \Omega$.

- Find the current flowing through the circuit.
- Find the terminal voltage of the battery.

- Calculate the power consumed in the resistors and the power provided by the battery to the circuit.

Answer

- We add the voltage changes over each circuit component as we make a complete clockwise cycle starting from the

negative terminal of the battery:

$$+\mathcal{E} - rI - RI = 0$$

$$I = \frac{\mathcal{E}}{R+r} = \frac{15}{7+0.5} = 2 \text{ A}$$

(b) The terminal voltage of the battery is defined in Eq. (18.10):

$$V_{ab} = \mathcal{E} - rI = 15 - 0.5 \times 2 = 14 \text{ V}$$

(c) We use the formula RI^2 to calculate the power consumed in the resistors:

$$P_R = RI^2 = 7 \times 2^2 = 28 \text{ W} \quad \text{and} \quad P_r = rI^2 = 0.5 \times 2^2 = 2 \text{ W}$$

The power provided by the battery is calculated using the formula (18.11):

$$P = VI = \mathcal{E}I = 15 \times 2 = 30 \text{ W}$$

Example 18.5

An electric kettle made of tungsten has 50Ω in resistance at room temperature (20°C) and operates under 240 V .

- (a) How much power does the kettle consume when it starts to heat water at 20°C ?
- (b) How much power does it consume when the water starts to boil?

Answer

(a) Using the power formula $P=VI$ and Ohm's law $V=RI$, the power of the heater can be written as follows:

$$P = RI^2 = V^2/R$$

We calculate the power by using the resistance at room temperature:

$$P = 240^2/50 = 1200 \text{ W} = 1.2 \text{ kW}$$

(b) As the boiling point of water is 100°C , we have to calculate the resistance at that temperature:

$$R = R_0 [1 + \alpha (T - T_0)]$$

We take the temperature coefficient α of tungsten from the table on page 305:

$$R = 50 [1 + 0.005 \times (100 - 20)] = 70 \Omega$$

We calculate the power consumed at this temperature:

$$P = \frac{V^2}{R} = \frac{240^2}{70} = 820 \text{ W} = 0.82 \text{ kW}$$

18.4 COMBINATION OF RESISTORS

A single resistor that performs the function of multiple resistors in a circuit is called an **equivalent resistor**. Very complex connections usually require advance circuit analysis. We shall only discuss circuits consisting of resistances connected in series and parallel.

Two simple principles are sufficient in circuit solutions:

- The same current flows through resistors on the same branch.
- The same voltage applies to resistors connected to the same two terminals.

Resistors in Series

Suppose that resistors R_1 and R_2 are connected in series under a potential difference V , as shown in Figure 18.13. The same current I will flow through these resistors. Therefore, we write the potential differences and Ohm's law:

$$V_{ac} = V_{ab} + V_{bc}$$

$$V = R_1I + R_2I$$

The equivalent resistor R_{eq} should draw the same current I under the same potential difference V :

$$V = R_{\text{eq}} I$$

Comparing the last two expressions, we find the series connected equivalent resistor:

$$R_{\text{eq}} = R_1 + R_2$$

This result is also valid for more than two resistors:

$$R_{\text{eq}} = R_1 + R_2 + \cdots + R_N \quad (\text{Resistors in series}) \quad (18.14)$$

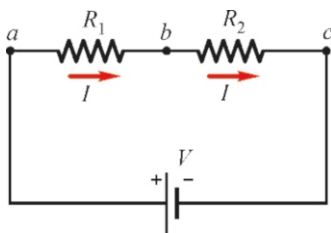


Figure 18.13: Resistors in series.

Resistors in Parallel

Two resistors with resistance R_1 and R_2 are connected to the same battery with potential difference V , as shown in Figure 18.14. In this case, each resistor will be subject to the same V potential difference, but the current flowing through each will be different. If we use I_1 and I_2 to indicate these currents, we get

$$I_1 = \frac{V}{R_1} \quad I_2 = \frac{V}{R_2}$$

The total current drawn from the battery will thus be $I = I_1 + I_2$.

The equivalent resistor placed between the same terminals should draw the same total current:

$$I = I_1 + I_2$$

$$\frac{V}{R_{\text{eq}}} = \frac{V}{R_1} + \frac{V}{R_2}$$

After simplifying, we get the equivalent resistance of resistors in parallel:

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2}$$

This result is valid for more than two resistors in parallel:

$$\boxed{\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_N} \quad (\text{Resistors in parallel})} \quad (18.15)$$

The equivalent resistance of a parallel connection is less than each resistance.

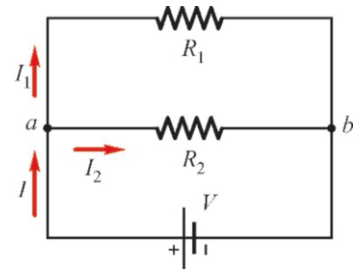
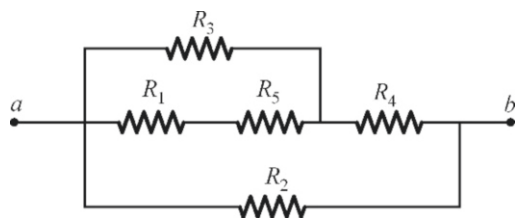


Figure 18.14: Resistors in parallel.

Example 18.6



In the figure, we have $R_1=1$, $R_2=2$, $R_3=3$, $R_4=4$ and $R_5=5 \Omega$.

- What is the equivalent resistance between ab ?
- How much current will flow through resistor R_4 when the terminals ab are connected to a 12 V source?

Answer

(a) The resistance of wires is taken to be zero in circuit problems. In other words, we can extend, shorten or even merge these wires at one point. It thus becomes easy to see series or parallel connections.

If we examine the circuit in the figure starting from the inside, we can observe that R_1 and R_5 are connected in series. Let us use R' to show their equivalent resistor:

$$R' = R_1 + R_5 = 1 + 5 = 6 \Omega$$

If we replace these two resistors with a single resistor R' , it will be observed to be in parallel connection with R_3 . Let us

calculate the equivalent resistance of these two:

$$R'' = \frac{R_3 R'}{R_3 + R'} = \frac{3 \times 6}{3 + 6} = 2 \Omega$$

This resistor R'' will be in series connection with R_4 :

$$R''' = R_4 + R'' = 4 + 2 = 6 \Omega$$

Finally, resistor R''' will be in parallel connection with R_2 :

$$R_{\text{eq}} = \frac{R_2 R'''}{R_2 + R'''} = \frac{2 \times 6}{2 + 6} = 1.5 \Omega$$

(b) We first calculate the current drawn by the equivalent resistor when the terminals ab are connected to the battery:

$$I = \frac{V_{ab}}{R_{\text{eq}}} = \frac{12}{1.5} = 8 \text{ A}$$

Let us assume that this current I enters through the terminal b . The current will separate here into two branches, which we will indicate with I_2 and I_4 :

$$I = I_2 + I_4 = 8 \text{ A}$$

We can immediately calculate the current I_2 using Ohm's law, because the resistor R_2 is also subject to the potential difference $V_{ab} = 12 \text{ V}$:

$$I_2 = \frac{V_{ab}}{R_2} = \frac{12}{2} = 6 \text{ A}$$

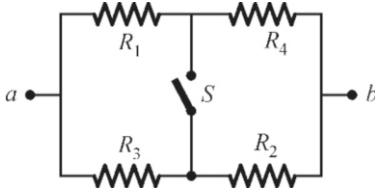
From here we calculate the current I_4 :

$$I_4 = I - I_2 = 8 - 6 = 2 \text{ A}$$

Example 18.7

In the circuit shown in the figure, $R_1=1$, $R_2=2$, $R_3=3$ and $R_4=4$. The terminals ab are connected to a 12 V source.

- (a) Calculate the equivalent resistor and the power consumed in the circuit when the switch S is open.
 (b) Calculate the equivalent resistor and the power consumed in the circuit when the switch S is closed and compare with item (a).

**Answer**

(a) The pairs (R_1, R_4) and (R_2, R_3) are in series connection with each other when the switch is open. Let us use R' and R'' to indicate their equivalent resistances:

$$R' = R_1 + R_4 = 1 + 4 = 5 \Omega$$

$$R'' = R_2 + R_3 = 2 + 3 = 5 \Omega$$

Then, these resistors R', R'' will be in parallel connection:

$$R_{eq} = \frac{R'R''}{R' + R''} = \frac{5 \times 5}{5 + 5} = 2.5 \Omega$$

We calculate the power consumed in the circuit from the potential difference:

$$P = \frac{V_{ab}^2}{R_{eq}} = \frac{12^2}{2.5} = 58 \text{ W.}$$

(b) The pairs (R_1, R_3) and (R_2, R_4) will be in parallel when the switch is closed. Let us use R' and R'' to indicate their equivalent resistances:

$$R' = \frac{R_1 R_3}{R_1 + R_3} = \frac{1 \times 3}{1 + 3} = \frac{3}{4} \Omega$$

$$R'' = \frac{R_2 R_4}{R_2 + R_4} = \frac{2 \times 4}{2 + 4} = \frac{4}{3} \Omega$$

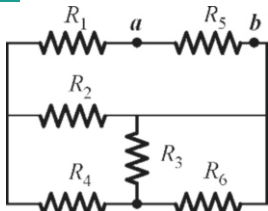
Then, these resistors R', R'' will be in series:

$$R_{eq} = R' + R'' = \frac{3}{4} + \frac{4}{3} = \frac{25}{12} = 2.1 \Omega$$

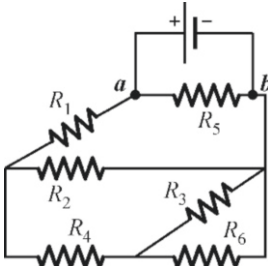
We calculate the power consumed in the circuit from the potential difference:

$$P = \frac{V_{ab}^2}{R_{eq}} = \frac{12^2}{2.1} = 69 \text{ W.}$$

The power consumed in the circuit is higher when the switch is closed.

Example 18.8

In the circuit shown in the figure, $R_1=1$, $R_2=2$, ... and $R_6=6 \Omega$. Find the equivalent resistance between points ab .



Answer In circuit problems, it is convenient to consider that

a potential difference is applied across the points between which equivalent resistance is requested. The circuit simplifies as shown above if we change the lengths of the connecting wires and merge them at one point.

It is easily seen from the figure that the resistors R_3 and R_6 are connected in parallel. We calculate the equivalent resistance of these two as follows:

$$R' = \frac{R_3 R_6}{R_1 + R_3} = \frac{3 \times 6}{3 + 6} = 2 \Omega$$

This resistor R' will be in series with R_4 :

$$R'' = R_4 + R' = 4 + 2 = 6 \Omega$$

This resistor R'' will be in parallel with R_2 :

$$R''' = \frac{R_2 R''}{R_2 + R''} = \frac{2 \times 6}{2 + 6} = 1.5 \Omega$$

This resistor R''' will be in series with R_1 :

$$R^{iv} = R_1 + R''' = 1 + 1.5 = 2.5 \Omega$$

Finally, this resistor R^{iv} will be in parallel with R_5 :

$$R_{eq} = \frac{R_5 R^{iv}}{R_5 + R^{iv}} = \frac{5 \times 2.5}{5 + 2.5} = \frac{5}{3} = 1.7 \Omega$$

18.5 ELECTRICAL MEASURING INSTRUMENTS

It is important to know how to connect electrical measuring instruments that measure resistance, current or emf. An instrument that measures current is called an **ammeter**, an instrument that measures potential difference is called a **voltmeter** and an instrument that measures emf is called a **potentiometer**.

The common aspect of all these measuring instruments is that they use the same instrument, called the **galvanometer** in different ways. So, let us first examine the galvanometer.

Galvanometer

When a current flows through a wire placed between the poles of a magnet, a magnetic force acts on the wire (Figure 18.15). As we shall discuss in Chapter 20, the magnetic force on the wire is proportional to the magnitude of the current. Also, a torque is exerted if the wire is wound like a coil and the coil gets deflected in the magnetic field. Likewise, the amount of deflection of the wire is proportional to the current flowing through it. Thus, the current can be calculated by measuring the deflection angle on a scale.

However, the wire inside of the galvanometer also has a resistance. This resistance is part of the circuit, and will thus change the current flowing through the circuit. *The internal resistance of the galvanometer should be very small so that this effect can be small.*

The galvanometer is the basis of all kinds of electrical measuring instruments. Other quantities can be measured by connecting resistors in series or in parallel to the galvanometer.

Ammeter

The ammeter used to measure current is actually a galvanometer with a very small resistor. The ammeter is connected in series by inserting it into the branch in which the current is to be measured. It is thus ensured that the whole current to be measured flows through the ammeter. However, the resistance of the ammeter should be very small, almost zero, such that the current flowing through this branch is not affected. The current flowing through the ammeter in the circuit shown in Figure 18.16 will be

$$I = \frac{\mathcal{E}}{R + R_A}$$

The change in the current I can be neglected if the resistance of the ammeter is $R_A \ll R$.

Voltmeter

The voltmeter used to measure the potential difference between two points is actually a galvanometer with very large resistance. The voltmeter is connected in parallel to points a, b of the circuit, and is thus subject to the potential difference to be measured (Figure 18.17).

This time, some of the current will flow through the voltmeter and the current flowing through the resistor R will decrease. We calculate the potential difference on resistor R :

$$V = \frac{RI}{1 + R/R_V}$$

Therefore, the resistance R_V of the voltmeter should be very large to ensure that the potential difference does not deviate too much from its correct value RI .

Potentiometer

A potentiometer is an instrument used to measure the emf voltage of a battery. As we have seen previously in Eq. (18.10), when a battery connected to the circuit starts to produce a current, the terminal voltage will be $V_{ab} = \mathcal{E} - rI$ and will be less than \mathcal{E} . Since we cannot eliminate the internal resistance r of the battery, can we take a measurement at zero current?

We can do this with the potentiometer assembly shown in Figure 18.18. The figure shows a battery with a known value of \mathcal{E} and a resistor R . The emf source

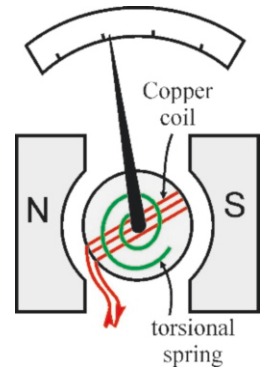


Figure 18.15: Galvanometer.

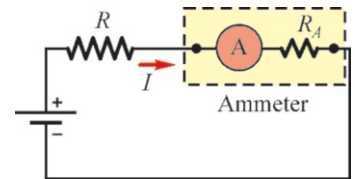


Figure 18.16: An ammeter is connected in series.

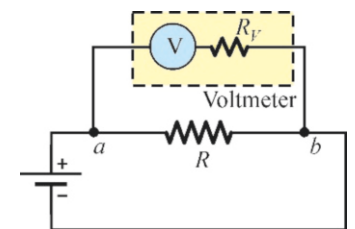


Figure 18.17: A voltmeter is connected in parallel.

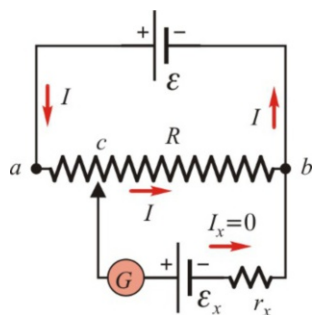


Figure 18.18: Potentiometer.



Figure 18.19: Multimeter.

whose \mathcal{E}_x value is to be measured is connected between point b and a variable point c on the resistor R .

The point c is varied such that a point is found where the current I_x flowing through \mathcal{E}_x becomes zero. This is possible when \mathcal{E}_x and \mathcal{E} are connected in reverse and $\mathcal{E} > \mathcal{E}_x$.

We calculate the same potential difference V_{cb} at two branches at this zero condition as follows:

$$V_{cb} = R_{cb} I \quad (R_{bc} \text{ part of resistor } R)$$

$$V_{cb} = \mathcal{E}_x \quad (\text{because } I_x = 0)$$

Therefore, the desired \mathcal{E}_x is equal to the product $R_{bc} I$. \mathcal{E}_x can be calculated by measuring the current flowing through the circuit and the resistance R_{bc} .

Today, current, voltage and resistance can all be measured using instruments called the **multimeter** or the **avometer**, which have all these properties in a digital medium. However, their operating principle is the same.

Multiple-choice Questions

- What will the current be if a charge of 6 C flows through the cross-section of a wire in 2 seconds?
 - 6 A
 - 8 A
 - 3 A
 - 1/3 A
- Which of the following are correct for electric current?
 - Electrons perform accelerated motion in conductors.
 - Electrons collide with ions in the conductor and slow down.
 - Electrons travel at a constant average speed due to collisions.
 - I & II
 - II & III
 - I & III
 - I, II & III
- Which of the following is correct for the current in conductors?
 - Positive ions move in the direction of the current.
 - Positive ions move in the opposite direction of the current.
 - Electrons move in the direction of the current.
 - Electrons move in the opposite direction of the current.
- Which of the following are correct for electromotive force (emf)?
 - It converts another type of power into electrical power.
 - It is an electrical force.
 - It provides a constant current to the circuit.
 - It provides a constant potential difference to the circuit.
- Which of the following are correct for the biological effects of electricity?
 - It is dangerous to complete a circuit through the body.
 - It is dangerous for a current to reach the ground through the body.
 - There will be no harm if the voltage is high but no current passes through.
 - Voltages higher than 10 V can be dangerous.
 - I & II
 - II & III
 - I & IV
 - All
- Which of the following are correct for the resistance of a wire?
 - Resistance increases with temperature.
 - Resistance decreases with temperature.
 - Resistance increases with the cross-section area.
 - Resistance decreases with the cross-section area.
 - I & III
 - I & IV
 - II & III
 - II & IV
- Both the length and the cross-section area of a wire are doubled. By what factor will its resistance increase?
 - No change
 - 2
 - 4
 - 8
- Which is the expression for Ohm's law?
 - $V = RI$
 - $I = VR$
 - $V = R/I$
 - $I = R/V$

9. Which of the following are correct for the resistance of a conductor?
 I. It is proportional to the resistivity.
 II. It is proportional to the length.
 III. It is proportional to the cross-section area.
 IV. It is inversely proportional to the cross-section area.
 (a) I & II (b) I, II & IV (c) I & III (d) II & IV
10. Which of the following will be correct when a 12 V emf source with an internal resistance of 1 Ω is connected to a circuit?
 I. It will provide a constant 12 V potential difference to the circuit.
 II. It will provide a potential difference less than 12 V to the circuit.
 III. It will provide a constant current to the circuit.
 (a) I & II (b) II & III (c) I & III (d) All
11. When you go along the current through a resistor R in which a current I is flowing, which of following happens?
 (a) The potential increases by RI .
 (b) The potential decreases by RI .
 (c) The potential remains constant.
 (d) The potential increases by RI^2 .
12. Which two of the following principles are correct for direct current circuits?
 I. The same current flows through resistors on the same branch.
 II. The potential difference is the same in resistors on the same branch.
 III. The same current flows through resistors connected between the same two points.
 IV. The potential difference in resistors connected between the same two points is the same.
 (a) I & II (b) II & III (c) I & III (d) I & IV
13. A 12 V emf source delivers 3 A when connected to a circuit. What is the power provided by the source?
 (a) 4 W (b) 15 W (c) 36 W (d) 108 W

14. How much energy does a light bulb with 5 W of power consume in 1 minute?
 (a) 5 J (b) 10 J (c) 25 J (d) 300 J
15. How much power is consumed in a 3 Ω resistor through which 2 A in current flows?
 (a) 5 W (b) 6 W (c) 12 W (d) 18 W
16. What is the equivalent resistance of the resistors with resistances of 3 and 6 Ω connected in series?
 (a) 2 Ω (b) 9 Ω (c) 18 Ω (d) 24 Ω
17. What is the equivalent resistance of the resistors with resistances of 3 and 6 Ω connected in parallel?
 (a) 2 Ω (b) 9 Ω (c) 18 Ω (d) 24 Ω



18. What is the equivalent resistance between ab in the figure?
 (a) 2 Ω (b) 3 Ω (c) 6 Ω (d) 9 Ω
19. Which of the following are correct for an ammeter?
 I. It is connected to the circuit in series.
 II. It is connected to the circuit in parallel.
 III. It has very low internal resistance.
 IV. It has very high internal resistance.
 (a) I & II (b) I & III (c) II & III (d) II & IV
20. Which of the following are correct for a voltmeter?
 I. It is connected to the circuit in series.
 II. It is connected to the circuit in parallel.
 III. It has very low internal resistance.
 IV. It has very high internal resistance.
 (a) I & II (b) I & III (c) II & III (d) II & IV

Problems

18.2 Ohm's Law and Resistance

(Use the table on Page 305 for the resistivities and temperature coefficients that may be required in the problems.)

18.1 A $3\ \Omega$ resistor is to be made using an aluminum wire with a cross-section of $0.2\ \text{mm}^2$. (a) What should the length of the wire be? (b) How many amperes of current will flow when this resistor is connected to a potential difference of $60\ \text{V}$? (c) What will the magnitude of the electric field inside of the wire be? [A: (a) $21\ \text{m}$, (b) $20\ \text{A}$, (c) $2.8\ \text{V/m}$.]

18.2 How much current passes through when a tungsten wire with a cross-section area of $1\ \text{mm}^2$ and a length of $10\ \text{m}$ is connected to a $12\ \text{V}$ battery? [A: $23\ \text{A}$.]

18.3 A wire with a resistance of $0.02\ \Omega$ is to be manufactured out of an aluminum block with a volume of $1\ \text{cm}^3$. What should the length and cross-section area of the wire be? [A: $85\ \text{cm}$ and $1.2\ \text{mm}^2$.]

18.4 A wire with resistance R is cut into 4 equal parts and joined side-by-side. What will the resistance of the new wire be in terms of R ? [A: $R/16$.]

18.5 An electric field with a magnitude of $80\ \text{V/m}$ exists inside of a wire with a cross-section of $3\ \text{mm}^2$ when a $5\ \text{A}$ current flows through. What is the resistivity of the wire? [A: $4.8 \times 10^{-5}\ \Omega\cdot\text{m}$.]

18.6 A tungsten wire lamp has $7\ \Omega$ in resistance at 20°C . (a) What will its resistance be at 500°C ? (b) The lamp has $50\ \Omega$ in resistance when it is producing light. Determine the temperature of the lamp. [A: (a) $24\ \Omega$, (b) 1250°C .]

18.7 At what temperature will the resistivity of tungsten be four times that of silver? [A: 300°C .]

18.3 Direct Current Circuits

18.8 An $18\ \text{V}$ battery is connected to a $8\ \Omega$ resistor. The internal resistance of the battery is $1\ \Omega$. (a) Find the current flowing through the circuit. (b) Find the terminal voltage of the battery. (c) Calculate the power spent on the resistors and the power provided by the battery to the circuit. [A: (a) $2\ \text{A}$, (b) $16\ \text{V}$, (c) 32 , 4 and $36\ \text{W}$.]

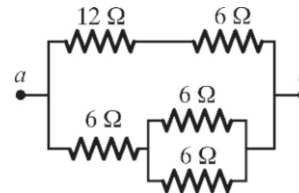
18.9 When a battery with an emf of $12\ \text{V}$ is connected to a circuit, a $5\ \text{A}$ current flows out and the terminal voltage is $11\ \text{V}$. What are the internal resistance of the battery and the connected resistance? [A: $0.2\ \Omega$ and $2.2\ \Omega$.]

18.10 An oven heater made of tungsten has a resistance of $40\ \Omega$ at room temperature (20°C) and operates under $240\ \text{V}$. (a) How much power does the oven consume when it starts to cook food at 20°C ? (b) How much power does the oven

consume when its temperature reaches 300°C ?

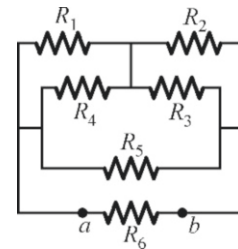
[A: (a) $1440\ \text{W}$, (b) $600\ \text{W}$.]

18.4 Combination of Resistors



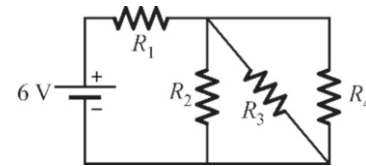
Problem 18.11

18.11 Calculate the equivalent resistance between the terminals ab shown in the figure. [A: $6\ \Omega$.]



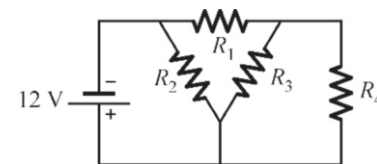
Problem 18.12

18.12 $R_1 = 1$, $R_2 = 2$, \dots , $R_6 = 6\ \Omega$ in the circuit shown in the figure above. Calculate the equivalent resistance between the terminals ab . [A: $15/13\ \Omega$.]



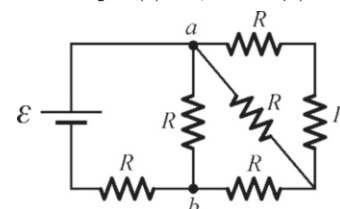
Problem 18.13

18.13 $R_1 = 1$, $R_2 = 2$, $R_3 = 3$ and $R_4 = 6\ \Omega$ in the circuit shown in the figure above. (a) Calculate the equivalent resistance. (b) Calculate the current flowing through each resistor. [A: (a) $2\ \Omega$, (b) $I_1 = 3$, $I_2 = 1.5$, $I_3 = 1$, $I_4 = 0.5\ \text{A}$.]



Problem 18.14

18.14 $R_1 = 1$, $R_2 = 2$, $R_3 = 3$ and $R_4 = 4\ \Omega$ in the circuit shown in the figure. (a) Calculate the equivalent resistance. (b) Calculate the current flowing through resistor R_1 . [A: (a) $21/16\ \Omega$, (b) $I_1 = 9\ \text{A}$.]

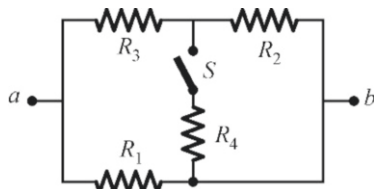


Problem 18.15

18.15 All of the resistors in the circuit above have the same value R . (a) Find the equivalent resistance. (b) Calculate the

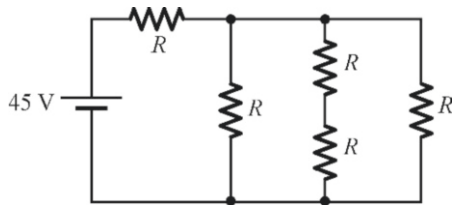
potential difference V_{ab} if $R=8\ \Omega$ and $\mathcal{E}=26\ \text{V}$.

[A: (a) $13R/8$, (b) $10\ \text{V}$.]



Problem 18.16

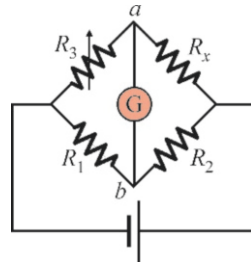
18.16 $R_1 = 1$, $R_2 = 2$, $R_3 = 3$ and $R_4 = 4\ \Omega$ in the circuit shown above and $6.5\ \text{V}$ in voltage is applied between ab . (a) How much current is drawn from the battery when switch S is open? (b) How much current is drawn from the battery when the switch is closed? [A: (a) $8\ \text{A}$, (b) $7.8\ \text{A}$.]



Problem 18.17

18.17 $58\ \text{W}$ power is consumed in the circuit shown in the figure when the resistors, each with resistance R , are connected to a $45\ \text{V}$ -battery. Calculate the value of R . [A: $25\ \Omega$.]

18.18 Wheatstone bridge. The circuit shown in the figure below is used to determine an unknown resistance R_x . After a battery is connected to the circuit, the value of the variable resistor R_3 is changed such that the current flowing through a galvanometer connected between ab is zero.



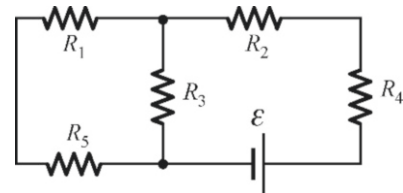
Problem 18.18

?

Show that

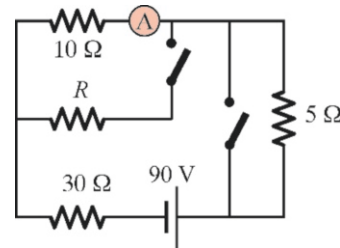
$$R_x = \frac{R_2 R_3}{R_1}$$

when the current on the galvanometer is zero. (*Hint:* This is an interesting case in which resistors can be in series and parallel at the same time. When the galvanometer current is zero, both the points a and b will have the same potential, and the current in the upper branch will continue without going to the lower branch. It is sufficient to write potential drops $R_i I_i$ equal for each half of the circuit.)



Problem 18.19

18.19 $R_1 = 1$, $R_2 = 2$, $R_3 = 3$, $R_4 = 4$ and $R_5 = 5\ \Omega$ in the circuit shown in the figure. Considering that the current flowing through R_1 is $1\ \text{A}$, what is the emf value \mathcal{E} of the battery? [A: $24\ \text{V}$.]

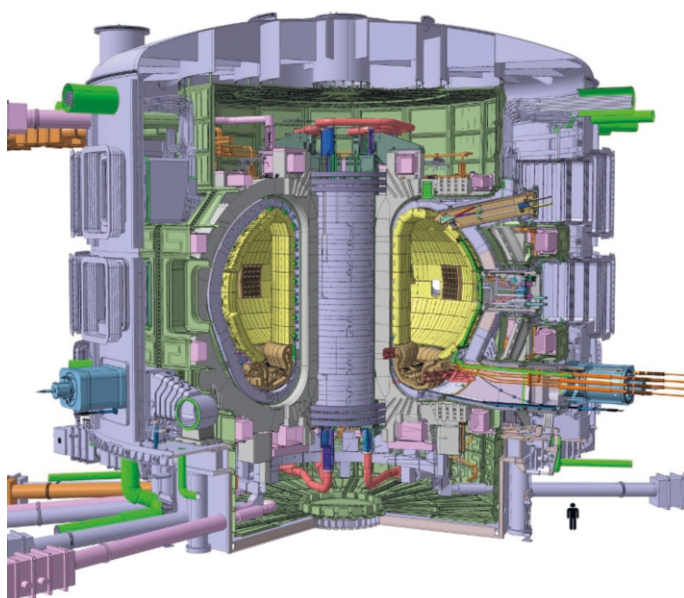


Problem 18.20

18.20 The ammeter A in the figure above shows the same value when both of the switches are closed and when both switches are open. Find the resistance R . [A: $60\ \Omega$.]

19

MAGNETIC FIELD



The ITER *Tokamak* reactor, under construction in France as the world's largest fusion reactor, is expected to produce electricity in 2045. (The man on the bottom right of the figure shows the scale.)

As no container can contain the charged particles at 100 million °C temperatures in the medium called *plasma*, they are suspended in the air with a very strong magnetic field (in the yellow donut-shaped area in the figure).

We can better understand magnetic field if we first understand how charged particles move in magnetic field.

The magnetic needle of a compass will always point north, regardless of the direction which you turn it. The Chinese and Indian civilizations were already cognizant of magnetism thousands of years ago. It is said that, in Ancient Greece, magnetized rocks were found in a region of Anatolia known as Magnesia. Sailors have used magnetized compasses for navigation for a thousand years.

Understanding magnetism was possible as late as the 17th century. In the 1600s, the English scientist William Gilbert claimed that the Earth was a giant magnet and that this was how it attracted the poles of a compass. In the 19th century, two scientists, the French André Ampere and the Danish Oersted, observed that a magnet would deflect a current-carrying wire. Later, Michael Faraday discovered that moving magnets generated emf.

All these discoveries indicated that there was a close relationship between electricity and magnetism. Ultimately, Scottish scientist James Clerk Maxwell merged electricity and magnetism under a single theory.

We shall discuss the properties of magnetism and its relation to electrical forces in this and the next two chapters.

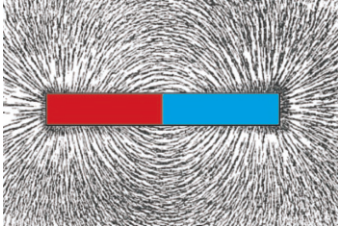


Figure 19.1: The magnetic field lines observed in iron dust around a magnet.

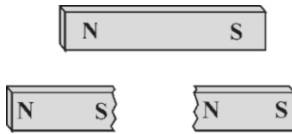


Figure 19.2: We cannot isolate magnetic poles even if we keep splitting the magnet.

19.1 MAGNETIC FORCE

We know how a horseshoe or bar magnet attracts nails or pins. This force cannot have an electrical nature, since both the magnet and the nails are neutral. From this observation, we can conclude the existence of a new “magnetic force.” Indeed, we can see the presence of a magnetic field around magnets in the patterns of iron dust scattered around a bar magnet, as seen in Figure 19.1.

The Coulomb force that we discussed in electrics was caused by electrical charges. Using the same logic to look for a “magnetic charge” that causes magnetic force, we observe that this is not possible. Consider that we take a magnet with poles indicated as being North-South ($N - S$) and divide it into two pieces. In such a case, it is observed that each of the new pieces turns into a magnet with $N - S$ poles (Figure 19.2). We cannot separate one of the poles even if we repeat this process and go down to the atomic scale. Each atom acts like a small magnet. Today, in modern physics, it is understood that **there is no single magnetic pole**.

So, what is the origin of the magnetic force? As we shall discuss later, magnetism has two sources:

- Currents.
- Magnetic dipole moments of elementary particles.

We shall learn these concepts in this and subsequent chapters.

In our study of electricity, we had adopted the following modern view: A charge distribution produces an electric field \vec{E} at every point in space and then a charge q placed in this field interacts with the electric field as $\vec{F} = q\vec{E}$.

We shall use the same outlook for magnetic field:

- A current generates a magnetic field \vec{B} at every point in space.
- A force \vec{F} is exerted upon moving charges q and currents I in this magnetic field.

We shall start with the second item about magnetism, setting aside how a magnetic field is generated and concentrating first on trying to understand the force that it exerts.

Magnetic Force Exerted on a Moving Electric Charge

A charge moving between the poles of a magnet is observed to deflect from its trajectory. The magnetic force exerted upon a charge q traveling with velocity \vec{v} inside of a magnetic field is observed to have the following properties:

- The force is proportional to the charge q and is in the opposite direction for $+q$ and $-q$ charges.
- The force is proportional to velocity v and perpendicular to it. The force is zero if the charge is at rest ($v = 0$).
- The force is proportional to the magnitude of the magnetic field B and perpendicular to it.

All these properties indicate that magnetic force can be expressed as a **vector product** as follows:

$$\vec{F} = q (\vec{v} \times \vec{B}) \quad (\text{Magnetic force on a charge}) \quad (19.1)$$

Now, let us get to know the force \vec{F} better by remembering the properties of a vector product:

- **Magnitude of the magnetic force.** The magnitude of a vector product $\vec{a} \times \vec{b}$ is $ab \sin \theta$. Therefore, the force on charge q produced by velocity vector \vec{v} and magnetic field vector \vec{B} with angle θ in between, will have the magnitude

$$F = qvB \sin \theta$$

It can be seen that the force will be zero if the velocity is zero ($v = 0$) or if it is parallel to the magnetic field ($\theta = 0^\circ$).

- **Direction of the magnetic force.** The orientation of a vector $\vec{c} = \vec{a} \times \vec{b}$ is perpendicular to both \vec{a} and \vec{b} . We use the right-hand rule to determine which of the two directions in this orientation is to be used:

According to the right-hand rule, when we point our four fingers towards the first vector (\vec{a}) and point our palm towards the second vector (\vec{b}), our thumb gives the direction of \vec{c} .

Likewise, for a positive charge q , the direction of the force is perpendicular to the plane formed by both \vec{v} and \vec{B} , and when we point the four fingers of the right-hand towards the vector \vec{v} and the palm towards the field vector \vec{B} , the thumb gives the direction of the force \vec{F} .

When the charge q is negative ($-$), the formula (19.1) gives a force in the direction of $-\vec{F}$. In other words, the magnetic forces exerted upon negative and positive charges are in opposite directions.

Unit of Magnetic Field

The force formula (19.1) allows us to determine the unit of magnetic field. If we compare both sides of the equation and substitute (coulomb/second = ampere), we get the unit of magnetic field in the SI system as

$$1 \frac{\text{N}}{\text{C} \times \text{m/s}} = 1 \frac{\text{N}}{\text{A} \cdot \text{m}} = 1 \text{ tesla} = 1 \text{ T}$$

The unit of magnetic field was named the **tesla** (T) in memory of the great Serbian scientist Nikola Tesla (1857–1943).

The unit of tesla is extremely large for normal magnetic fields. Another unit of magnetic field is the **gauss** and

$$1 \text{ gauss} = 10^{-4} \text{ T}$$

For example, Earth's magnetic field is approximately 1 gauss.

Motion of Charged Particles in a Magnetic Field

The fact that magnetic force is perpendicular to velocity \vec{v} causes particles to move in a specific trajectory. In the simplest case, let us consider a particle with charge $+q$ entering with velocity \vec{v} perpendicularly into a medium in which the magnetic field \vec{B} is *into* the plane of the paper, as shown in Figure 19.4. (In the following figures in this chapter, the \times symbol means into the paper and the \odot symbol means out of the paper.)

According to the formula $\vec{F} = q(\vec{v} \times \vec{B})$, the force exerted upon the particle will be perpendicular both to the velocity \vec{v} and the magnetic field \vec{B} . As force is perpendicular to velocity, it cannot change the magnitude of the velocity, but can only change its direction. (In other words, magnetic force does not perform work

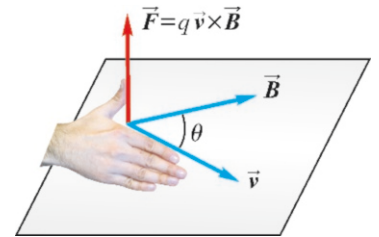


Figure 19.3: Direction of the magnetic force according to the right-hand rule.

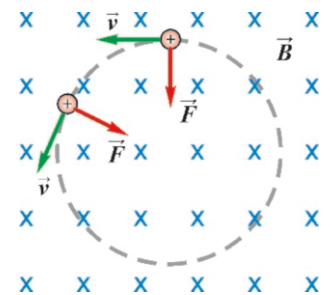


Figure 19.4: Magnetic force is always perpendicular to trajectory.

on the particle.) Even when the particle is slightly deflected, the force will still be perpendicular to the velocity \vec{v} .

We know already what kind of motion has this property in which the force is always perpendicular to the velocity vector. This is the circular motion that we discussed in Section 3.3. In this case, the magnetic force provides the centripetal force of the circular motion:

$$F_r = qvB = m \frac{v^2}{r}$$

From here, we find the radius of the circular motion as follows:

$$r = \frac{mv}{qB} \tag{19.2}$$

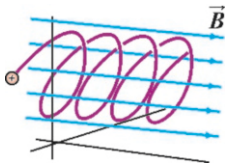


Figure 19.5: The most general spiral motion of a charged particle.

Note that the numerator of this formula is $p = mv$, the momentum of the particle. In scientific research and technological applications, the radius of the circular motion of a particle is measured inside of a known magnetic field and either the velocity v , the charge q or the m of the particle can be determined from this.

If the particle does not enter the magnetic field as fully perpendicular, the component of velocity along the magnetic field does not change. In such a case, it performs circular motion on one hand and continues to move forward on the other hand. This is the spiral motion.

Let us briefly introduce the instrument called the **mass spectrograph**, which is an important application of this formula.

In this instrument, shown in Figure 19.6, a charge $+q$ generated at a source of ion is first accelerated to a known velocity v under a potential difference V in an electric field. The charge then enters perpendicularly into a uniform magnetic field B and exits by following a semicircular path. The distance $2r$ of the point at which it exits is determined using detectors. This data can be used to determine the mass m of a particle with a known charge q .

The mass spectrograph was used to determine the masses of atoms very precisely and prove the existence of isotopes. Today, it is frequently used in medicine and engineering, in addition to physics, chemistry and biology researches.

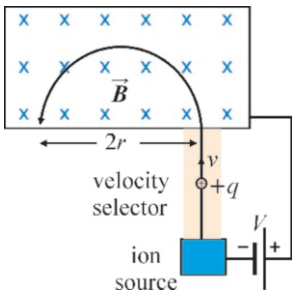
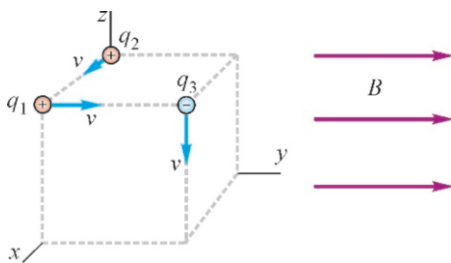


Figure 19.6: Mass spectrograph.

Example 19.1



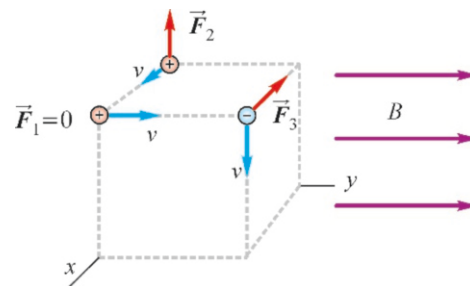
Determine only the directions of the forces exerted upon charges q_1 , q_2 and q_3 thrown with velocity v from the corners of a cube, in the directions shown in the figure. The magnetic field vector \vec{B} is in the $+y$ direction.

Answer We use the formula (19.1) for the magnetic force:

$$\vec{F} = q (\vec{v} \times \vec{B})$$

The velocity vector \vec{v} of charge q_1 is given in the direction of the $+y$ axis as the vector \vec{B} . The vector product of two parallel vectors is zero:

$$\vec{F}_1 = 0 \quad (\vec{v} \parallel \vec{B})$$



The velocity of charge q_2 is in the $+x$ direction. If we point the four fingers of our right hand in the $+x$ direction and point our palm in the direction of \vec{B} ($+y$), our thumb will

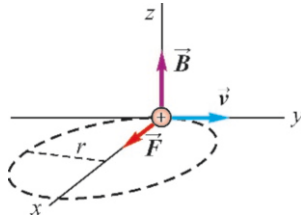
point in the $+z$ direction. Therefore, the force \vec{F}_2 is in the $+z$ direction.

Note that charge q_3 is negative. In other words, a force will be exerted in the opposite direction of the one to be found using the right-hand rule. Again, if we point the four fingers of our

right hand in the \vec{v} ($-z$) direction and point our palm in the direction of \vec{B} ($+y$), our thumb will point in the $+x$ direction. The force will be opposite to this direction. Therefore, the force \vec{F}_3 is in the $-x$ direction.

The results are shown in the figure above:

Example 19.2



In an area where the magnetic field has a magnitude of $B=0.1$ T in the direction of the $+z$ -axis, a proton is thrown at a velocity of 10^6 m/s in the direction of the $+y$ -axis. (The proton charge is $e=1.6 \times 10^{-19}$ C and mass is $m_p=1.7 \times 10^{-27}$ kg.)

- (a) Find the magnitude and direction of the force exerted upon the proton.
- (b) Find the radius of the circular trajectory of the proton.

Answer (a) The force exerted upon the proton will be in the $+x$ direction according to the formula $\vec{F}=q(\vec{v} \times \vec{B})$, as shown in the figure. We calculate its magnitude:

$$F = qvB \sin 90^\circ = evB$$

$$F = 1.6 \times 10^{-19} \times 10^6 \times 0.1 = 1.6 \times 10^{-14} \text{ N}$$

(b) As force is always perpendicular to velocity, the proton follows a circular trajectory in the xy -plane. Since magnetic force provides the centripetal force, we find that

$$m \frac{v^2}{r} = qvB \quad \longrightarrow \quad r = \frac{mv}{qB}$$

We substitute the values and calculate the radius as follows:

$$r = \frac{1.7 \times 10^{-27} \times 10^6}{1.6 \times 10^{-19} \times 0.1} = 0.11 \text{ m}$$

19.2 MAGNETIC FORCE ON A CURRENT-CARRYING WIRE

A current-carrying wire remains neutral because there are positive charges in the background of the moving electrons. Therefore, no electric force is exerted upon a current-carrying wire in an electric field.

However, the case is different when a current-carrying wire is placed in a magnetic field. Positive ions are not affected by the magnetic field, as they are at rest; however, a magnetic force is exerted upon the moving electrons, and a net magnetic force is thus generated on the wire.

We can calculate the force exerted upon a current I starting from the formula $\vec{F}=q(\vec{v} \times \vec{B})$ exerted upon a charge q . Consider the amount of charge flowing through the cross-section A during time interval dt (Figure 19.7). According to the definition $I = dq/dt$ of current, we get

$$dq = I dt$$

In section 18.1, we showed that electrons moved with an average drift velocity v_d in conductors. These charges will travel the distance $L = v_d dt$ during the time interval dt . As all these velocities are in the same direction and equal in magnitude, we can assume that the total charge dq also moves with velocity v_d . Accordingly, the force exerted upon the total charge dq with velocity \vec{v}_d in magnetic field \vec{B} is

$$\vec{F} = dq (\vec{v}_d \times \vec{B}) = I dt (\vec{v}_d \times \vec{B})$$

If we define $\vec{L} = \vec{v}_d dt$ as a displacement vector along the wire and in the direction of the current, we can find the expression for the force exerted upon a piece of wire with length L in a magnetic field:

$\vec{F} = I (\vec{L} \times \vec{B})$
(Magnetic force on a current)
(19.3)

Note that the force is perpendicular to both the wire and the magnetic field.

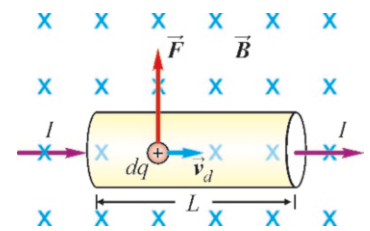
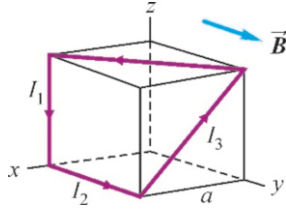


Figure 19.7: Force exerted on a charge dq in a conductor.

Example 19.3

The wires extending along the sides of a cube with side length a , as shown in the figure, carry the currents with the same magnitude: $I_1=I_2=I_3=I$. The magnetic field \vec{B} is in the $+y$ -direction. Calculate the forces exerted upon the currents I_1, I_2 and I_3 .



Answer

We use the formula (19.3) to calculate the magnetic force exerted upon a current-carrying wire:

$$\vec{F} = I (\vec{L} \times \vec{B})$$

Here, \vec{L} is a vector with the length of the wire and in the direction of the current.

The force on I_1 : In the figure, the vector \vec{L} is given in the $-z$ direction and the vector \vec{B} in the $+y$ direction. The force \vec{F}_1 will be in the $+x$ direction according to the right-hand rule.

We calculate its magnitude as follows:

$$F_1 = I L B \sin 90^\circ = I a B$$

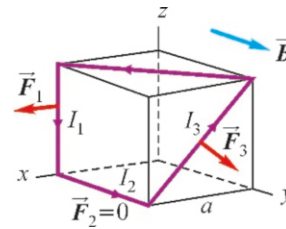
The force on I_2 : This time, the vector product is zero, as the vector \vec{L} is parallel to vector \vec{B} :

$$F_2 = 0$$

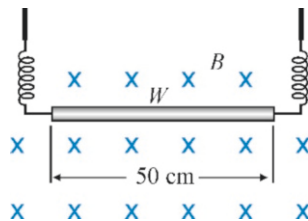
The force on I_3 : As the vector \vec{B} in the figure is in the $+y$ direction, the vector \vec{F}_3 perpendicular to it will be parallel to the xz -plane, in other words, inside that plane of the cube. Likewise, the vector inside this plane that is perpendicular to the vector \vec{L} must be along its diagonal. The right-hand rule gives the direction of the force as downward. We take the value $L = \sqrt{2}a$ and calculate the magnitude of the force:

$$F_3 = I L B \sin 90^\circ = \sqrt{2} I a B$$

All three forces are shown in the figure below.



Example 19.4



A conducting rod with weight $W=0.3\text{ N}$ and a length of 50 cm is suspended with conducting springs, as shown in the figure. The region has a magnetic field with a magnitude of 0.2 T into the paper. What should the magnitude and direction of the

current flowing through the rod be such that the force on the springs is zero?

Answer The magnetic force should be upward to balance the weight of the rod. As the rod and the magnetic field are perpendicular, we directly write the magnetic field and set it equal to the weight W :

$$F = W \rightarrow I L B = W$$

We calculate the current I from here as follows:

$$I = \frac{W}{L B} = \frac{0.3}{0.5 \times 0.2} = 3\text{ A}$$

The current must flow from the left to the right so that the force is upward according to the right-hand rule.

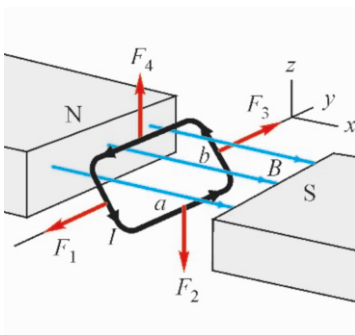


Figure 19.8: A current loop in a magnetic field.

19.3 MAGNETIC TORQUE ON A CURRENT LOOP – THE ELECTRIC MOTOR

A conducting loop placed between the poles of a magnet starts to rotate when a current flows through it. This is the electric motor, which is the instrument that ensured one of the greatest technological advances in history. Electric motors are used everywhere, from tiny dentist drills to large cranes, electrical watches to water pumps, CD drives to submarine propellers.

Figure 19.8 shows a rectangular loop placed inside of a magnetic field B . The side lengths of the loop are a and b and it can freely rotate about the y -axis.

When a current I flows through the loop, a force $\vec{F} = I (\vec{L} \times \vec{B})$ will be exerted upon each side. The directions of these forces according to the right-hand rule

are shown in Figure 19.8. And their magnitudes are

$$\begin{aligned} F_1 &= I b B \cos \theta && \text{(in the } -y \text{ direction)} \\ F_2 &= I a B && \text{(in the } -z \text{ direction)} \\ F_3 &= I b B \cos \theta && \text{(in the } +y \text{ direction)} \\ F_4 &= I a B && \text{(in the } +z \text{ direction)} \end{aligned}$$

The net force on the loop is zero, as these forces are equal and opposite. Therefore, the loop performs no translational motion.

However, the total torque of the forces is different from zero, and it may thus perform a rotational motion. Let us calculate the torques of these forces with respect to the rotation axis y .

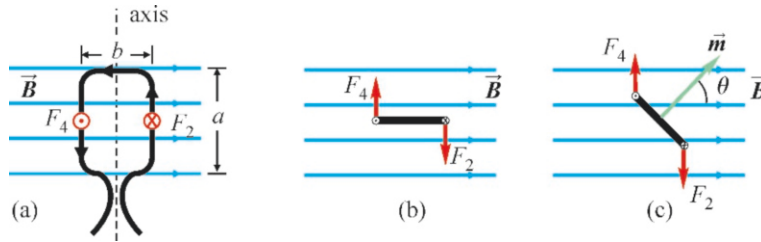


Figure 19.9: The forces on the sides from different angles: Views from (a) above, (b) front. (c) The dipole moment \vec{m} in the direction of the normal of the loop as it makes angle θ with \vec{B} .

Let θ be the angle between the normal of the loop and the magnetic field at any time. Accordingly, we can write the sum of the moments as follows:

$$\begin{aligned} \tau &= F_1 \cdot 0 + F_2 (b/2) \sin \theta + F_3 \cdot 0 + F_4 (b/2) \sin \theta \\ \tau &= I a b B \sin \theta = I A B \sin \theta \end{aligned} \quad (19.4)$$

In this expression, the moments of the forces F_1 and F_3 are zero, because they are parallel to the axis. $A = ab$ is the area of the loop.

The product IA here is defined as the **magnetic dipole moment** of a current loop and is defined as a vector perpendicular to the plane of the loop, in the direction of the normal vector \hat{n} (Figure 19.9c):

$$\vec{m} = IA \hat{n}$$

More generally, if the loop has N windings, the magnitude of the dipole moment will be:

$$m = N IA \quad \text{(Magnetic dipole moment)} \quad (19.5)$$

The aforementioned torque calculation is also valid for geometric shapes other than a rectangle. As a result, the amount of torque exerted on a current loop with dipole moment \vec{m} in a magnetic field \vec{B} is found as follows:

$$\vec{\tau} = \vec{m} \times \vec{B} \quad \text{(Magnitude: } \tau = mB \sin \theta) \quad (19.6)$$

Electric Motor

The torque given with the expression (19.4) above, will try to rotate the loop counterclockwise with respect to the directions given in the figure, as it is positive. However, after the loop makes a half rotation, in other words, when the bottom side goes to the top, the torque of forces F_2 and F_4 will be negative this time, as the currents are in the reverse direction. In this case, the loop will try to rotate in the reverse direction.



Figure 19.10: Hybrid automobiles have two motors, one gasoline and one electric (right). The electric motor activates at startup and when the speed changes.

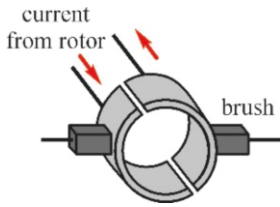


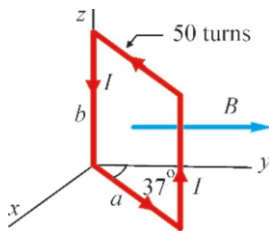
Figure 19.11: The commutator reverses the current at each half rotation.

Continuous rotation in the same direction cannot be achieved in this mechanism. It is necessary to reverse the direction of the current at each half rotation.

There are two ways to achieve continuous rotation in electric motors:

1. In alternating current (AC) motors, the current varies as a sinusoidal wave. After the loop makes a half rotation, the direction of the current reverses automatically and ensures that the torque remains in the same direction.
2. In direct current (DC) motors, the current that always flows out of the battery in the same direction passes through a mechanism called the **commutator** before reaching the loop (Figure 19.11). A commutator consists of two half conductor rings, and the side connected to the + charged pole of the battery changes at each half rotation. The direction of the current thus changes at each half rotation of the loop and the torque is ensured to remain in the same direction.

Example 19.5

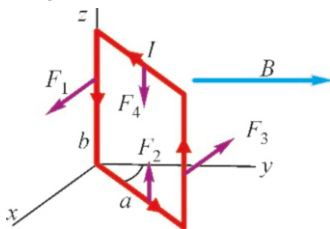


The loop shown in the figure consists of 50 windings and can rotate around the z -axis. A current of $I=3$ A is flowing through each winding. The dimensions of the loop are $a=1$ m and $b=2$ m. The magnetic field $B = 2$ T is towards the y -axis.

- (a) Calculate the net torque on the loop by calculating the torques acting on each side.
- (b) This time, calculate the net torque by using the magnetic dipole moment of the loop.

Answer

(a) The forces acting on each side are as shown in the figure according to the right-hand rule:



Among these forces, only the moment of \vec{F}_3 is different from zero with respect to the z -axis. As \vec{F}_1 is along the axis, and as \vec{F}_2 and \vec{F}_4 are parallel to the axis, they do not cause rotation and have zero torques.

The remaining force \vec{F}_3 is in the $-x$ direction and its magnitude is calculated using the formula (19.3) and multiplied by the number of windings N :

$$F_3 = N I b B \sin 90^\circ = 50 \times 3 \times 2 \times 2 \times 1 = 600 \text{ N}$$

The moment arm $a \cos 37^\circ$ is used when finding the torque of force F_3 with respect to the z -axis:

$$\tau = \tau_3 = F_3 a \cos 37^\circ$$

$$\tau = 600 \times 1 \times 0.8 = 480 \text{ N}\cdot\text{m}$$

The direction of the torque will be so as to turn from the x - to the y -axis.

(b) The magnetic dipole moment of the loop is calculated with the formula (19.5):

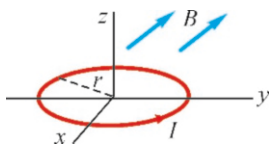
$$m = N I A = 50 \times 3 \times (1 \times 2) = 300 \text{ A}\cdot\text{m}^2$$

We then calculate the torque from the formula (19.6). Here, the angle between the magnetic field and the surface normal should be taken:

$$\tau = m B \sin \theta = 300 \times 2 \times \sin 53^\circ = 480 \text{ N}\cdot\text{m}$$

As it can be seen, both calculations give the same result, but it is easier to work with magnetic moments.

Example 19.6



A current of 25 A is flowing through a circular loop with a radius of 40 cm on the xy -plane. This region has a uniform magnetic field of 0.3 T in a direction with a 45° angle with the xy -plane. Find the magnetic moment of the loop and the

torque that is trying to rotate it.

Answer

The magnetic dipole moment of the loop is calculated with the formula (19.5):

$$m = I A = I \pi r^2 = 25 \times 3.14 \times 0.40^2$$

$$m = 12.6 \text{ A}\cdot\text{m}^2$$

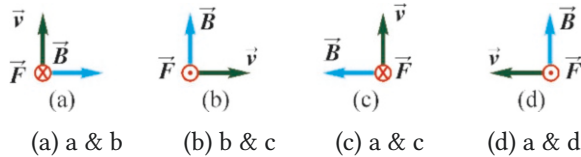
We then calculate the torque from the formula (19.6).

$$\tau = m B \sin \theta = 12.6 \times 0.3 \times \sin 45^\circ$$

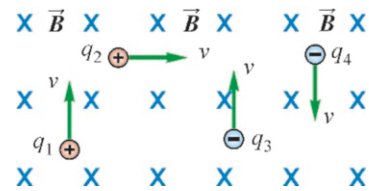
$$\tau = 2.7 \text{ N}\cdot\text{m}$$

Multiple-choice Questions

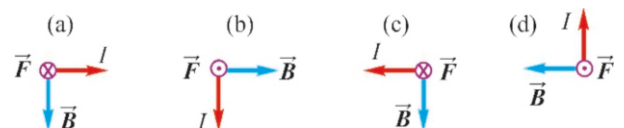
- Which is the unit of magnetic field in the SI unit system?
(a) tesla (b) gauss (c) ampere (d) einstein
- What will the direction of the force be exerted upon a positively charged particle thrown in the $+y$ direction in a magnetic field that is in the $+x$ direction?
(a) $+x$ (b) $+y$ (c) $+z$ (d) $-z$
- Which of the following figures correctly shows the force acting on charge $+q$ with the velocity \vec{v} in the magnetic field \vec{B} ?



- If a magnetic force towards the East is exerted upon a positively charged particle thrown towards the North, in which direction could the magnetic field be?
(a) Up (b) Down (c) South (d) West
- What will the magnitude and direction of the force be exerted on a particle with charge 4 C thrown upwards from the ground with a velocity of 3 m/s in a 2 T magnetic field towards the North?
(a) 24 N to the West
(b) 24 N to the East
(c) 6 N to the West
(d) 6 N to the East
- By what factor does the magnetic force acting on an object increase if its velocity and the magnetic field are both doubled?
(a) No change (b) 2 (c) 4 (d) 8
- In which of the following cases will the magnetic force exerted on a charge be zero?
I. If the velocity is perpendicular to the magnetic field,
II. If the velocity is parallel to the magnetic field,
III. If the velocity is zero,
IV. If the magnetic field is constant.
(a) I & II (b) II & III (c) I & III (d) II & IV
- If the magnetic force exerted upon a charge $+q$ is 5 N in the $+y$ direction, what will the force exerted upon a charge $-2q$ be?
(a) 10 N in the $+y$ direction
(b) 10 N in the $-y$ direction
(c) 2.5 N in the $+y$ direction
(d) 50 N in the $-y$ direction

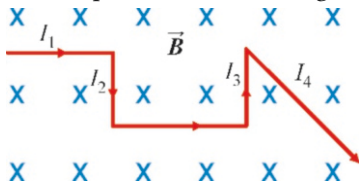


- Which of the forces exerted upon the charges shown in the figure above is to the right? (The magnetic field is into the paper.)
(a) q_1 (b) q_2 (c) q_3 (d) q_4
- What will the direction of the force be exerted upon a current in the $+y$ direction in a magnetic field in the $+x$ direction?
(a) $+x$ (b) $+y$ (c) $+z$ (d) $-z$
- By what factor does the magnetic force exerted upon a current-carrying wire increase if its length and the magnetic field are both doubled?
(a) No change (b) 2 (c) 4 (d) 8
- By what factor does the magnetic force exerted upon a wire increase if the current and the magnetic field are both doubled?
(a) No change (b) 2 (c) 4 (d) 8
- In which of the following cases will the magnetic force exerted upon a current-carrying wire be zero?
I. If the current is perpendicular to the magnetic field,
II. If the current is parallel to the magnetic field,
III. If the current is zero,
IV. If the magnetic field is constant.
(a) I & II (b) II & III (c) I & III (d) II & IV
- What will the magnitude and direction of the force be exerted upon a wire with a length of 4 m and carrying 3 A in current upward from the ground in a 2 T magnetic field towards the North?
(a) 24 N to the West
(b) 24 N to the East
(c) 6 N to the West
(d) 6 N to the East



- Which of the above figures correctly show the force acting on the current I in the magnetic field \vec{B} ?
(a) a & b (b) b & c (c) a, b & d (d) a & d

16. Which section of the wire shown in the figure below has a force exerted upon it towards the right?



- (a) I_1 (b) I_2 (c) I_3 (d) I_4

17. A charged particle is in circular motion in a magnetic field. If its charge and mass are doubled, by what factor will the radius of the circle change?

- (a) No change (b) 2 (c) 4 (d) 1/2

18. Two charged particles with the same velocity are in circular motion with the same radius in a magnetic field. Which of the following is correct?

- (a) The charges of the particle are equal.
 (b) The masses of the particles are equal.
 (c) The charge/mass ratios of the particles are equal.
 (d) The product charge \times mass of the particles are equal.

19. What is the maximum torque exerted upon a loop with a cross-section area of 1 m^2 and carrying 2 A in current in a 3 T magnetic field?

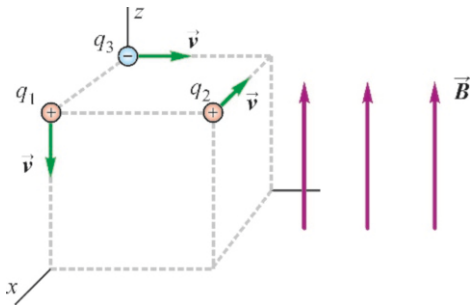
- (a) $5 \text{ N}\cdot\text{m}$ (b) $6 \text{ N}\cdot\text{m}$ (c) $2 \text{ N}\cdot\text{m}$ (d) $4 \text{ N}\cdot\text{m}$

20. In what orientation should a current loop be placed in a magnetic field for the torque exerted upon it to be zero?

- (a) Parallel to the magnetic field
 (b) Perpendicular to the magnetic field
 (c) 45° with the magnetic field.
 (d) It is never zero.

Problems

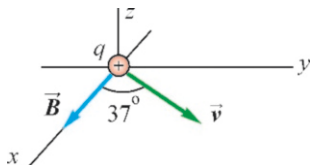
19.1 Magnetic Force on Charges



Problem 19.1

- 19.1 Determine only the directions of the forces exerted upon charges q_1 , q_2 and q_3 thrown with velocity v from the corners of a cube, in the directions shown in the figure. The magnetic field vector \vec{B} is in the $+z$ direction.

[A: $F_1 = 0$, F_2 : in the $+y$ direction; F_3 in the $-x$ direction.]

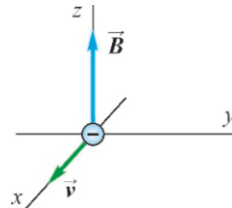


Problem 19.2

- 19.2 In a region with a magnetic field with a magnitude of 0.4 T in the $+x$ direction, a $q=3 \mu\text{C}$ charge is thrown with a velocity of $5 \times 10^6 \text{ m/s}$ and at a 37° angle with the magnetic field. What are the magnitude and direction of the force exerted upon the charge? [A: 3.6 N in the $-z$ direction.]

- 19.3 When a $3 \mu\text{C}$ charge with a velocity of 10^6 m/s is thrown into a $B = 0.1 \text{ T}$ magnetic field, a 0.15 N force is

- exerted upon it. What is the angle between the velocity and the magnetic field? [A: 30° .]

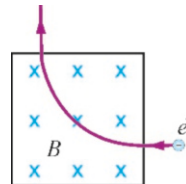


Problem 19.4

- 19.4 In a region where the magnetic field has magnitude $B=0.1 \text{ T}$ in the direction of the $+z$ -axis, an electron is thrown at a velocity of $2 \times 10^8 \text{ m/s}$ along the $+x$ -axis.

- (a) Find the magnitude and direction of the force exerted upon the electron. (b) Calculate the radius of the circular trajectory of the electron. (The electron's charge is $-e = -1.6 \times 10^{-19} \text{ C}$ and its mass is $m_e = 9.1 \times 10^{-31} \text{ kg}$.)

[A: (a) $3.2 \times 10^{-12} \text{ N}$ in $+y$ -direction, (b) 0.01 m .]



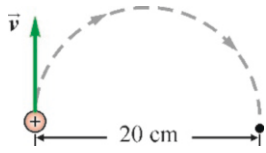
Problem 19.5

- 19.5 An electron enters the region of a magnetic field shown in the figure with a velocity of $2 \times 10^8 \text{ m/s}$ and exits perpendicular to its incoming direction after traveling a quarter of a circle 1.57 cm long. Calculate the magnitude of the magnetic field in the region. (Hint: Calculate the radius from the length of the quarter circle.) [A: 0.11 T .]

- 19.6 An electron at rest is first accelerated under an electric potential difference of 200 kV , after which it enters a region

where a perpendicular magnetic field of $B=0.1\text{ T}$ is present. (a) What is velocity of the electron? (b) What will the radius of the circular path in the magnetic field be?

[A: (a) $2.7 \times 10^8\text{ m/s}$, (b) 1.5 cm .]

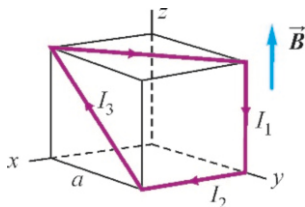


Problem 19.7

19.7 The proton shown in the figure is thrown into a region of unknown magnetic field with a velocity of $2 \times 10^6\text{ m/s}$. The proton travels a semicircle and exits 20 cm away. Calculate the magnitude and direction of the magnetic field in the area ($m_p = 1.7 \times 10^{-27}\text{ kg}$). [A: 0.21 T out of the paper.]

19.2 Magnetic Force on Currents

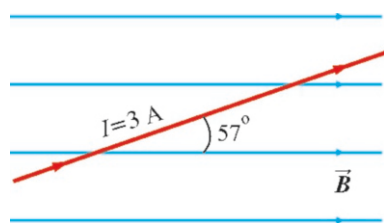
19.8 When a current-carrying wire with a length of 75 cm is placed perpendicularly in a magnetic field with a magnitude of 0.2 T , a 3 N force is exerted upon it. What is the current flowing through the wire? [A: 20 A .]



Problem 19.9

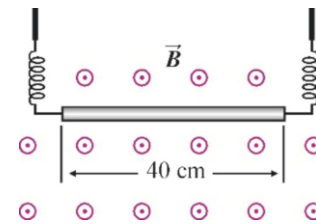
19.9 The wires extending along the sides of a cube with side length a , as shown in the figure, carry the currents with the same magnitude: $I_1=I_2=I_3=I$. The magnetic field \vec{B} is in the $+z$ -direction. Calculate the forces exerted upon the currents I_1 , I_2 and I_3 .

[A: $F_1=0$, $F_2=IaB$: in the $-y$ direction, $F_3=IaB$: in the $-x$ direction]



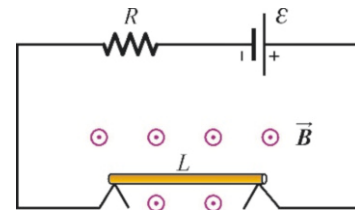
Problem 19.10

19.10 A 3 A current is flowing through a wire that makes a 53° angle with the field lines in an area with a 0.5 T magnetic field in the plane shown in the figure. Calculate the magnitude and direction of the force exerted upon a unit length of the wire. [A: 1.2 N into the paper.]



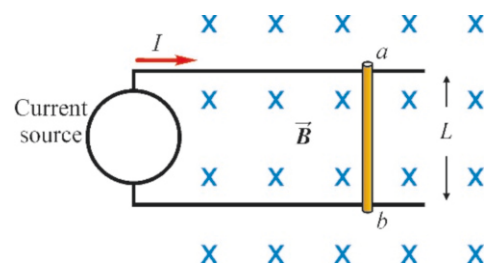
Problem 19.11

19.11 A conducting rod with weight $W = 0.5\text{ N}$ and a length of 40 cm is suspended with conducting springs, as shown in the figure. The region has a magnetic field with a magnitude of 0.1 T out of the paper. What should the magnitude and direction of the current flowing through the rod be such that the force on the springs is zero. [A: 12.5 A to the left.]



Problem 19.12

19.12 A metal rod with a length of 40 cm , a weight of 0.5 N and zero resistance is freely placed on two supports, as shown in the figure. The area has a magnetic field of 0.3 T out of the paper. What is the minimum value of the resistance R such that the rod does not jump into the air when the circuit is connected to a 12 V -battery? [A: $2.9\ \Omega$.]



Problem 19.13

19.13 A constant current of 3 A is given to the circuit in the figure above from a source. A metal rod with a mass of 100 g and a length of 50 cm is freely placed on the frictionless points ab . There is a magnetic field with a magnitude of 0.2 T into the paper. What will the acceleration of the rod be when the current is flowing? [A: 3 m/s^2 .]

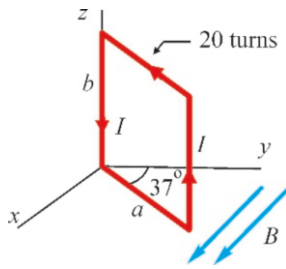
19.3 Magnetic Torque



Problem 19.14

19.14 A 5 A current is flowing through a circular loop with radius 10 cm in the xy -plane. This region has a magnetic field of 0.2 T in a direction at a 37° angle with the xy -plane.

Find the magnetic moment of the loop and the torque that is trying to rotate it. [A: $m = 0.16 \text{ A}\cdot\text{m}^2$, $\tau = 0.25 \text{ N}\cdot\text{m}$.]



Problem 19.15

19.15 A rectangular loop made of 20 windings can rotate around the z -axis and a current of $I=5 \text{ A}$ is flowing through

each winding. The dimensions of the loop are $a = 1 \text{ m}$ and $b = 2 \text{ m}$. The magnetic field in the region is $B=3 \text{ T}$ towards the x -axis. (a) Calculate the magnetic moment of the loop, (b) Calculate the torque exerted on the loop.

[A: (a) $200 \text{ A}\cdot\text{m}^2$, (b) $360 \text{ N}\cdot\text{m}$.]

19.16 In the Bohr model of the Helium atom, an electron rotates with a velocity of $6.6 \times 10^6 \text{ m/s}$ around the nucleus in an orbit with a radius of $1.8 \times 10^{-11} \text{ m}$. (a) In how many seconds does the electron complete one circle? (b) If we consider the electron as the charge flowing through a given cross-section, what will the current I be? (c) What is the magnetic moment generated by this orbital motion of the electron?

[A: $1.7 \times 10^{-17} \text{ s}$, (b) 9.3 mA , (c) $9.5 \times 10^{-24} \text{ A}\cdot\text{m}^2$.]

?

20

SOURCES OF MAGNETIC FIELD



The 2000-ton CMS detector is being inserted into its socket at the CERN European Nuclear Research Center. The superconductor magnet at the center of the detector (the gray area) can generate a magnetic field up to 4 T.

We started our study of magnetism in Chapter 19 by examining its effects on moving charges and currents. But we did not discuss the source of this magnetic field. Now, we shall examine how a magnetic field is generated.

The source of magnetic field is moving charges and currents. This was first discovered by the Danish scientist Oersted in 1819 through his observation of the deflection of a compass needle near a current-carrying wire. Soon afterwards, the French scientists Biot and Savart were able to establish the form of the magnetic field produced by any current.

In electrostatics, we had used the Gauss law as a shortcut for calculating electric field. There is a similar law for magnetic field: We shall examine Ampère's law in this chapter. Then, we will look at the magnetic properties of matter on the atomic scale.

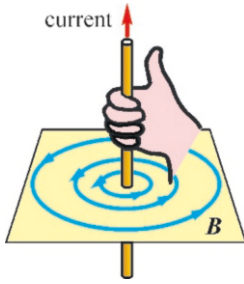


Figure 20.1: Magnetic field of a straight wire current.

20.1 MAGNETIC FIELD OF A CURRENT

As the simplest case, let us consider the magnetic field generated by a straight wire carrying a current I (Figure 20.1). Observations have revealed the following:

- Magnetic field lines are circles centered around the wire on a plane perpendicular to the wire.
- According to the right-hand rule, the direction of the magnetic field is in the direction in which the four fingers fold when the thumb points in the direction of the current.
- The magnitude of the magnetic field decreases as inversely proportional to the radius r .

The French scientists Jean-Baptiste Biot and Felix Savart performed meticulous experiments on the magnetic field of current-carrying wires and discovered the expression of magnetic field for all kinds of currents. Known as the **Biot-Savart law**, this formula gives the contribution of a small current element $I d\ell$ with length $d\ell$ to the magnetic field (Figure 20.2):

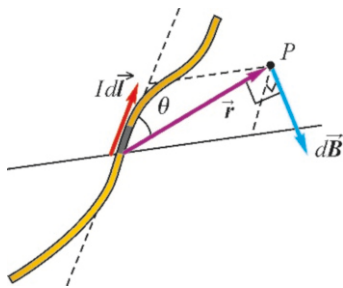


Figure 20.2: The contribution dB of a piece $d\ell$ of a current-carrying wire to the magnetic field at point P located at \vec{r} is perpendicular to the plane formed by these two vectors.

The Biot-Savart Law

The contribution of a small current element $I d\ell$ with length $d\ell$ and carrying a current I to the magnetic field at a point at distance r is

$$dB = k' \frac{I d\ell \sin \theta}{r^2} \quad (20.1)$$

The angle θ here is the angle between the position vector \vec{r} and the direction of $d\ell$.

The constant k' in this formula has the following value in the SI system:

$$k' = 10^{-7} \text{ T}\cdot\text{m/A} \quad (20.2)$$

In electrics, we had expressed the constant k in terms of ϵ_0 , the electric permittivity of free space. Likewise, in magnetism, a constant μ_0 called the **magnetic permeability of free space** is defined as follows:

$$k' = \frac{\mu_0}{4\pi} \quad \longrightarrow \quad \mu_0 = 4\pi \times 10^{-7} \text{ T}\cdot\text{m/A} \quad (20.3)$$

Let us emphasize the important points of the Biot-Savart law:

- The magnetic field of each small current element $I d\ell$ is perpendicular to the plane formed by the element $I d\ell$ and the position r .
- The right-hand rule gives the direction of the magnetic field among the two possible perpendicular directions: The thumb is directed towards the current, the four fingers are pointed towards r and the palm shows the direction of the vector $d\vec{B}$.
- The magnetic field contribution is directly proportional to the current I and inversely proportional to r^2 , the square of the distance.
- Magnetic field is directly proportional to the sine of the angle between the element $I d\ell$ and r .

Vector Product Expression of the Biot-Savart Law

In the Biot-Savart formula above, $\sin \theta$ and the lengths $d\ell$ and r indicate that this is actually a vector product. Indeed, if we write the piece $d\ell$ as a vector $d\vec{\ell}$ in the direction of the current I and take \hat{r} as the unit vector in the direction of r , we get the vector expression of the Biot-Savart law as follows:

$$d\vec{B} = k' \frac{I d\vec{\ell} \times \hat{r}}{r^2} \quad (\text{Biot-Savart: vector expression}) \quad (20.4)$$

It can be seen that this expression is equivalent to the magnetic field Eq. (20.1) in terms of both magnitude and direction.

Magnetic Field of a Finite Wire

Every current flows through a finite wire consisting of the sum of small $d\ell$ parts. The magnetic field of a whole current will be the limit sum, in other words, the integral of the small $d\vec{B}$ vectors above:

$$\vec{B} = k' \int \frac{I d\vec{\ell} \times \hat{r}}{r^2} \quad (20.5)$$

This integral will be taken over the whole wire.

20.2 MAGNETIC FIELD CALCULATIONS

Let us calculate the magnetic fields of the most essential current distributions.

Magnetic Field of an Infinite Straight Wire

Consider a straight wire extending along the x -axis chosen upward, and carrying current I . Let us calculate the magnetic field at a point P as shown in Figure 20.3. We choose the origin at the point on the wire that is closest to point P at a distance a . (This choice will not affect the generalization, as the wire is infinite.)

Let us choose the length element $d\ell$ on the wire at distance x and with length dx . If we write the contribution of this piece to the magnetic field at point P located at distance r according to Eq. (20.1) of the Biot-Savart law, we get

$$dB = k' \frac{I dx \sin \theta}{r^2}$$

The direction of the magnetic field dB will be into the plane of the paper. The contributions of all other dx elements on the wire to this point P will always be into the paper, in other words, in the same direction. Therefore, the algebraic sum of these dB contributions can be directly integrated. Considering that the wire ranges from $(-\infty)$ to $(+\infty)$, we get

$$B = k' I \int_{-\infty}^{\infty} \frac{dx \sin \theta}{r^2}$$

Let us write all the variables in the integral in terms of x . If we use the angle $(\pi - \theta)$, the complement of angle θ , we get

$$\sin \theta = \sin(\pi - \theta) = \frac{a}{r} = \frac{\sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}}$$

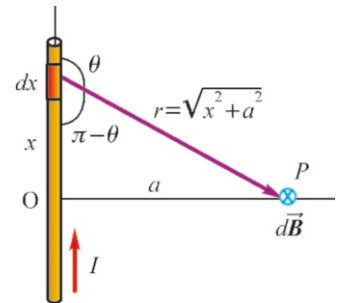


Figure 20.3: The dB contribution of a length element dx on an infinite wire to the magnetic field at point P .

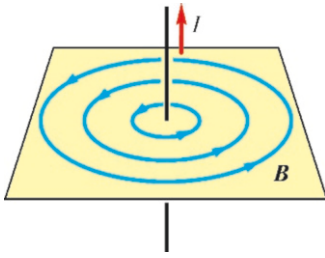


Figure 20.4: Magnetic field of a straight wire.

When we use these values,

$$B = k' I \int_{-\infty}^{\infty} \frac{dx (a / \sqrt{x^2 + a^2})}{x^2 + a^2} = k' I a \underbrace{\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^{3/2}}}_{2/a^2}$$

Let us state here that the result is $2/a^2$ without going into the details of the calculation. (It can be calculated by a change of variable $x = -a \cot \theta$.) Accordingly, the magnetic field of an infinite straight wire is

$$B = \frac{2k' I}{a} \quad (\text{Magnetic field of a straight wire}) \quad (20.6)$$

The magnetic field lines form concentric circles with the wire as the axis. The vector \vec{B} will be tangent to these circles (Figure 20.4).

Magnetic Field of a Current Loop

We wish to find the magnetic field at a point along the axis of a circular loop with radius a and carrying a current I . Let us choose the small length element $d\ell$ along the wire as the arc of circle ds (Figure 20.5). Let us place this arc ds on the y -axis and in the $+z$ direction. Let point P be located on the x -axis at distance h . These choices do not affect the generality because of the symmetry of the loop.

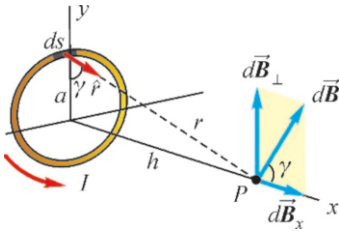


Figure 20.5: The contribution dB of a piece of arc ds to the magnetic field.

As seen in the figure, the contribution dB of this current element $I ds$ will be perpendicular to both ds and the unit vector \hat{r} , and therefore will be located in the xy -plane. Also, the angle between ds and \hat{r} will be $\theta = 90^\circ$, because every vector located on the plane perpendicular to ds will be perpendicular to ds .

If we write this contribution dB according to the Biot-Savart law, we get

$$dB = k' \frac{I ds \sin 90^\circ}{r^2} = k' I \frac{ds}{a^2 + h^2}$$

Let us separate the vector $d\vec{B}$ into two components, one in the x -direction and the other perpendicular to it:

$$dB_x = dB \cos \gamma$$

$$dB_\perp = dB \sin \gamma$$

If we move the arc ds around the loop and add each contribution $d\vec{B}$, the contributions dB_\perp will make a circle around point P and give zero contribution due to symmetry:

$$\int dB_\perp = 0 \quad (\text{due to symmetry})$$

Therefore, the contributions in the direction of the x -axis will give a total magnetic field in the x -direction:

$$B = \int dB_x = \int dB \cos \gamma$$

The γ angles shown in two places in the figure are equal because they are angles with perpendicular sides. Accordingly, we get $\cos \gamma = a/r = a/\sqrt{h^2 + a^2}$ and the

integral simplifies as follows:

$$\begin{aligned} B &= \int dB \cos \gamma = k' I \int \frac{ds}{a^2 + h^2} \frac{a}{\sqrt{h^2 + a^2}} \\ &= \frac{k' I a}{(h^2 + a^2)^{3/2}} \oint ds \end{aligned}$$

All of the constants are taken outside of the integral here and the \oint sign is used to indicate that the elements ds form a closed curve around the loop.

The sum of the pieces of arc around a complete loop gives the circumference of the circle. Therefore, if the value of the integral is taken as $2\pi a$, we find the result as follows:

$$B = \frac{2\pi k' I a^2}{(h^2 + a^2)^{3/2}} \quad (\text{Magnetic field on the axis of a loop}) \quad (20.7)$$

The magnetic field at the center of the loop is found by taking $h = 0$ in this formula:

$$B = \frac{2\pi k' I}{a} \quad (\text{Magnetic field at the center of a loop}) \quad (20.8)$$

Figure 20.6 shows the magnetic field lines of a current loop. The lines are not parallel to the axis but curve over it at points outside of the axis.

Magnetic Field of a Magnetic Dipole

In section 19.3, we defined the magnetic dipole moment of a current loop as follows (Eq. 19.5):

$$m = I A$$

A is the surface area of the loop. Let us return to Eq. (20.7), which we found above for the magnetic field on the axis of a current loop:

$$B = \frac{2\pi k' I a^2}{(h^2 + a^2)^{3/2}}$$

We can easily form a dipole moment in this expression. As the surface area of the circle is πa^2 , we get

$$B = \frac{2k'(I\pi a^2)}{(h^2 + a^2)^{3/2}} = \frac{2k'm}{(h^2 + a^2)^{3/2}}$$

At distances very far from the magnetic dipole, in other words, when $h \gg a$, the approximate expression of magnetic field becomes as follows:

$$B \approx \frac{2k'm}{h^3} \quad (\text{Field of a magnetic dipole at large } h) \quad (20.9)$$

Considering a current loop as a magnetic dipole has many advantages: An electron rotating around the nucleus in an atom generates a circular current loop and has an orbital magnetic dipole moment. As we shall discuss later in this chapter, the magnetic properties of matter can be explained by taking atoms to be very small magnetic dipoles.

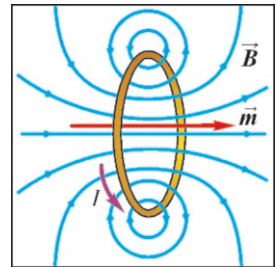


Figure 20.6: Magnetic field lines of the current loop and magnetic dipole moment.

Example 20.1

The Earth's magnetic field is $5 \times 10^{-5} \text{ T}$.

- (a) At what distance from a straight wire carrying a current of 1 A will the magnetic field be equal to this value?
 (b) How much current should pass through a circular loop with a radius of 1 m for the magnetic field at its center to be equal to this value?

Answer

(a) We use Eq. (20.6) for the magnetic field of a straight wire:

$$B = \frac{2k'I}{a}$$

We set this expression equal to the Earth's magnetic field and

find the distance a :

$$a = \frac{2k'I}{B} = \frac{2 \times 10^{-7} \times 1}{5 \times 10^{-5}}$$

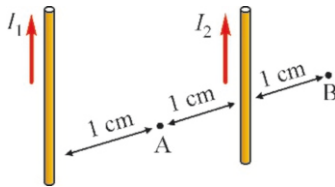
$$a = 0.004 \text{ m} = 4 \text{ mm}$$

(b) We use Eq. (20.6), which gives the magnetic field at the center of a circular loop:

$$B = \frac{2\pi k'I}{a}$$

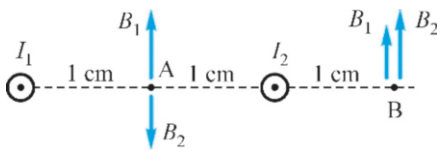
We again set this expression equal to Earth's magnetic field and calculate the current:

$$I = \frac{aB}{2\pi k'} = \frac{1 \times 5 \times 10^{-5}}{2 \times 3.14 \times 10^{-7}} = 80 \text{ A}$$

Example 20.2

In the figure, there is a 2 cm distance between the parallel currents $I_1 = 100 \text{ A}$ and $I_2 = 200 \text{ A}$.

- (a) Calculate the total magnetic field at the midpoint A between the wires.
 (b) Calculate the magnetic field at point B located 1 cm away from the wire I_2 .

**Answer**

(a) First, let us determine the directions of the magnetic fields B_1 and B_2 at both points. Using the right-hand rule, the

magnetic fields will appear as viewed from the top of the paper in the figure above.

Then, we use the formula $B = 2k'I/a$ to calculate the magnetic field of each current:

At point A:

$$B_1 = \frac{2k'I_1}{a} = \frac{2 \times 10^{-7} \times 100}{0.01} = 0.002 \text{ T} = 2 \text{ mT}$$

$$B_2 = \frac{2 \times 10^{-7} \times 200}{0.01} = 4 \text{ mT}$$

We take the resultant vector of the two opposite vectors at point A:

$$B_A = B_1 - B_2 = 0.002 - 0.004 = -2 \text{ mT}$$

At point B:

$$B_1 = \frac{2 \times 10^{-7} \times 100}{0.03} = 0.00067 \text{ T} = 0.67 \text{ mT}$$

$$B_2 = \frac{2 \times 10^{-7} \times 200}{0.01} = 0.004 \text{ T} = 4 \text{ mT}$$

We take the resultant vector of the two vectors in the same direction at point B:

$$B_B = B_1 + B_2 = 0.67 + 4 = 4.7 \text{ mT}$$

Example 20.3

1 A of current is flowing through a circular conducting loop with a radius of 1 m.

- (a) What is the magnitude of the magnetic field at the center of the loop?
 (b) At what point along the axis will the magnetic field drop down to half of its value at the center?

Answer

(a) We use Eq. (20.8) to find the magnetic field at the center of the current loop:

$$B = \frac{2\pi k'I}{a}$$

We substitute the numerical values and calculate the magnetic field:

$$B = \frac{2 \times 3.14 \times 10^{-7} \times 1}{1} = 6.3 \times 10^{-7} \text{ T}$$

(b) There is no need to recalculate the value of the magnetic field here. We only find the ratios of the expressions at the center and at distance h . The expression for the magnetic field at point h along the axis of a current loop was given with in Eq. (20.7):

$$B = \frac{2\pi k'I a^2}{(h^2 + a^2)^{3/2}}$$

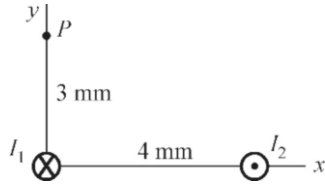
Let us use B_0 to indicate the expression that we wrote above for the field at the center of the loop, and take the ratio of these two fields:

$$\frac{B}{B_0} = \frac{2\pi k'I a^2}{(h^2 + a^2)^{3/2}} \times \frac{a}{2\pi k'I} = [(h/a)^2 + 1]^{-3/2}$$

We substitute the values $B=B_0/2$ and $a=1$ and solve for h :

$$[h^2 + 1]^{3/2} = 2 \rightarrow h = 0.77 \text{ m}$$

Example 20.4



In the figure, the straight wires perpendicular to the plane of the paper have currents $I_1 = 60\text{ A}$ and $I_2 = 50\text{ A}$. Calculate the components of the total magnetic field at point P.

Answer

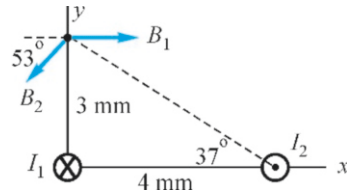
We first use Eq. (20.6) to calculate the magnitudes of the fields B_1 and B_2 at point P:

$$B = \frac{2k'I}{a}$$

$$B_1 = \frac{2 \times 10^{-7} \times 60}{0.003} = 0.004\text{ T} = 4\text{ mT}$$

$$B_2 = \frac{2 \times 10^{-7} \times 50}{0.005} = 0.002\text{ T} = 2\text{ mT}$$

We then use the right-hand rule to find the directions of these magnetic field vectors. The direction of both fields will be as follows:



Lastly, we find the sum of these two vectors in terms of their components:

$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

$$B_x = B_1 - B_2 \cos 53^\circ = 4 - 2 \times 0.6 = 2.8\text{ mT}$$

$$B_y = -B_2 \sin 53^\circ = -2 \times 0.8 = -1.6\text{ mT}$$

The magnitude and direction of the total magnetic field can be calculated from these components if required.

20.3 FORCE BETWEEN PARALLEL CURRENTS – AMPERE UNIT

In Chapter 19, we calculated the force exerted upon a current placed inside of a magnetic field. We now know the magnetic field produced by current-carrying wires. Therefore, we can understand the interaction between two current-carrying wires: The magnetic field generated by one of the wires will exert a force upon the second wire.

Let us consider two straight parallel wires with distance d in between (Figure 20.7). Suppose currents I_1 and I_2 are flowing through these wires in the same direction. The magnetic field generated by the current I_1 at distance d is given by the equation (20.6):

$$B_1 = \frac{2k'I_1}{d}$$

In Chapter 19, we saw that the force exerted upon the length L of the second wire in this magnetic field B_1 is $\vec{F} = I(\vec{L} \times \vec{B})$ (equation 19.1):

$$\vec{F} = I_2 (\vec{L} \times \vec{B}_1)$$

As shown in Figure 20.7, the magnetic field B_1 is perpendicular to the other wire with current I_2 . In such a case, the force will also be perpendicular to the current I_2 and the field B_1 and be towards the wire I_1 . In other words, the two wires will attract each other. Let us calculate the magnitude of the force:

$$F = LI_2 B_1 \sin 90^\circ = LI_2 \frac{2k'I_1}{d} = 2k'L \frac{I_1 I_2}{d}$$

We could have done this calculation for the force exerted upon the current I_1 in the magnetic field generated by the current I_2 . In such a case, we would have found the same force to be equal and in the opposite direction in accordance with Newton's third law.

If the currents were taken as opposite to each other (anti-parallel), we would have found that the wires repelled each other. Therefore, *parallel currents attract and anti-parallel currents repel each other*.

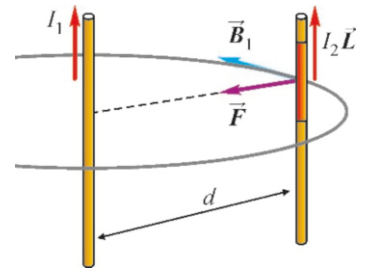


Figure 20.7: The magnetic field of two parallel currents exerts a force upon the other.

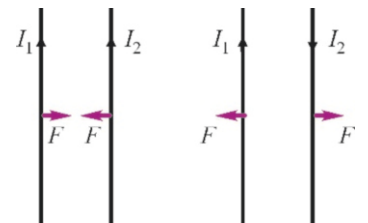


Figure 20.8: Parallel currents attract and anti-parallel currents repel each other.

The magnitude of the force in both cases is as follows:

$$F = \frac{2k' I_1 I_2}{d} L \quad (\text{Force between parallel currents}) \quad (20.10)$$

Definition of the Ampere Unit

Let us calculate the expression for force between parallel currents that we found above, between two equal currents $I_1 = I_2 = 1$ A separated by $d = 1$ m. The force exerted upon a unit length of wire will be as follows:

$$\frac{F}{L} = \frac{2k' I_1 I_2}{d} = \frac{2 \times 10^{-7} \times 1 \times 1}{1 \text{ m}} = 2 \times 10^{-7} \text{ N/m}$$

This expression is the definition of the unit of current **ampere** (A), which is one of the basic units in the SI system:

When a force of 2×10^{-7} N/m is exerted per unit length between two parallel wires separated by 1 m and carrying identical currents, the current flowing through the wires is 1 ampere (A).

This definition allows for the most accurate measuring assembly that can be installed in a laboratory. An instrument called the *Watt balance* (Figure 20.9) is used to keep a current-carrying weight in balance with the same current, and it is thus ensured that the current in both wires is identical.

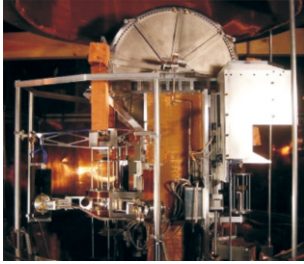
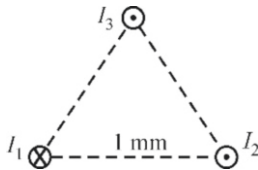


Figure 20.9: The Watt balance used to measure the force between currents.

Example 20.5



In the figure, wires perpendicular to the plane of the paper constitute the corners of an equilateral triangle with side lengths of 1 mm. Since $I_1 = I_2 = I_3 = 100$ A, calculate the total force exerted upon 1 m of length on the wire with current I_3 .

Answer

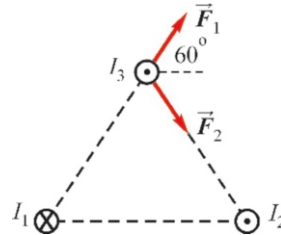
The force between parallel currents is given in Eq. (20.10):

$$F = \frac{2k' I_1 I_2}{d} L$$

The magnitude of the forces exerted by the currents I_1 and I_2 are equal, because the magnitude of the currents and distances are equal:

$$F_1 = F_2 = \frac{2 \times 10^{-7} \times 100^2}{0.001} \times 1 = 2 \text{ N}$$

We use the practical rule to find the directions of these two forces: Parallel currents attract and anti-parallel currents repel each other. Accordingly, the forces on current I_3 are as follows:

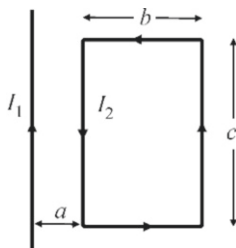


The sum of two equal forces making equal angles with the horizontal will be towards their angle bisector, in other words, horizontal:

$$F = F_{1x} + F_{2x} = 2F_1 \cos 60^\circ$$

$$F = 2 \text{ N}$$

Example 20.6



A current of $I_2 = 4$ A flows through the rectangular loop laying on the same plane as the straight wire carrying a current of $I_1 = 25$ A, as shown in the figure. The distance of the loop to the near side is $a = 1$ cm, its width is $b = 3$ cm and length is $c = 5$ cm. Calculate the net force exerted upon the loop.

Answer

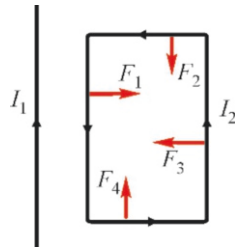
We first write the magnetic field of the straight wire:

$$B_1 = \frac{2k'I_1}{r}$$

Its direction is into the plane of the paper inside of the loop.

Eq. (19.1) is used to find the force exerted upon the piece of current I_2 with length L in this magnetic field:

$$\vec{F} = I_2 (\vec{L} \times \vec{B}_1)$$



Using the right-hand rule to calculate this vector product, the

forces exerted upon four sides are in the directions shown in the figure:

The forces F_2 and F_4 do not contribute to the total force, as they are equal and in the opposite direction. Therefore, it is sufficient to calculate the forces F_1 and F_3 :

$$F_1 = I_2 L B_1 = 2k' I_1 I_2 \frac{c}{a}$$

$$= 2 \times 10^{-7} \times 25 \times 4 \frac{5}{1} = 1 \times 10^{-4} \text{ N}$$

$$F_3 = 2 \times 10^{-7} \times 25 \times 4 \frac{5}{1+3} = 0.125 \times 10^{-4} \text{ N}$$

The difference of these two forces gives the total force as being towards the right:

$$F = F_1 - F_3 = 0.875 \times 10^{-4} \text{ N}$$

20.4 AMPERE'S LAW

In using the Biot-Savart law, we have to take an integral to find the magnetic field of a current distribution. These integrals are often complicated. However, we can find the magnetic field without having to integrate if the current distribution has some symmetry. This is possible with Ampère's law.

Let us consider a simple case to understand the essence of Ampère's law. We had previously calculated the magnetic field of an infinite straight wire. We had found the magnetic field at distance r from a wire carrying a current I in Eq. (20.6):

$$B = \frac{2k'I}{r}$$

Now, let us take the integral of the magnetic field's component that is tangent to the circle with radius r . As shown in Figure 20.10, we multiply the tangent component of the magnetic field B at each point with the small element of arc ds at that point and add them over the circle. This is different from the familiar integral that we take along the x -axis. It is called a **line integral**, because it is taken along a curved line (circle). As the magnetic field of an infinite wire is tangent to the circle with radius r anyway, it is sufficient to take its integral:

$$\oint B ds = B \oint ds$$

The magnetic field B was taken outside of the integral, as it has the same value everywhere on the circle. The integral of small elements of arc ds over the full circle will be the circumference $2\pi r$. If we also use the expression for magnetic field, we get

$$\oint B ds = B 2\pi r = \frac{2k' I}{\chi} 2\pi \chi = 4\pi k' I = \mu_0 I$$

In the last expression, we replaced $k' = \mu_0/4\pi$ with the magnetic permeability of vacuum μ_0 .

It is surprising that the result is independent of radius r . Of course, in this particular case, this resulted from the fact that the magnetic field of an infinite wire is inversely proportional to r . However, advanced integration techniques can be used to prove that the integral of any current distribution over any closed curve gives the same result in the most general case.

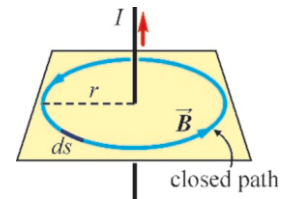


Figure 20.10: Closed circular curve around a straight wire.

The result would have been zero if this integral had been taken over any curve that left the current I outside. If we accept this without a proof, Ampère's law is expressed as follows:

Ampère's Law

The line integral of the tangential component of the magnetic field over any closed curve is proportional to the net current crossing any surface bounded by the closed curve:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{encl}} \quad (20.11)$$

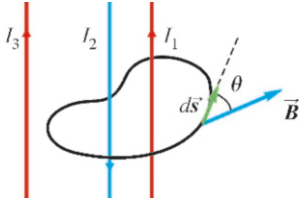


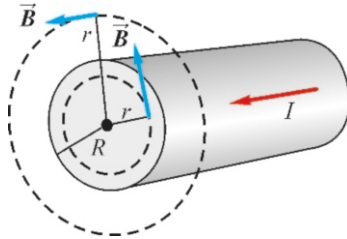
Figure 20.11: The currents I_1 , I_2 are taken into consideration according to Ampère's law. The current I_3 outside of the loop is not taken into consideration.

In this formula, I_{encl} indicates the net current enclosed by the closed loop, in other words, if the current in one direction is positive, the current in the negative direction is included as negative. Currents outside of the loop are not taken into consideration. The scalar product $\vec{B} \cdot d\vec{s}$ shows that the projection of the magnetic field along the path is to be taken.

It is not necessary to carry out this integral when applying Ampère's law. We take the magnetic field B outside of the integral by examining the symmetry of the problem, and this makes it easier to calculate the left-hand side. It is only required to add the currents on the right-hand side.

Example 20.7

The current I flowing through an infinite cylindrical conductor with radius R is evenly distributed throughout its cross-section. Find the magnetic field outside and inside of the cylinder.



Answer

It is always necessary to find a symmetry to apply Ampère's law. The most basic symmetry in magnetism is the magnetic field of a straight wire current. The magnetic field of a straight wire is generated as tangent to circles centered around the wire. This should be the starting point if nothing disrupts this symmetry.

Outside of the wire: Let us consider an imaginary circle with radius $r > R$. Looking from a point on this circle, both halves of the wire will give the same contribution because of the symmetry. Therefore, the magnetic field should be in a plane perpendicular to the wire. Again, looking from a point on this circle, the half sections along the wire would give the same contributions that would result in a magnetic field along the tangent of the circle. (We could also deduce that dB is in the tangential direction from the Biot-Savart law, dB being

perpendicular to both $d\ell$ and r .)

According to Ampère's law Eq. (20.11), the integral of this tangential magnetic field along the circle with radius r should be equal to the enclosed net current:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{encl}}$$

As B is parallel to the path, we have $\vec{B} \cdot d\vec{s} = B ds$ and take the constant value B outside of the integral. The remaining integral is equal to the circumference of the circle $2\pi r$. The current on the right-hand side is the total current I :

$$B(2\pi r) = \mu_0 I \rightarrow B = \frac{\mu_0 I}{2\pi r} = \frac{2k'I}{r}$$

This result is the same as the straight wire expression.

Inside of the wire: Let us again consider an imaginary circle with radius $r < R$. Again, according to symmetry, the magnetic field on the circle should have equal value at every point on the circle and be tangent to the circle. We write Ampère's law as follows:

$$\oint B ds = B(2\pi r) = \mu_0 I_{\text{encl}}$$

This time, the current inside of the circle is less. As the current is uniformly distributed over the cross-section of the cylinder with radius R , we use proportions to find the amount enclosed inside of the radius r :

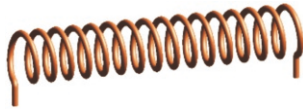
$$I_{\text{encl}} = \frac{I}{\pi R^2} \cdot \pi r^2$$

We substitute this expression and find B :

$$B = \frac{\mu_0 I r}{2\pi R^2} = \frac{2k'I}{R^2} r$$

Example 20.8

Solenoid. Calculate the magnetic field inside and outside of a solenoid with infinite length and carrying a current I . The number of windings per unit length is n .



Answer

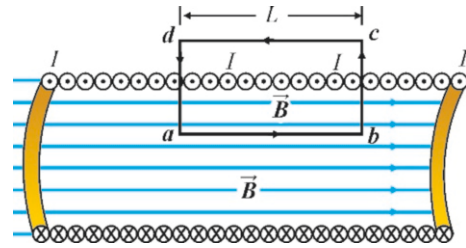
We obtain a solenoid by taking a conducting wire and winding it spirally around a long cylinder. Recall from the topic of electric fields that a plane capacitor generated uniform electric field. Likewise, we use a solenoid to generate a uniform magnetic field.

In Section 20.2, we found the magnetic field of a circular current loop and showed that it had the following distribution:



Now, let us consider that we place a second identical current loop next to it. Performing vector summation of the magnetic field at every point, we observe that lines in the interior region get more and more parallel to the axis and exit the loop much later.

When infinitely many such loops are placed side by side, the magnetic field inside of the solenoid becomes parallel to the axis and approaches zero outside. As a result, we obtain the following structure:



Now, let us apply Ampère's law along the rectangular path $abcd$ shown in the figure above:

$$\oint \vec{B} \cdot d\vec{s} = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a \right] \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{encl}}$$

Only the first of these integrals will be non-zero. This is because, in the 2nd and 4th integrals (paths bc and da), the vector \vec{B} will be perpendicular to the path $d\vec{s}$ and the scalar product will be zero. The third integral is completely outside of the solenoid where it was shown that $B = 0$. As the value of B is the same along ab , we take it outside of the integral. If we use L to indicate the distance ab , the left-hand side becomes as follows:

$$BL = \mu_0 I_{\text{encl}}$$

Now let us find the value of I_{encl} on the right-hand side. Since n is the number of windings of the solenoid per unit length, the number of windings inside of the rectangle will be nL . As current I is passing through each one, the amount of current inside will be nLI :

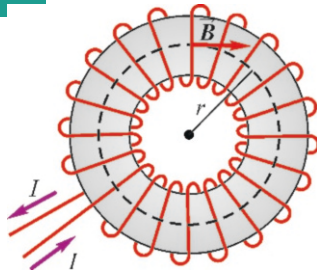
$$BL = \mu_0 nLI$$

From here, we find the expression for the magnetic field inside of the solenoid:

$$B = \mu_0 nI \quad (\text{Magnetic field of a solenoid}) \quad (20.12)$$

Note that the result is not dependent on radius. The magnetic field has this uniform value at every point inside of the volume of the solenoid.

Example 20.9



Toroid. Calculate the magnetic field at a distance r inside of a solenoid consisting of N total windings wound in the shape of a toroid (donut) carrying a current I .

Answer Let us consider the toroid as consisting of N current loops. As the magnetic field of each loop on its axis is along that axis, the structure of the toroid will not change this symmetry and the magnetic field inside will circulate

around the toroid in the direction of the axes of loops.

Now let us consider an imaginary loop with radius r inside of the toroid and apply Ampère's law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{encl}}$$

As \vec{B} is parallel to the path along the circle, the scalar product becomes $B ds$ and we take B outside of the integral, because it does not vary along the path. Also, I_{encl} on the right-hand side is the sum of N currents with value I :

$$B(2\pi r) = \mu_0 NI$$

From here, we find the magnetic field of the toroid:

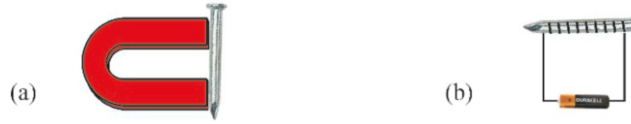
$$B = \frac{\mu_0 NI}{2\pi r} = \frac{2k'NI}{r}$$

If we had performed the same operation for a circle outside of the toroid, we would have found the magnetic field there to be zero.

20.5 MAGNETIC PROPERTIES OF MATTER

It is possible to turn a non-magnetic iron nail into a magnet. This can be performed in two ways: first, by keeping the nail in contact with the poles of a horseshoe magnet for a while, as shown in Figure 20.12-a. When we detach it later, we observe that the nail also starts to attract needles, in other words, that it has gained permanent magnetism.

Figure 20.12: We magnetize a nail through two methods: (a) Contact with a permanent magnet, (b) Placing it inside of the magnetic field of a solenoid.



The second way is to place a non-magnetic iron nail in the gap between the windings of a solenoid and send a current through the solenoid for a while (Figure 20.12-b). This nail is again observed to gain magnetic property later.

These observations show that magnetism is intrinsic to matter. What is the source of this magnetism? Why can magnetism be permanent in some materials, such as iron and nickel? We need to examine the atomic structure of matter in order to answer these questions.

Magnetic Dipole Moment of Atoms

Negatively charged electrons (e^-) in an atom rotate around a positive nucleus (Figure 20.13). In classic physics, they are assumed to rotate in a circular path with radius r at speed v . Accordingly, we can consider that the electron generates a current around the nucleus. Since the current is the amount of charge passing through a cross-section per unit time, one pass of the charge e taking place in one period $T=2\pi r/v$ will give a current of:

$$I = \frac{\Delta q}{\Delta t} = \frac{e}{T} = \frac{e}{2\pi r/v} = \frac{ev}{2\pi r}$$

(In modern physics, this model is not exactly correct, because neither the speed nor the orbit radius of the electron are well-defined quantities. Nevertheless, the classical model is useful to have a rough idea.)

We had studied this current loop previously (in Section 20.2) and calculated its magnetic field and its magnetic dipole moment. If we write the definition of the magnetic dipole moment here for the electron, we get

$$m = IA = \left(\frac{ev}{2\pi r}\right) \pi r^2 = \frac{1}{2} evr$$

Therefore, atoms have a magnetic dipole moment caused by the rotation of electrons in their orbit.

In Chapter 7, we defined the angular momentum of a point mass as $L=mv r$. If we use m_e here to indicate the mass of the electron, we get

$$m_L = \frac{e}{2m_e} L$$

We thus see that there is a relation between angular momentum and magnetic dipole moment: Each rotating particle generates an orbital magnetic moment that depends on its angular momentum.

Electrons also have another angular moment called the **spin**, which is intrinsic to them (Figure 20.14). The spin angular moment has no classic explanation, but

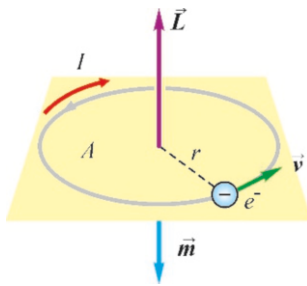


Figure 20.13: The current and magnetic moment \vec{m} generated by the orbital motion of an electron.

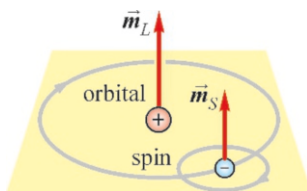


Figure 20.14: The magnetic moment of the atom consists of the contributions of spin and orbit.

you may consider the electron as a spinning top rotating about its own axis. Therefore, in addition to the orbital magnetic moment, it is necessary to add a **spin magnetic moment**, indicated with m_s :

$$\vec{m} = \vec{m}_L + \vec{m}_s$$

Consequently, every atom generates a magnetic field caused by its total magnetic dipole moment \vec{m} . When placed in an external magnetic field, it contributes to and changes the net surrounding magnetic field.

Magnetization (\vec{M})

In section 19.3, we found the torque exerted upon the magnetic dipole moment of a current-carrying loop in an external magnetic field (Eq. 19.6):

$$\tau = mB \sin \theta$$

The atoms of a material at the macroscopic scale usually have magnetic moments in random directions, and therefore do not produce a net macroscopic magnetic moment. However, when this object is placed in an external magnetic field \vec{B}_0 , the torque exerted upon the magnetic moments of the atoms try to rotate them. The object thus gains a macroscopic magnetic moment.

The net magnetic moment per unit volume is called **magnetization** :

$$\vec{M} = \frac{\sum_i \vec{m}_i}{V} \quad (20.13)$$

The magnetic field generated by this magnetization is found to have the value $\vec{B}_M = \mu_0 \vec{M}$. Here, μ_0 is the magnetic permeability of free space that we defined earlier.

Therefore, the net magnetic field for a material medium is as follows:

$$\vec{B} = \vec{B}_0 + \mu_0 \vec{M}$$

Let us write this expression as follows to bring the contributions to magnetic field in the same dimension:

$$\vec{B} = \mu_0 \left(\frac{\vec{B}_0}{\mu_0} + \vec{M} \right)$$

The first term in the brackets is the contribution of external currents to the magnetic field inside of the matter, but it is written in the magnetization dimension. This term is called the **magnetic field strength vector** and is indicated with \vec{H} :

$$\vec{H} = \frac{\vec{B}_0}{\mu_0} \quad (20.14)$$

Accordingly, the magnetic field inside the material can be written as follows:

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \quad (20.15)$$

If there is no magnetization in the medium, in other words, if $\vec{M} = 0$, this expression gives the magnetic field \vec{B}_0 in free space.

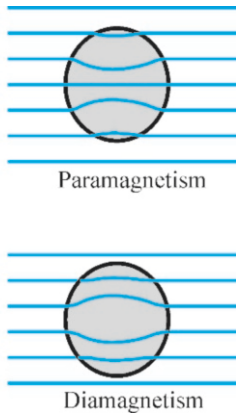


Figure 20.15: The change of magnetic field in paramagnetic and diamagnetic media.

Paramagnetism and Diamagnetism

The magnetic moments of the electrons in a given atom usually cancel each other out and do not generate a net magnetic moment. However, some materials (aluminum, platinum, calcium, sodium, etc.) have a non-zero atomic magnetic moment. The torque generated when these types of materials are placed in an external magnetic field will try to rotate the magnetic moment in the direction of the field, and a contribution that *increases the external magnetic field* will thus be generated. This is called **paramagnetism**.

On the other hand, in certain other materials (gold, silver, copper, lead, etc.), although the magnetic moment is zero, an opposing reaction is given when they are placed in an external magnetic field. The orbit of the electrons changes and a magnetic moment is generated again. However, this magnetic moment gives a contribution that *reduces the external magnetic field*. This is called **diamagnetism**.

Figure 20.15 shows the change in magnetic field in these two types of medium. Magnetic field increases in a paramagnetic medium (dense field lines) and decreases in a diamagnetic medium (sparse field lines).

In both types of magnetization, the magnetization vector \vec{M} of the medium is proportional to the magnetic field strength vector \vec{H} :

$$\vec{M} = \chi_m \vec{H}$$

The dimensionless constant χ_m here is called **magnetic susceptibility**. χ_m is positive for paramagnetic media and negative for diamagnetic media.

Accordingly, if we write Eq. (20.15), which we found above for magnetic field \vec{B} ,

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1 + \chi_m) \vec{H}$$

Just as we defined ϵ to replace ϵ_0 for the electric field, here we define the **magnetic permeability of the medium**, indicated with μ , as follows:

$$\mu = (1 + \chi_m) \mu_0 \quad (20.16)$$

Depending on the type of medium,

$$\begin{aligned} \text{in a paramagnetic medium :} & \quad \mu > \mu_0 \\ \text{in a diamagnetic medium :} & \quad \mu < \mu_0 \end{aligned}$$

We thus find the expression for \vec{B} that takes into account the magnetic properties of the medium:

$$\vec{B} = \mu \vec{H} \quad (20.17)$$

Ferromagnetism

The magnetism of paramagnetic and diamagnetic materials occurs only when an external magnetic field is applied. The magnetization disappears when the external magnetic field is removed.

However, in five metals (iron, nickel, cobalt, gadolinium and dysprosium) and certain oxide alloys, magnetism does not disappear when the external magnetic field is removed. This permanent magnetization is called **ferromagnetism**. Ferromagnetic materials are used in the structure of credit cards, computer memories, speakers, motor cores, compasses, etc.

Ferromagnetic materials have very strong magnetism. Their magnetization values M can be more than 1000 times compared to paramagnetic materials.

However, this magnetization continues until a certain critical temperature, called the **Curie temperature**. (Curie temperature for iron is $770\text{ }^\circ\text{C}$.) When this temperature is exceeded, the material again returns to the paramagnetic phase with a sudden phase transition.

There is no satisfactory classic explanation for ferromagnetism; one needs quantum theory for that. Let us briefly provide some essential concepts here as information.

When the microscopic structure of ferromagnetic materials is examined, it is observed that it consists of small regions where magnetic moments of the atoms are aligned together. These regions are called **domains** (Figure 20.16). Although these domains have a net magnetism, they do not lead to a net magnetism, because each one is in a random direction. When placed in an external magnetic field, the domains in the direction of the field are observed to enlarge and those in the other directions are observed to shrink. In other words, the domain walls move so as to increase the net magnetization.

A very interesting change occurs in magnetism when we place a ferromagnetic material in an external magnetic field B_0 (Figure 20.17). As the external field B_0 is increased, the magnetization M also increases (curve ab in the figure). This increase continues up until a value called the **saturation** (point b). At this point, the magnetic moments of all of the atoms are in parallel with the external field and no further increase takes place.

Then, when B_0 is decreased, this time the value of M decreases along a different path (curve bc). As shown in the figure, although $B_0 = 0$ at point c , a permanent magnetization value M_r , called **remanence**, remains. If the external magnetic field is reversed, (curve cd), magnetization also decreases and forms in the reverse direction, slightly delayed. When we start to reduce the magnitude of the external field in the negative direction (curve de), this time, a permanent magnetization occurs in the reverse direction (point e). We thus observe a closed curve with different forward and backward paths. This is called a **hysteresis curve**.

The hysteresis curve also explains why ferromagnetic materials are used in memory chips. As seen in the curve, the magnetization value of the ferromagnetic material has the value M_r if the magnetic field is reduced in one direction and the value $-M_r$ if reduced in the other direction. In other words, it has a memory and remembers how it reached zero magnetic field.

Earth's Magnetic Field

Earth's magnetic field is what directs the needle of a compass towards the North Pole. Measurements show that the Earth has an average magnetic field of 10^{-4} T . Examining the distribution of this field from space, we can see that it acts as if a giant magnet was placed at the center of the earth. However, note that the North pole of this magnet (N) is at the geographic South pole (Figure 20.18). In other words, the magnetic field lines come out of Earth's South pole and merge at the North pole. The North pole of a compass can only be attracted to the geographical North in such a case. The magnetic field of the Earth is not exactly parallel to the surface and also has a component that is perpendicular to the surface.

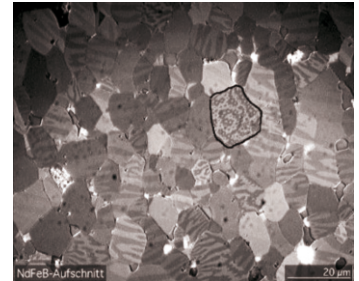


Figure 20.16: Ferromagnetic domain structure in the NdFeB crystal.

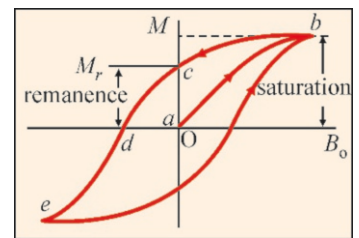


Figure 20.17: Hysteresis curve of ferromagnetic magnetization as a function of applied magnetic field B_0 .

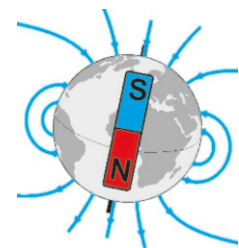


Figure 20.18: The Earth is a giant magnet, with the N-pole located at the south.



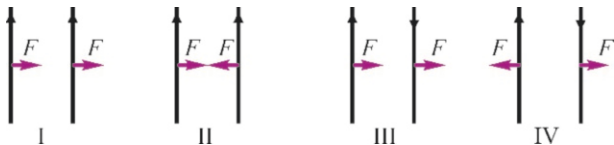
Figure 20.19: Homing pigeons can find their home from thousands of miles away. It has been established that they use small magnetic crystals in their beaks to detect Earth's magnetic field.

The magnetic poles where the field lines converge do not coincide with the geographical North and South Poles. The location of the magnetic pole varies with time. Research on rocks show that they vary over 1000-year periods. For example, according to measurements conducted in 2005, the magnetic North Pole was located around Ellesmere Island in Northern Canada, with latitude 82.7°N (north) and longitude 114.4°W (west), has been shifting towards Russia since then. Magnetic field also shows local variations on Earth. The magnetic field of every region has been mapped for use in maritime and air transportation.

The source of Earth's magnetic field is one of the greatest mysteries in science. At first, it was considered that layers of iron and nickel at the center of the Earth were the source; however, we know that ferromagnetism disappears at very high temperatures. Today, it is considered to be caused by convection currents generated by electrically charged underground molten lava.

Multiple-choice Questions

- Which of the following are the source of magnetic field?
 - Magnetic charges.
 - Currents.
 - Magnetic moments of atoms.
 (a) All (b) I & II (c) I & III (d) II & III
- Which of the following are correct?
 - Parallel currents repel each other.
 - Anti-parallel currents attract each other.
 - Parallel currents attract each other.
 - Anti-parallel currents repel each other.
 (a) I & II (b) III & IV (c) I & IV (d) II & III
- By what factor does the magnetic field increase when the current flowing through an infinite wire is doubled?
 (a) No change (b) 2 (c) 4 (d) $1/2$
- The radius of a current-carrying loop is doubled. By what factor will the magnetic field at the center of the loop increase?
 (a) No change (b) 2 (c) 4 (d) $1/2$
- What is the magnetic field at a 1 m distance from a straight wire carrying 1 A in current?
 (a) 1×10^{-7} T (b) 2×10^{-7} T (c) 3×10^{-7} T (d) 4×10^{-7} T
- What is the expression for the magnetic field at the center of a circular current loop?
 (a) $k'I/a$ (b) $k'I/a^2$ (c) $2k'I/a$ (d) $2\pi k'I/a$
- In which of the following figures are the forces between parallel currents shown correctly?
 -
 -
 -
 -
- What is the force exerted upon 1 m in length of one of two parallel wires separated by 1 m and each carrying 1 A of current?
 (a) 1×10^{-7} N (b) 2×10^{-7} N (c) 3×10^{-7} N
- Which of the following is Ampère's law?
 - The integral of the magnitude of the magnetic field over any closed curve is proportional to the net current enclosed by the curve.
 - The integral of the tangential component of the magnetic field over a straight line is proportional to the net current in that wire.
 - The integral of the magnetic field over any volume is proportional to the net current enclosed by the volume.
 - The integral of the tangential component of the magnetic field over any closed curve is proportional to the net current enclosed by the curve.
- By what factor does the magnetic field increase when the current of a solenoid is doubled?
 (a) No change (b) 2 (c) 4 (d) $1/2$
- By what factor does the magnetic field increase when the number of windings per unit length of a solenoid is doubled?
 (a) No change (b) 2 (c) 4 (d) $1/2$
- Which of the following is correct?
 - Magnetic field is zero inside of a solenoid.
 - Magnetic field is zero outside of a solenoid.
 - Magnetic field is constant outside of a solenoid.
 - Magnetic field is infinite outside of a solenoid.

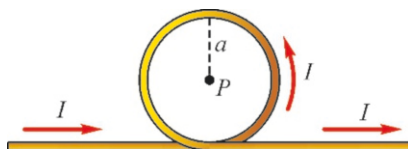


13. What is the expression for the magnetic field of a solenoid?
 (a) $\frac{1}{2}\mu_0 n I^2$ (b) $\mu_0 n I^2$ (c) $\mu_0 n I$ (d) $\mu_0 n^2 I$
14. Which of the following produces a uniform magnetic field?
 (a) Straight wire current.
 (b) Circular current loop.
 (c) Solenoid.
 (d) Toroid.
15. Which of the following is correct in terms of the magnetization properties of matter?
 (a) For paramagnetic materials: $\mu > \mu_0$
 (b) For diamagnetic materials: $\mu < \mu_0$
 (c) For ferro magnetic materials: $\mu \gg \mu_0$
 (d) All of the above.
16. Which of the following is the source of the magnetization property of materials?
 I. The orbital motion of atomic electrons.
 II. The spin property of electrons.
 III. The merging of magnetized domains.
 (a) I (b) I & II (c) I & III (d) I, II & III
17. Which of the following are correct?
 I. Magnetic field increases inside paramagnets.
 II. Magnetic field increases inside diamagnets.
 III. Magnetic field decreases inside paramagnets.
 IV. Magnetic field decreases inside diamagnets.
- (a) I (b) I & II (c) I & III (d) I & IV
18. Which of the following are correct for ferromagnetism?
 I. Magnetism increases with temperature.
 II. Magnetism decreases with temperature.
 III. Magnetism disappears above a certain temperature.
 (a) I (b) II (c) I & III (d) II & III
19. Which of the following is correct when a ferromagnetic material is placed inside of an external magnetic field?
 (a) It gets magnetized in the direction of the external magnetic field.
 (b) Magnetization increases with the external magnetic field.
 (c) Magnetization does not increase after a certain saturation value.
 (d) All of the above.
20. Which of the following are correct for Earth's magnetism?
 I. The geographic North Pole is the magnetic North Pole.
 II. The geographic South Pole is the magnetic North Pole.
 III. The locations of the magnetic poles vary over time.
 IV. The locations of magnetic poles do not vary over time.
 (a) I & III (b) II & III (c) I & IV (d) III & IV

Problems

20.2 Magnetic Field Calculations

20.1 (a) What is the magnetic field at a distance of 1 cm from a straight wire carrying a current of 10 A? (b) How much current should be flowing through a loop current with a radius of 1 cm such that a 1 mT magnetic field is generated at its center?
 [A: (a) 0.2 mT, (b) 16 A.]

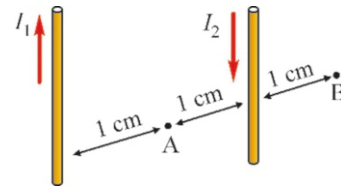


Problem 20.2

20.2 A part of an infinite wire carrying a current $I = 50$ A is bent into a circle with radius $a = 1$ cm, as shown in the figure. Find the magnitude and direction of the magnetic field

at point P located at the center of the circle.

[A: 4 mT, out of the paper.]



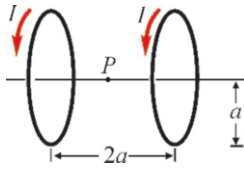
Problem 20.3

20.3 In the figure, there is a 2 cm distance between the anti-parallel currents $I_1 = 300$ A and $I_2 = 50$ A. (a) Calculate the total magnetic field at the midpoint A between the wires. (b) Calculate the magnetic field at point B located 1 cm away from the wire I_2 .
 [A: (a) 7 mT, (b) 1 mT.]

20.4 A current of 1 A is flowing through a conducting circular loop with a radius of 1 m. (a) What is the magnitude

of the magnetic field located at the center of the loop? (b) At what point along the axis will the magnetic field drop down to a quarter of its value at the center?

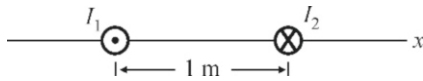
[A: (a) 6.3×10^{-7} T, (b) $h = 1.2$ m.]



Problem 20.5

20.5 Helmholtz coils. Two identical circular loops, each with radius $a=1$ cm and carrying the same $I=100$ A-currents in the same direction, are placed on an axis with $2a$ distance in between, as shown in the figure. Calculate the magnetic field at the midpoint between the loops. (This assembly is used to obtain uniform magnetic field in a small region.)

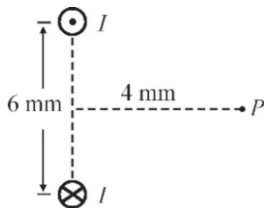
[A: 4.4 mT.]



Problem 20.6

20.6 The currents on two straight wires perpendicular to the plane of the paper, as shown in the figure, are $I_1 = 1$ A and $I_2 = 5$ A, respectively, and are separated by 1 m. At what point on the x -axis (left, center, right) will the total magnetic field be zero?

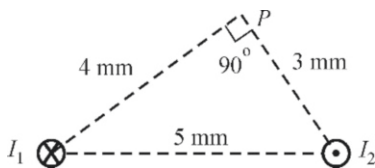
[A: 0.25 m on the left.]



Problem 20.7

20.7 The two straight wires perpendicular to the plane of the paper, as shown in the figure above, are separated by 6 mm and have equal currents of $I = 100$ A in opposite directions. Calculate the magnitude and direction of the total magnetic field at point P at 4 mm in distance on the perpendicular bisector of the currents.

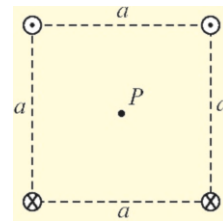
[A: 6.4 mT to the right.]



Problem 20.8

20.8 The two straight wires perpendicular to the plane of the paper, as shown in the figure above, are separated by 5 mm and have equal currents of $I = 60$ A. Calculate the magnitude and direction of the total magnetic field at point P , which forms a right triangle with the currents.

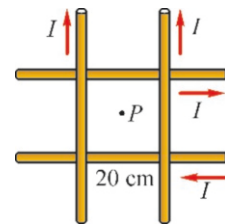
[A: 5 mT.]



Problem 20.9

20.9 The straight wire currents perpendicular to the plane of the paper and located on the corners of a square with side $a = 1$ mm are equal and have $I = 10$ A magnitude. Calculate the magnitude and direction of the total magnetic field at point P located at the center of the square. (Hint: The problem will not be as lengthy as it looks if you first determine the direction of each magnetic field.)

[A: 8 mT to the right.]

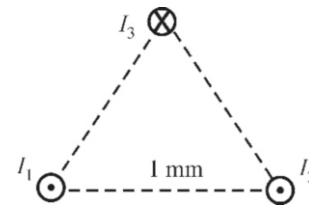


Problem 20.10

20.10 The four straight wire currents on the plane of the paper are equal and have magnitude $I = 50$ A. Calculate the magnitude and direction of the total magnetic field at point P located at the center of the square with a side length of 20 cm.

[A: 0.2 mT into the paper.]

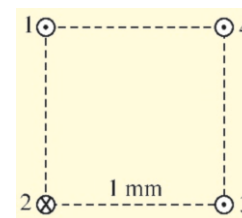
20.3 Force Between Parallel Currents



Problem 20.11

20.11 In the figure, wires perpendicular to the plane of the paper constitute the corners of an equilateral triangle with a side length of 1 mm. Since $I_1 = I_2 = I_3 = 60$ A, calculate the magnitude and direction of the total force exerted upon 1 m of length on the wire with current I_3 .

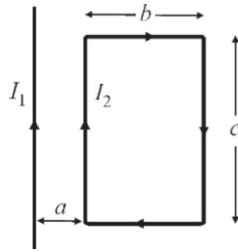
[A: 1.2 N upward.]



Problem 20.12

20.12 In the figure, wires perpendicular to the plane of the paper constitute the corners of a square with a side length of 1 mm. Since the magnitude of the current passing through each wire is $I = 100$ A, calculate the magnitude and direction

of the total force exerted upon 1 m of length on the 4th wire.
 [A: 1.4 N towards the inside from the diagonal.]



Problem 20.13

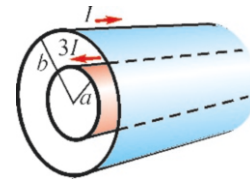
20.13 A current $I_2=10$ A flows through the rectangular loop laying on the same plane as the straight wire carrying a current of $I_1=50$ A as shown in the figure. The distance of the loop to the near side is $a=1$ cm, its width is $b=2$ cm and length is $c=60$ cm. Calculate the net force exerted upon the loop.
 [A: 0.004 N to the left.]

20.4 Ampère's Law

20.14 A current I flows through an infinite cylindrical shell with radius R . Calculate the magnetic field at a distance r inside and outside of the cylinder.

[A: $B = 0$ inside and $B = 2k'I/r$ outside.]

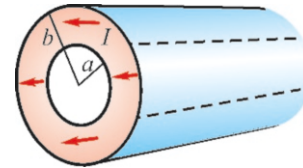
?



Problem 20.15

20.15 Of the two coaxial infinite cylindrical shells shown in the figure, the one with radius a carries a current $3I$ and the one with radius b carries a current I in the opposite direction. Calculate the magnetic field in all three regions.

[A: $B = 0$ for $r < a$, $B = 6k'I/r$ for $a > r > b$, $B = 4k'I/r$ for $r > b$.]



Problem 20.16

20.16 The total current I is evenly distributed inside of the infinite cylindrical region with inner radius a and outer radius b as shown in the figure. Calculate the magnetic field in all three regions.

[A: 0 for $r < a$, $B = 2k'I \frac{r^2 - a^2}{(b^2 - a^2)r}$ for $a < r < b$,

$B = 2k'I/r$ for $r > b$.]

21

FARADAY'S LAW – INDUCTION



Solar-powered aircraft *Helios* cruising the Hawaiian skies. Developed by NASA, the solar batteries located on the wings of this plane supply each of its 14 electric motors with 1.5-kilowatt (2 HP) of power.

What physics laws govern the operation of electric motors and generators? We will find the answer to these questions in Faraday's law.

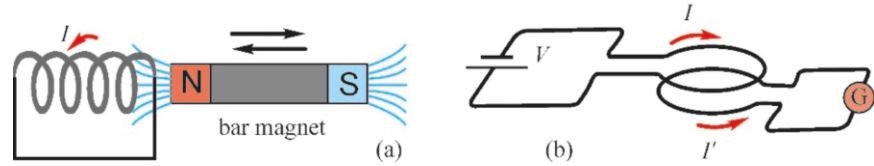
Towards the mid-19th century, it was established that electric currents would produce magnetic fields. Therefore, many people naturally asked the question, "Could magnetic field also lead to electric field?". Finally, in 1831, the British scientist Michael Faraday and the American scientist Joseph Henry succeeded in producing electric current through a changing magnetic field. No revolution has, to date, been greater than this one. Motors, generators, transformers, wireless energy, signal transmission, etc. were all developed as a result of this discovery.

The discovery of magnetic induction completed another great circle in science. The British scientist James Clerk Maxwell was able to unify electricity and magnetism in a single **electromagnetic theory** in which electric and magnetism induce and affect each other.

21.1 FARADAY'S LAW

A current is observed to be induced in a conducting loop when we move a bar magnet near it (Figure 21.1a). No current is induced if the magnet is stationary, regardless of its strength. This experiment shows us that current flows when *there is a change* in the magnetic field lines passing through the loop.

Figure 21.1: Two methods of inducing current: (a) A current is induced when a magnet moves near a solenoid. (b) A current is induced in one of two adjacent circuits when the current changes in one.



Moving a bar magnet is not the only way to change the magnetic field lines. Two circuits are shown in Figure 21.1b. The first circuit has a loop connected to a battery and, in the other circuit, there is a loop connected solely to a galvanometer. When the switch in the first circuit is closed, a brief current is observed in the second circuit. Here, a change in the magnetic field of the first loop also produces an electric current in the second loop.

Induction of current means that an emf is induced as if a battery were connected to the loop. In order to understand this effect, let us consider that we are moving a conducting rod with velocity \vec{v} towards the right on a plane perpendicular to the magnetic field (Figure 21.2). Recall the expression for magnetic force:

$$\vec{F} = q(\vec{v} \times \vec{B})$$

As the force \vec{F} exerted upon a free charge $+q$ inside of the conductor will be perpendicular to both the magnetic field and the velocity, it will be towards the top of the conductor. The $+q$ charges will thus gather at the top of the conductor and leave a negative end at the bottom. A potential difference will thus be induced between the two ends of the conductor.

Faraday's Law

We had discussed the concept of flux in examining Gauss' law in electric. Magnetic flux is also defined here as the number of magnetic field lines crossing an area A enclosed by a conducting wire, as follows:

$$\Phi_B = \int_A B dA \cos \theta \quad (\text{Magnetic flux}) \quad (21.1)$$

The angle θ here is the angle between the magnetic field vector and the surface normal \hat{n} (Figure 21.3).

Faraday's law states that a variation in magnetic flux induces an emf:

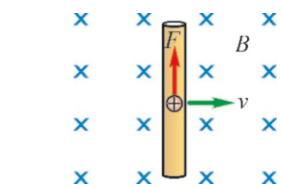


Figure 21.2: Conducting rod moving in a magnetic field.

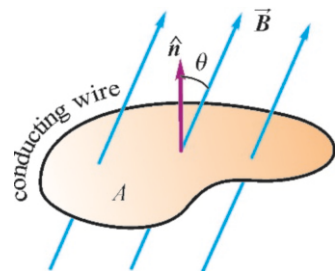


Figure 21.3: Magnetic flux crossing a surface enclosed by a conductor.

Faraday's Law

The potential difference induced in a circuit is directly proportional to the time rate of change of the number of magnetic field lines passing through the circuit:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad (21.2)$$

We shall explain the meaning of the negative sign in this expression using Lenz's law.

Faraday's law includes several ways in which the magnetic flux may vary:

- The magnetic field B may vary.
- The surface area A may vary.
- The angle θ between the surface and the magnetic field may vary.
- All or some of the above may vary simultaneously.

Lenz's Law

The negative sign ($-$) in Faraday's law is placed symbolically in the equation. This sign tells us the direction of the emf or that of the current to flow through the circuit. This direction is determined by *Lenz's Law*:

Lenz's Law

The current produced in a loop by the induced emf flows in a direction such that its magnetic field opposes the change in original magnetic flux.

In order to apply Lenz's law, let us first recall the magnetic field at the center of a circular loop carrying a current. According to the right-hand rule, when the four fingers are curved in the direction of current I , the thumb points in the direction of the magnetic field B .

Now let us consider the bar magnet in Figure 21.4a. The magnetic flux crossing the surface of the loop increases as the magnet approaches. The current opposing this must be induced in such a direction that its magnetic field B' is generated in the opposite direction, and thus decreases the increasing flux. According to the right-hand rule, this current I' should be in the direction shown in the figure. Likewise, the current produced will be in the opposite direction as the bar magnet moves away.

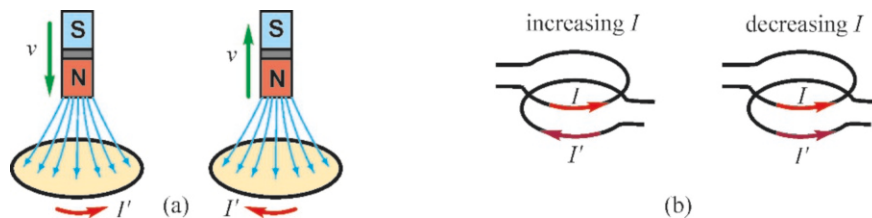
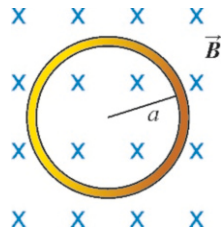


Figure 21.4: (a) Magnetic flux increases as the bar magnet approaches, therefore the induced current should be in the direction that generates an opposing magnetic field. (b) The current I' in the bottom circuit will be induced in the opposite direction as the current I increases in the top circuit, and in the same direction as I decreases.

Let us again consider two loops affecting each other, as shown in Figure 21.4b. If the current I increases in the loop connected to the battery, its magnetic field B will increase. The induced current to oppose this in the second loop will generate a current I' with a magnetic field B' in the opposite direction. Likewise, if the current I and field B decrease, the magnetic field in the second loop should be in the same direction so as to compensate for this decrease. Therefore, a current I' is induced in the same direction.

Let us now look at some examples to see how Faraday's law and Lenz's law are applied.

Example 21.1



The conducting loop with resistance $R = 1 \Omega$, as shown in the figure, is located in a magnetic field B directed into the plane of the paper.

- (a) The magnetic field starts to increase as $B = 0.1t$ (tesla), while the radius of the loop is constant at $a = 3$ m. Determine the value and direction of the current induced in the loop.
- (b) When the magnetic value is constant at $B = 0.1$, the radius of the loop starts to increase as $a = 3t$ (meters). Find the value and the direction of the current at $t = 1$ s.

Answer

We write Eq. (21.2) for Faraday's law:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

The magnetic flux Φ_B here is calculated using Eq. (21.1). As B is constant and parallel to the surface normal, we get

$$\Phi_B = BA \cos 0^\circ = \pi a^2 B$$

We write the magnetic field as $B = 0.1t$, as a function of time, and take its derivative (we ignore the negative sign):

$$\mathcal{E} = \frac{d[\pi a^2(0.1t)]}{dt} = 0.1\pi a^2$$

We use Ohm's law to find the current generated in the circuit by this electromotor force:

$$I = \mathcal{E}/R = 0.1\pi a^2/R$$

$$I = 0.1 \times 3.14 \times 3^2/1 = 2.8 \text{ A}$$

The direction of the current should be such as to oppose the increase in the magnetic field, in other words, its own magnetic field B' should be out of the paper. According to the right-hand rule, the current on the loop should be counter-clockwise.

(b) This time, the variation in magnetic flux is caused by the increase in the cross-section area A :

$$\Phi_B = BA = B(\pi a^2) = \pi B(3t)^2$$

The induction emf is calculated using Faraday's formula:

$$\mathcal{E} = \frac{d\Phi_B}{dt} = \frac{d(9\pi Bt^2)}{dt} = 18\pi Bt$$

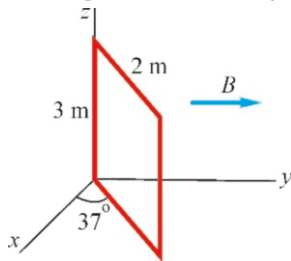
We use Ohm's law to find the induced current at time $t = 1$ s:

$$I = \mathcal{E}/R = 18\pi Bt/R = 18 \times 3.14 \times 0.1 \times 1/1 = 5.7 \text{ A}$$

Again, according to Lenz's law, the current will be counter-clockwise.

Example 21.2

The rectangular loop with the dimensions given in the figure is located in a region where a uniform magnetic field $B = 1$ T in the $+y$ direction is present. The loop plane has a 37° angle with the x -axis. The magnitude of the magnetic field is reduced down to zero in 0.3 s. Calculate the value and direction of the current induced in the loop. The resistance of the loop is $R = 4 \Omega$.



Answer We write the expression of the magnetic flux cross-

ing A :

$$\Phi_B = BA \cos \theta$$

Here, θ is the angle between the magnetic field and the surface normal. As seen in the figure, the angle between \vec{B} and the loop surface is 53° , and therefore the angle with the normal perpendicular to the surface is $\theta = 37^\circ$. We therefore calculate the magnetic flux at the start:

$$\Phi_B = 1 \times (2 \times 3) \times \cos 37^\circ = 5.2$$

We apply Faraday's law:

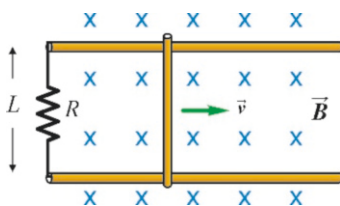
$$\mathcal{E} = \Delta\Phi_B/\Delta t = (5.2 - 0)/0.3 = 17.4 \text{ V}$$

We use Ohm's law to find the induced current:

$$I = \mathcal{E}/R = 17.4/4 = 4.4 \text{ A}$$

As magnetic flux is decreasing, according to Lenz's law, the direction of the magnetic field induced on the loop should be in a direction that prevents the decrease in \vec{B} , in other words, in the same direction. Therefore, the current will be downward in the front side.

Example 21.3



The rectangular loop in the figure has a resistance $R = 2 \Omega$ and is located in a region where $B = 0.2$ T in magnetic field directed

into the plane of the paper is present. The side of the loop with length $L = 50$ cm is pulled to the right at velocity $v = 4$ m/s. Find the value and direction of the induced current.

Answer

In this problem, the variation in the magnetic flux crossing the loop is caused by the change in the surface area of the loop. The side with length L of the rectangular loop is constant and the length of the other side varies as $x = vt$. Let us write the magnetic flux:

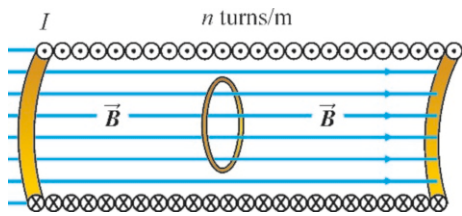
$$\Phi_B = BA = BL(vt)$$

We take the derivative of this expression with respect to time t and calculate the current from the emf using Ohm's law:

$$I = \frac{\mathcal{E}}{R} = \frac{1}{R} \frac{d(BLv t)}{dt} = \frac{BLv}{R}$$

$$I = 0.2 \times 0.5 \times 4/2 = 0.2 \text{ A}$$

Example 21.4



The number of windings per unit length of the solenoid shown in the figure is $n=500$ turns/meter and the current it carries varies with time as $I=5e^{-t}$ (ampere). A conducting loop with radius $a=50$ cm placed inside of the volume of the solenoid has resistance $R=1 \Omega$. Determine the value and direction of the current flowing through the loop at $t=0$ s.

Answer

As we discussed in Chapter 20, the magnetic field inside of a solenoid is uniformly distributed and given in Eq. (20.12):

$$B = \mu_0 n I$$

The magnetic flux crossing the loop with radius a is:

The magnetic flux crossing the loop is increasing. Therefore, the magnetic field generated by the induction current shall be in the direction that decreases this, thus out of the plane of the paper. The current that gives such a magnetic field should be counterclockwise.

$$\Phi_B = BA = \mu_0 n I (\pi a^2) = \mu_0 n \pi a^2 I$$

We use Faraday's law to find the induction emf:

$$\mathcal{E} = \frac{d\Phi_B}{dt} = \mu_0 n \pi a^2 \frac{dI}{dt}$$

The current I decreases exponentially. Its derivative will be equal to itself:

$$\frac{dI}{dt} = \frac{d}{dt} (5e^{-t}) = -5e^{-t}$$

The value of this derivative at $t = 0$ s will be 5 (we ignore the negative sign). We substitute this value and calculate the current I' induced in the loop:

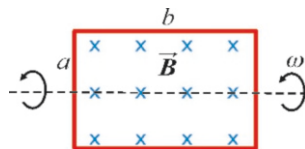
$$I' = \frac{\mathcal{E}}{R} = \frac{\mu_0 n \pi a^2}{R} \left| \frac{dI}{dt} \right|_{t=0}$$

$$I' = \frac{4\pi \times 10^{-7} \times 500 \times \pi \times 0.5^2 \times 5}{1}$$

$$I' = 0.0025 \text{ A} = 2.5 \text{ mA}$$

The direction of this current I' should be in the direction that compensates the decrease in the magnetic field of the solenoid, in other words, in the same direction as the current I .

Example 21.5



In the figure, a rectangular loop with dimensions $a=1$ m and $b=2$ m is placed in a uniform magnetic field $B=0.1$ T directed into the paper. The loop is rotating around its axis with an angular velocity of $\omega=3$ rad/s. Calculate the current induced on the loop with resistance $R=1 \Omega$ as a function of time t .

Answer

We use the formula for the magnetic flux crossing a loop:

$$\Phi_B = BA \cos \theta$$

B and A are constant here, but the angle θ varies as $\theta = \omega t$ depending on the angular velocity ω . We find the expression for the induction emf using Faraday's law:

$$\mathcal{E} = \frac{d\Phi_B}{dt} = BA \frac{d(\cos \omega t)}{dt} = -\omega BA \sin \omega t$$

We use Ohm's law to find the current without considering the negative sign:

$$I = \frac{\mathcal{E}}{R} = \frac{\omega BA}{R} \sin \omega t$$

We substitute the numerical values:

$$I = \frac{3 \times 0.1 \times (1 \times 2)}{1} \sin 3t$$

$$I = 0.6 \sin 3t$$

21.2 GENERATORS AND TRANSFORMERS

Generators

A device that converts mechanical energy into electric energy is called a generator or a dynamo. The energy of water falling from a height in hydroelectric power plants and the energy of steam in thermoelectric and nuclear power plants are converted into electric power.

When examining the electric motor in Chapter 16, we discussed how a torque was exerted on a current-carrying loop inside of a magnetic field. Generators perform the opposite of that which motors do, in other words, electricity is generated by a loop rotating in a magnetic field.

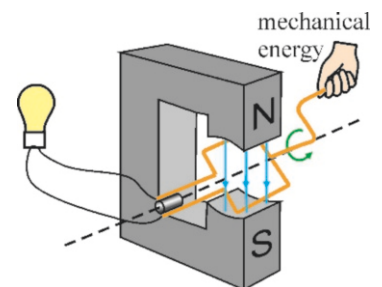


Figure 21.5: Diagram of a simple generator.

Figure 21.5 shows the diagram of a simplified alternating current generator. A rectangular loop with cross-sectional area A is rotated between the poles of a magnet. The magnetic field B of the magnet is constant in the indicated direction. The variation in the magnetic flux crossing the loop is caused only by the change in angle θ due to the rotational motion. Let us write the magnetic flux for N windings:

$$\Phi_B = NBA \cos \theta$$

If the loop rotates with angular momentum ω , the angle will increase over time as $\theta = \omega t$. The magnetic flux thus varies sinusoidally:

$$\Phi_B = NBA \cos \omega t$$

Applying Faraday's law, the derivative of flux Φ_B gives the induction emf:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = NBA\omega \sin \omega t \quad (21.3)$$

The sinusoidal variation of this current in time is given in Figure 21.7, with the various directions of the loop plane indicated. This is an alternating current, in other words, it varies between the values $\pm NBA\omega$ during a full 360° rotation of the loop, positive in the first half and negative in the second half.

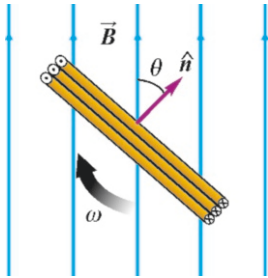


Figure 21.6: The angle θ between the loop plan and the magnetic field.

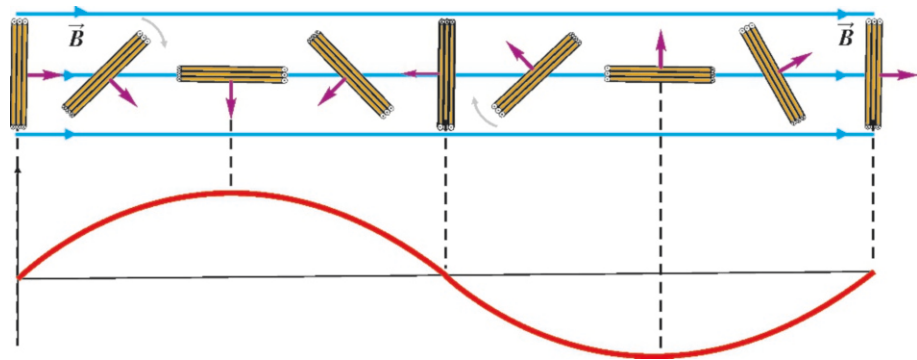


Figure 21.7: The values of the sinusoidal current depending on the direction of the loop in the generator.

Alternating current is the most practical type of current in energy transmission, as it can be supplied to circuits directly as it is generated. A commutator must be used as it is in motors, if direct current needs to be generated from this mechanism.

Transformers

The electricity generated in all kinds of power plants must be transmitted to the point of use. Heat losses must be minimized during transmission. If the resistance on the transmission lines is R , the heat loss will be RI^2 , proportional to the square of the current. Therefore, it is necessary to transmit the power with lower currents, or higher voltages, according to the power expression $P=VI$. Hence, very high values of voltage such as 100 000–500 000 Volts are carried by high voltage lines coming from power plants. This voltage is then further reduced in cities and, in the end, brought down to the 220 or 120 volts used in homes and the industry. Another advantage of alternating current is that it is easy to reduce or increase the voltage. The device that performs this function is called a **transformer**.

Figure 21.8 shows the structure of a transformer. Two coils of wire are wound around a core of iron ring. The first coil, called the **primary**, consists of N_1 windings and the second coil, called the **secondary**, consists of N_2 windings.

Suppose that a current I_1 is flowing through the first circuit. A very high percentage of the magnetic field to be generated by this current will be concentrated inside of the core and almost all of it will pass through the secondary coil on the other side. Accordingly, the flux crossing each winding will be the same everywhere:

$$\Phi_{B_1} = \Phi_{B_2} = \Phi_B$$

Now, if we write the total emf induced by the varying current for both circuits according to Faraday's law, we get:

$$\mathcal{E}_1 = N_1 \frac{d\Phi_{B_1}}{dt} \quad \mathcal{E}_2 = N_2 \frac{d\Phi_{B_2}}{dt}$$

As $\Phi_{B_1} = \Phi_{B_2}$ at each time t , their derivatives will also be equal. From here, we obtain the following formula between the emf of both circuits:

$$\mathcal{E}_2 = \frac{N_2}{N_1} \mathcal{E}_1 \quad (21.4)$$

This formula shows how the emf varies in the secondary circuit depending on the number of windings. For example, if we wish to double the voltage, we need to have $N_2 = 2N_1$, in other words, the number of windings in the second circuit should be double. Likewise, the number of windings in the second circuit should be lower to reduce the voltage.

Energy losses are inevitable in a real transformer. Transformers therefore heat up when operating and must be cooled. There are several causes for these losses. First, the flux will be less in the secondary circuit if the magnetic flux is not fully inside of the ferromagnetic core. Transformers are wound very densely and without any gaps to prevent this.

Another cause of energy losses is **eddy currents**. Eddy currents in random directions are induced in the ferromagnetic iron core when the transformer operates. These moving charges also take away some of the energy. To prevent the forming of eddy currents, the iron core is *laminated*, in other words, made of many insulated iron sheets.

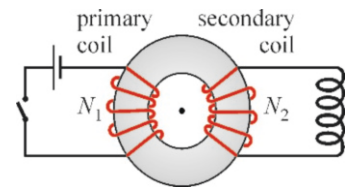


Figure 21.8: The primary and secondary windings of a transformer.

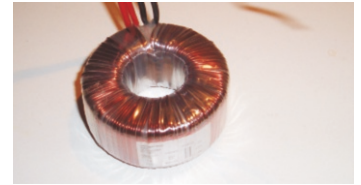


Figure 21.9: A transformer densely wound to prevent magnetic flux losses.

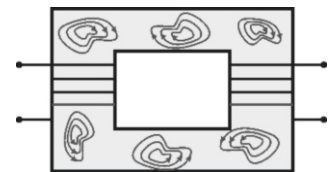


Figure 21.10: Eddy currents induced inside the transformer.

Example 21.6

The rotating coil of a generator is made of N windings, each with cross-section area of 20 cm^2 , and is rotating with an angular speed of 6000 rpm inside of a 0.5 T magnetic field. What is the minimum number of windings required such that the maximum voltage generated by the generator is 12 V ?

Answer We use Eq. (21.3) to find the emf produced by the generator:

$$\mathcal{E} = NBA\omega \sin \omega t$$

We calculate the number of windings N from the maximum value ($\sin \omega t = 1$) of this sinusoidal voltage:

$$N = \frac{\mathcal{E}}{BA\omega} = \frac{12}{0.5 \times 0.0020 \times (6000 \times 2\pi/60)} \\ N = 19.1$$

If we convert this into a whole number, we get the number of required windings as $N = 20$.

Example 21.7

The primary coil of a transformer has 200 windings and its secondary coil has 50 windings. What will a 240-volt and 20-

ampere current entering the transformer be at the output?

Answer The relation between the input and output voltages of the transformer is found using Eq. (21.4):

$$\mathcal{E}_2 = \frac{N_2}{N_1} \mathcal{E}_1$$

We calculate the output voltage from this formula:

$$\mathcal{E}_2 = \frac{50}{200} \times 240 = 60 \text{ V}$$

Example 21.8

Power loss on transmission lines. 1 MW (megawatt) in power generated at a dam is transmitted to a city using two different methods:

- (a) What will the power converted into heat be on the 1Ω resistance of the line when this power is transmitted at a voltage of $5000 \text{ V} = 5 \text{ kV}$?
- (b) What will the power converted into heat be on the 1Ω resistance of the line when this same power is first increased to a voltage of $500\,000 \text{ V} = 500 \text{ kV}$ using a transformer?

Answer

(a) The general power formula was found in Section 18.4:

$$P = VI$$

Also, the power converted into heat on a resistor R was found to be

$$P' = RI^2$$

Considering that an ideal transformer has no power loss, the power at the input and output are equal:

$$\mathcal{E}_1 I_1 = \mathcal{E}_2 I_2$$

From here, we find the output current to be $I_2 = 80 \text{ A}$.

Accordingly, the current I_1 when the power P is transmitted at the voltage V_1 is:

$$I_1 = \frac{P}{V_1} = \frac{10^6}{5000} = 200 \text{ A}$$

We calculate the power converted into heat with this current on the resistor R :

$$P'_1 = RI_1^2 = 1 \times 200^2 = 40\,000 \text{ W} = 40 \text{ kW}$$

(b) We repeat the same calculation when the same power is transmitted at voltage $V_2 = 500 \text{ kV}$:

$$I_2 = \frac{P}{V_2} = \frac{10^6}{5 \times 10^5} = 2 \text{ A}$$

We calculate the power converted into heat with this current on the resistor R :

$$P'_2 = RI_2^2 = 1 \times 2^2 = 4 \text{ W}$$

The power loss is less with transmission at higher voltage.

21.3 INDUCTANCE – MAGNETIC ENERGY

Let us consider two coils that are very close to each other. Let us focus on the magnetic flux Φ_{B_2} passing through the second coil and due to the current I_1 in the first coil. When the current I_1 is varied, the flux Φ_{B_2} will also vary and, according to Faraday's law, an emf \mathcal{E}_2 will be induced in the second coil:

$$\mathcal{E}_2 = -\frac{d\Phi_{B_2}}{dt}$$

If the geometric shapes and positions of the coils are constant, the induced magnetic flux will be proportional only to the current I_1 :

$$\Phi_{B_2} = M_{21} I_1 \quad (21.5)$$

We can see this feature from the expressions for the magnetic field of a coil and solenoid that we found earlier. In this case, the value of the emf induced in the second circuit can be written as follows, in terms of I_1 instead of Φ_{B_1} :

$$\mathcal{E}_2 = -M_{21} \frac{dI_1}{dt}$$

The coefficient M_{21} here, into which we gathered all of the constants is called the mutual induction coefficient or **mutual inductance**. Its unit is $1 \text{ T} \cdot \text{m}^2 / \text{A} = 1 \text{ Henry (H)}$. This coefficient is dependent on the geometric structure of the given current loops, the number of windings and the μ permeability of the medium.

We can use the same reasoning for the flux of the second coil on the first one. The emf value \mathcal{E}_1 induced on the first coil by the variation in the current I_2 flowing through the second coil, will be,

$$\mathcal{E}_1 = -M_{12} \frac{dI_2}{dt}$$

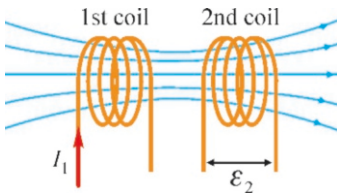


Figure 21.11: Magnetic flux between two coils.

It can be proven that these two coefficients M_{21} and M_{12} are equal even if the coils are not identical. We accept this result here without proof and define a single mutual inductance for both coils:

$$M_{21} = M_{12} = M \quad (\text{Mutual inductance})$$

Thus, instead of the induction flux Φ_B , we express the induction emf values in terms of the directly measurable current:

$$\mathcal{E}_1 = -M \frac{dI_2}{dt} \quad \text{and} \quad \mathcal{E}_2 = -M \frac{dI_1}{dt} \quad (21.6)$$

Self-inductance

When you shut down the current of a building by turning off the main lever switch, you will notice that a spark jumps between the two poles of the switch. How can an emf of about 1000 Volts be induced in a circuit that is switched off, when it is not possible for a spark to jump at 220 Volts? Faraday's law holds the answer to this. As the circuit is switched off, during that small duration, an emf opposed to the decrease in magnetic flux is induced *on itself*. We describe this with the concept of self-induction.

Consider the simple circuit in Figure 21.12 consisting of a coil with only two windings. When the switch is closed, the current and its magnetic field are constant. You will note that the magnetic flux of each winding passes through the other winding. Now, suppose that we open the switch and cut the current. The magnetic field of the second winding will decrease, and thus its magnetic flux crossing the first winding will also decrease. According to Faraday's law, an opposite emf will be induced in the first winding against this decrease. As both windings are part of the same circuit, we can consider this induction effect as the **self-induction** of the circuit.


Every current loop opposes the variation in the current that it carries. It does this by producing an opposing induction emf. This emf value is proportional to the variation in the current I flowing through the coil. Likewise, if we gather all of the geometric factors together and write the magnetic flux flowing through the coil as proportional to current I , we get

$$\Phi_B = LI \quad (21.7)$$

Thus, the **self-induced emf** generated in the circuit will be proportional to the time rate of change of the current:

$$\mathcal{E} = -L \frac{dI}{dt} \quad (L : \text{Self-inductance}) \quad (21.8)$$

L is called the self-induction coefficient or simply the **inductance**. Its unit is, again, the Henry (H), as in mutual inductance.

In alternating current circuits, windings with a fixed inductance value are used as a circuit component called the **inductor** and indicated with the symbol . The potential difference between the terminals of an inductor carrying a current I is

$$V_{ab} = -L \frac{dI}{dt} \quad (21.9)$$

The potential difference will be opposite to the current if the current is increasing and in the same direction if it is decreasing.

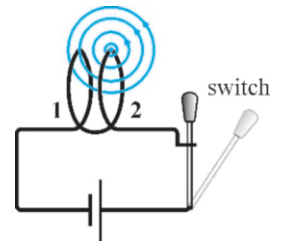


Figure 21.12: A circuit with two windings to understand self-induction.

Magnetic Energy

In Chapter 18, the most general expression for energy in electric circuits was found to be (equation 18.11):

$$P = VI$$

Using the expression that we found above for the potential difference between the terminals of a coil carrying current I , we get

$$P = LI \frac{dI}{dt}$$

The work performed on this coil during the time interval dt will be stored as an increase in the potential energy of the coil:

$$\begin{aligned} dU &= dW = P dt = LI \frac{dI}{dt} dt \\ dU &= LI dI \end{aligned}$$

When the current goes from an initial value of zero to a final value of I , the magnetic energy stored in the coil will be the integral of this expression:

$$U = L \int_0^I I dI = \frac{1}{2} LI^2 \quad (\text{Magnetic energy of a coil}) \quad (21.10)$$

This magnetic energy is stored in the coil and may later be returned to the circuit.

Example 21.9

- (a) Find the expression for the inductance of a solenoid.
 (b) Calculate the inductance of a solenoid with 500 windings, 20 cm in length and a 10 cm^2 cross-section area.
 (c) Calculate the energy stored when a 40 A current flows through this solenoid.

Answer

(a) We had found the expression for the magnetic field of a solenoid in Chapter 20:

$$B = \mu_0 n I$$

n is the number of windings per unit length and, for a solenoid with length ℓ and N windings, it is $n = N/\ell$.

As the magnetic field is constant inside of the cross-section of the solenoid, the magnetic flux is calculated for N windings:

$$\Phi_B = NBA = N(\mu_0 NI/\ell)A = \frac{\mu_0 N^2 A}{\ell} I$$

The inductance was defined in Eq. (21.8) as:

$$\Phi_B = LI$$

We find the expression for inductance by comparing the last two expressions:

$$L = \frac{\mu_0 N^2 A}{\ell}$$

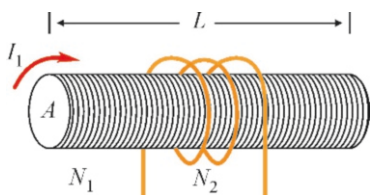
(b) We substitute the numerical values in the expression L as follows:

$$\begin{aligned} L &= \frac{4\pi \times 10^{-7} \times 500^2 \times 0.0010}{0.20} \\ L &= 0.0016 \text{ H} = 1.6 \text{ mH} \end{aligned}$$

(c) The energy stored in a coil was given with the formula (21.10):

$$U = \frac{1}{2} LI^2 = \frac{1}{2} \times 0.0016 \times 40^2 = 1.3 \text{ J}$$

Example 21.10



A wire is wound N_2 times around a solenoid with length ℓ , cross-section area A and N_1 windings.

- (a) Find the mutual inductance of the solenoid and the coil.
 (b) Find the coefficient M for $N_1=500$, $N_2=20$, $\ell=10 \text{ cm}$ and $A=40 \text{ cm}^2$.

Answer

(a) We write the magnetic field of a solenoid carrying a current I_1 :

$$B = \mu_0 n I_1 = \mu_0 (N_1/\ell) I_1$$

The flux of this magnetic field flowing through a coil with N_2 windings is

$$\Phi_{B_2} = N_2 BA = \frac{\mu_0 N_1 N_2 A}{\ell} I_1$$

$$M = \frac{\mu_0 N_1 N_2 A}{\ell}$$

The mutual inductance was defined in Eq. (21.5) as follows: (b) M is calculated using the given numerical values:

$$\Phi_{B_2} = M I_1$$




$$M = \frac{4\pi \times 10^{-7} \times 500 \times 20 \times 0.0040}{0.10}$$

We find the coefficient M by comparing the last two formulas:

$$M = 0.0005 \text{ H} = 0.5 \text{ mH}$$

21.4 RLC CIRCUITS

Let us recall the circuit components that we have examined thus far:

Resistor :	R		$V_{ab} = RI$
Capacitor :	C		$V_{ab} = \frac{q}{C}$
Coil :	L		$V_{ab} = L \frac{dI}{dt}$

L and C components are not used in a direct current circuit, because no direct current can flow through a capacitor and the coil gives no reaction to a direct current. However, these components are activated in alternating current circuits and generate a potential difference when a current flows through them.

The topic of alternating currents is very broad and simply cannot be examined thoroughly in this course. We shall merely briefly summarize some results here using what we have learned.

Figure 21.13 shows L , R , C circuit components connected in series to an emf source. In order to find the current flowing through this circuit, we add the potential differences on each component by going from the positive pole of the generator:

$$V_{ad} = V_{ab} + V_{bc} + V_{cd}$$

$$\mathcal{E} = L \frac{dI}{dt} + RI + \frac{q}{C}$$

We cannot use the equation in this form, as it includes both the unknown current I and charge q . We take the derivative of the equation, and recalling that $dq/dt = I$, we get

$$\frac{d\mathcal{E}}{dt} = L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} \underbrace{\frac{dq}{dt}}_I$$

Rearranging the terms, we get:

$$LI'' + RI' + \frac{1}{C} I = \frac{d\mathcal{E}}{dt} \quad (21.11)$$

This expression is a differential equation that gives the unknown current $I(t)$. We can solve this equation and find the current if the emf function $\mathcal{E}(t)$ on the right-hand side is given.

Before solving this equation, note a very important similarity: Recall the equation $x(t)$ giving the position of the mass when we examined damped and driven oscillatory motion in mechanics (Eq. 9.30):

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F(t)$$

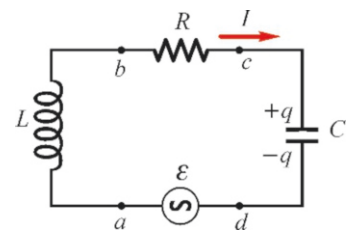


Figure 21.13: An RLC circuit.

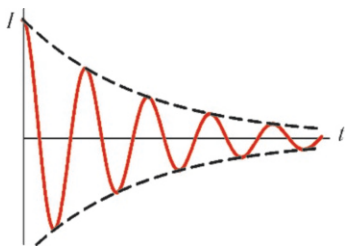


Figure 21.14: Damped motion in an RLC circuit.

Comparing the two equations, we notice that the mass m in mechanics is replaced by the coil L , the friction b is replaced by the resistance R and the spring k is replaced by the capacitor $1/C$. Let us remember the solutions that we found in mechanics: The system makes a damped oscillation motion if there is no external force ($F = 0$). Likewise, **resonance** was observed when the frequency ω of the external force was near the natural frequency.

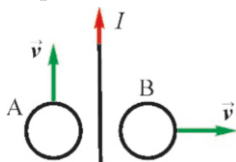
The RLC circuit also has damped oscillation motion (Figure 21.14) and resonance solutions, depending on the applied potential difference $\mathcal{E}(t)$. This topic is examined in more detail in courses on circuit analysis.

Multiple-choice Questions

- According to Faraday's law, the emf induced in a coil is:
 - Proportional to the magnetic field crossing the coil.
 - Proportional to the magnetic flux crossing the coil.
 - Proportional to the time rate of change of the magnetic flux crossing the coil.
 - Proportional to the cross-sectional area of the coil.

- According to Lenz's law, the direction of the current produced by an induction emf is:
 - Opposite to the current generating it.
 - In the same direction as the current generating it.
 - In opposite direction to the change in the current generating it.
 - In the same direction as the change in the current generating it.

- The current I flows through the linear wire in the figure below. In which of the two conductor coils will an induction emf be produced?



- (a) A (b) B (c) A and B (d) None
- In which of the following situations will an induction emf be produced in a coil?
 - If the magnetic field crossing the coil varies.
 - If the surface area of the coil varies.
 - If the magnetic field is changing direction.
 - All of the above.

(a) I & II (b) II & III (c) I & III (d) IV

- The maximum value of the induction emf induced on a rotating coil in a magnetic field depends on which of the following?

- The magnitude of the magnetic field.
- The surface area of the coil.
- Angular momentum.

IV. All of the above.

(a) I & II (b) II & III (c) I & III (d) IV

- The magnetic flux crossing a coil is varying as $\Phi_B = t^3 + 2t^2$ ($\text{T}\cdot\text{m}^2$). What is the emf induced at time $t = 1$ s?
 - 0
 - 3
 - 5
 - 7

- What will happen to the generated emf if the coil of a generator rotates faster?
 - It will increase.
 - It will decrease.
 - It will remain the same.
 - It is impossible to tell.

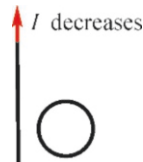
- When will an induction emf be induced in a coil?
 - When a constant current flows.
 - When the current is maximum.
 - When the current varies over time.
 - When the current is zero.

- Which of the following are correct for the inductance of a solenoid?

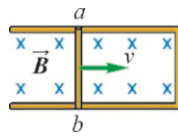
- It is directly proportional to the cross-section area.
- It is directly proportional to the number of windings.
- It is directly proportional to the square of the number of windings.
- It is inversely proportional to the length.

(a) I, III & IV (b) I & IV (c) II & IV (d) I & II

- In which direction will a current be induced on the coil when the current in the straight wire shown in the figure below is decreased?



- Clockwise.
- Counterclockwise.
- No current is induced.
- It is impossible to tell.



11. In which direction will the induction current flow on the branch ab of the circuit shown in the figure above, when it moves to the right?
 - (a) Upward.
 - (b) Downward.
 - (c) The current will be zero.
 - (d) It is impossible to tell.

12. Which of the following are correct for the number of windings in a transformer?
 - I. The secondary emf will be larger if $N_2 > N_1$.
 - II. The primary emf will be larger if $N_2 > N_1$.
 - III. The secondary current will be larger if $N_2 > N_1$.
 - IV. The primary current will be larger if $N_2 > N_1$.
 - (a) I & III
 - (b) II & IV
 - (c) I & IV
 - (d) II & III

13. Energy losses are caused by which of the following in a transformer?
 - I. Resistance of the wire.
 - II. Loss of magnetic flux.
 - III. Eddy currents.
 - IV. All of the above.
 - (a) I & III
 - (b) II & III
 - (c) I & II
 - (d) IV

14. Which of the following are correct for the magnetic energy stored in a coil?
 - I. It is directly proportional to the flowing current.
 - II. It is directly proportional to the square of the flowing current.

- III. It is directly proportional to the inductance.
- IV. All of the above.
 - (a) I & III
 - (b) II & III
 - (c) I & II
 - (d) IV

15. The magnetic flux crossing a coil decreases from $6 \text{ T}\cdot\text{m}^2$ to zero in 2 seconds. What will the induction emf be?
 - (a) 2
 - (b) 3
 - (c) 8
 - (d) 12

16. By what factor will the inductance increase when the number of windings of a solenoid is doubled?
 - (a) 2
 - (b) 4
 - (c) 1/4
 - (d) 1/2

17. By what factor will the magnetic energy increase if the inductance of a coil is doubled and the current flowing is tripled?
 - (a) 6
 - (b) 16
 - (c) 18
 - (d) 36

18. The number of primary windings of a transformer is doubled and the number of secondary windings is increased by a factor of 4. By what factor will the secondary voltage increase?
 - (a) 2
 - (b) 4
 - (c) 8
 - (d) 1/2

19. How much energy will be stored when a 10 A current flows through a coil with inductance $L = 0.5 \text{ H}$?
 - (a) 2 J
 - (b) 10 J
 - (c) 25 J
 - (d) 50 J

20. Which of the following is the expression for mutual inductance?
 - (a) $\frac{\mu_0 N_1 N_2 A}{\ell}$
 - (b) $\frac{\mu_0 N_1 A}{N_2 \ell}$
 - (c) $\frac{\mu_0 A}{N_1 N_2 \ell}$
 - (d) $\frac{\mu_0 A^2}{N_1 N_2 \ell}$

Problems

21.1 Faraday's law

21.1 The magnetic flux crossing a coil varies as

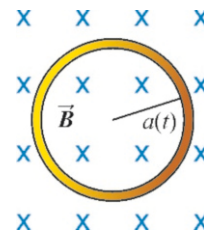
$$\Phi_B = t^3 + 2t \text{ (tesla}\cdot\text{m}^2\text{)}$$

Calculate the induction emf in the coil at time $t=1 \text{ s}$.

[A: 5 V.]

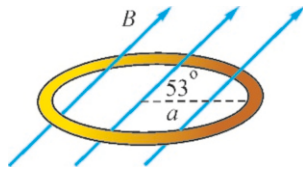
21.2 The magnetic field crossing a coil with a cross-section area of 50 cm^2 and 200 windings, is raised from zero to 0.4 T in 2 seconds. What is the emf induced on the coil?

[A: 0.2 V.]



Problem 21.3

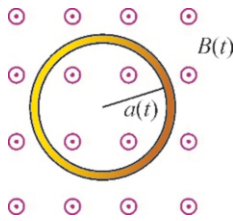
21.3 The conducting coil with resistance $R=2 \Omega$, as shown in the figure above, is located in a magnetic field $B=0.2 \text{ T}$ directed into the plane of the paper. The radius of the coil starts to decrease as $a = 3/t$ (meters). Find the value and the direction of the current at $t=2 \text{ s}$. [A: 0.24 A, clockwise.]



Problem 21.4

21.4 The coil with radius $a=50\text{ cm}$ and resistance $R=10\ \Omega$ in the figure above is placed inside of a magnetic field that varies as $B=0.2t^5\text{ T}$. The magnetic field makes a 53° angle with the plane of the coil. Find the value and the direction of the current induced in the coil at time $t=1\text{ s}$.

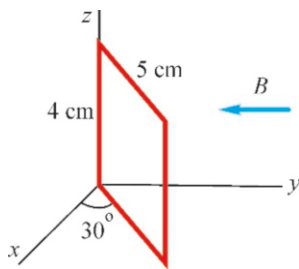
[A: 0.063 A , clockwise.]



Problem 21.5

21.5 The conducting coil with resistance $R=10\ \Omega$ in the figure is placed in a magnetic field directed out of the plane of the paper. The radius of the coil decreases as $a(t)=2e^{-3t}$ (meters), while the magnetic field increases as $B(t)=0.1 e^t$ (tesla). Calculate the value and direction of the induction current in the coil at time $t=0$.

[A: 0.63 A , counterclockwise.]



Problem 21.6

21.6 The rectangular loop with the dimensions given in the figure above is located in a region where a uniform magnetic field $B=0.5\text{ T}$ in the $-y$ direction is present. The loop plane makes a 30° angle with the x -axis. The magnitude of the magnetic field is reduced down to zero in 0.1 s . Calculate the value and direction of the current induced in the loop. The resistance of the loop is $R=1\ \Omega$.

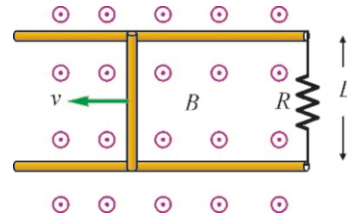
[A: 8.7 mA , upward on the front side.]

21.7 A coil with 100 windings has a cross-section area of 40 cm^2 and is placed perpendicularly to a $B=0.4\text{ T}$ magnetic field. The direction of the coil is suddenly reversed by 180° . Considering that this rotation takes place in 0.1 s , what will the average induction emf induced on the coil be?

[A: 3.2 V .]

21.8 The rectangular coil in the figure below has resistance $R=10\ \Omega$ and is located inside of a $B=0.4\text{ T}$ magnetic field directed out of the plane of the paper. The side of the coil with length $L=40\text{ cm}$ is pulled to the left with a velocity of $v=6\text{ m/s}$. Find the value and direction of the induction current in the coil.

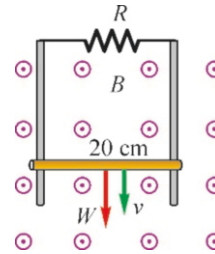
[A: 0.1 A , clockwise.]



Problem 21.8 and Problem 21.9

21.9 How much force is required to pull the mobile side of the rectangular coil shown in the figure above with the same constant velocity of $v=6\text{ m/s}$? (*Hint:* It is sufficient to overcome the force exerted upon the current-carrying wire in a magnetic field.)

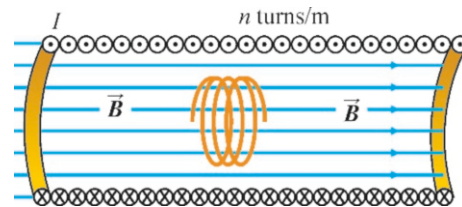
[A: $F = 0.016\text{ N}$.]



Problem 21.10

21.10 The mobile side with a length of 20 cm of a coil located vertically in a $B=0.5\text{ T}$ magnetic field directed out of the plane of the paper, as shown in the figure, has weight $W=0.1\text{ N}$. The total resistance of the coil is $R=10\ \Omega$. What is the limit velocity of the mobile side when it is released for from rest? (*Hint:* The mobile side travels with constant velocity when the magnetic force exerted upon its current is equal to its weight.)

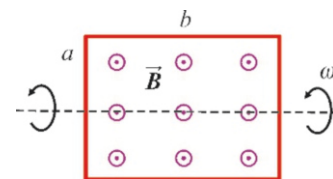
[A: 100 m/s .]



Problem 21.11

21.11 The number of windings per unit length of the solenoid shown in the figure is $n=800\text{ turns/meter}$ and the current it carries decreases as $I_1=5/t^2$ (ampere). A coil with radius $a=20\text{ cm}$ and 25 windings placed inside of the volume of the solenoid has resistance $R=1\ \Omega$. Determine the value and direction of the current flowing through the inner coil at $t=1\text{ s}$.

[A: 0.032 A , in the same direction as I_1 .]



Problem 21.12

21.12 In the figure above, a rectangular coil with resistance $R=10\ \Omega$ and dimensions $a=1\text{ m}$ and $b=3\text{ m}$ is placed inside of a uniform magnetic field $B=0.5\text{ T}$ directed out of the paper. The coil is rotating about its axis with an angular velocity

of $\omega=5$ rad/s. Calculate the current induced in the coil as a function of time t . [A: $I = 0.75 \sin 5t$.]

21.2 Generators and Transformers

21.13 The rotating coil of a generator is made of N windings, each with cross-sectional area 40 cm^2 and is rotating with an angular velocity of 100 rad/s inside of a 0.2 T magnetic field. What is the minimum number of windings required such that the maximum voltage generated by the generator is 24 V ? [A: $N = 300$ windings.]

21.14 The primary circuit of a transformer has 100 windings and its secondary circuit has 40 windings. What will a 120-volt and 10-ampere current entering the transformer be at the output? [A: 48 V and 25 A .]

21.15 A 24 V and 20 A current supplied to the primary circuit of a transformer is required to generate a 1 A current. What will the ratio N_2/N_1 and the output voltage be? [A: $N_2/N_1 = 20$ and $\mathcal{E}_2 = 480 \text{ V}$.]

21.16 The 1000 W in power generated in a generator is transmitted using two methods. (a) What will the power converted into heat be on the 1Ω of resistance of the line when transmitted at a voltage of 200 V ? (b) What will the power

converted into heat be on the 1Ω of resistance of the line when this same power is first increased to a voltage of 500 V using a transformer? [A: (A) 25 W , (b) 4 W .]

21.3 Inductance – Magnetic Energy

21.17 Of two coaxial solenoids, the current on the first one decreases from 5 A to zero in 0.01 s and a 2000 V emf is generated on the second solenoid. Determine the mutual inductance of this system. [A: $M = 4 \text{ H}$.]

21.18 A 3 A current is flowing through a coil with 2 H in inductance. (a) How much energy is stored in the coil? (b) The current increases by 40 A/s each second. What is the emf induced in the coil? [A: (a) 9 J , (b) 80 V .]

21.19 (a) Calculate the inductance of a solenoid with 600 windings, 50 cm in length and a 20 cm^2 cross-sectional area. (b) Calculate the energy stored when a 10 A current flows through this solenoid. [A: (a) 1.8 mH , (b) 0.09 J .]

21.20 A conducting wire is wound $N_2=50$ times around a solenoid with length $\ell=20 \text{ cm}$, cross-section area $A=100 \text{ cm}^2$ and $N_1=800$ windings. Calculate the mutual inductance of the system. [A: $M = 2.5 \text{ mH}$.]

?

22

GEOMETRIC OPTICS



The **Hubble Space Telescope**, with the Earth in the background. This 11-ton spacecraft contains a telescope 2.4 m in diameter. It is able to clearly see the furthest celestial objects in space, as its orbit at 560 km in altitude is located outside of Earth's atmosphere. The Hubble telescope will remain in service until 2030.

In this chapter, we shall introduce Optics, one of the most important branches of physics. All human activity in nature starts with the sense of sight. The sense of sight uses a property of materials called light. The light emitted from the objects around us forms an image in our eyes. We thus know that such an object is there even if we are unable to touch it.

We will first learn the structure of light. Understanding that light is an electromagnetic wave was one of the most important developments in the history of science. Many technical and optic instruments were able to be developed after this wave structure was understood.

We will also discuss the laws governing the way in which light behaves when passing from one medium into another or when being reflected. We will work out to calculate the formation of images by mirrors and lenses, which facilitates daily life and technology.

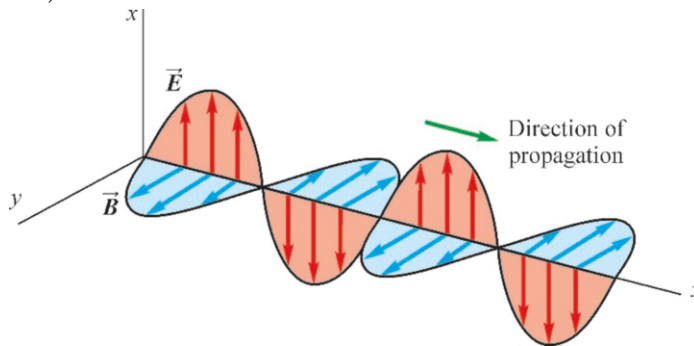
22.1 THE WAVE NATURE OF LIGHT

Various answers have been given throughout history to the question of “What is light?” The road of the theories of light is very interesting. The particle model set forth by Newton was accepted until the 19th century. Newton asserted that sources of light would emit particles too small to be seen, which would then cause images to be formed when they reached the eye, and he could explain reflection and refraction of light accordingly. Despite the fact that, during the same years, Huygens and Fermat had claimed that light could be a wave, they did not receive much recognition.

However, the wave-model of light began to stand out in the early 19th century, when Young and Fresnel revealed light’s interference and diffraction properties. This was because interference and diffraction properties cannot be explained with the particle model.

Later, in 1873, James Clerk Maxwell published his electromagnetic theory, merging electricity and magnetism into a single theory. He suggested the concept of the electromagnetic wave for the first time. According to Maxwell, the electric field E and the magnetic field B are both jointly included in an electromagnetic wave (Figure 22.1), as transverse sinusoidal waves (perpendicular to the direction of propagation).

Figure 22.1: The fields E and B constituting an electromagnetic wave are in transverse sinusoidal wave motions perpendicular to the direction of propagation.



Maxwell found the following expression for the propagation speed of electromagnetic waves in free space in terms of the electric permittivity and magnetic permeability of free space:

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

When calculated, this expression was found to be equal to the speed of light:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.9979 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s}$$

After coming up with this result, Maxwell went one step further and asserted that light was also an electromagnetic field. His ideas were soon proven experimentally. In 1887, Heinrich Hertz managed to generate an electromagnetic wave in laboratory.

Today, we know of all kinds of electromagnetic waves, from radio waves to microwaves, infrared rays to X- and gamma rays. Let us show these waves on a graph called a spectrum, sorted according to their frequencies (Figure 22.2):

- **Radio waves** are waves whose wavelength ranges from 10 cm to 10 km and that can easily cross the atmosphere. They are used in radio and TV broadcasts.

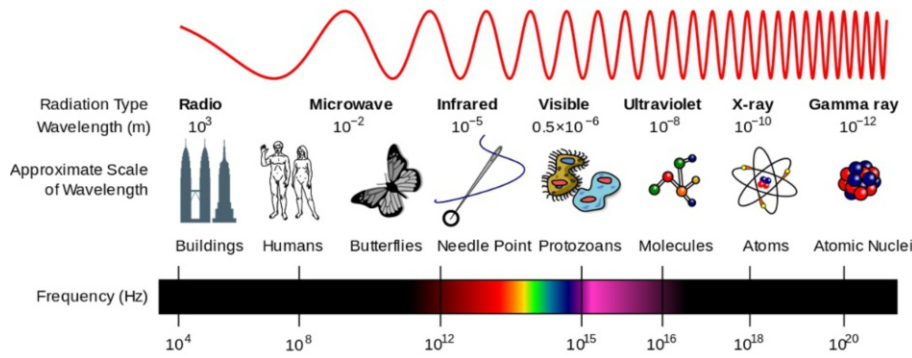


Figure 22.2: Electromagnetic wave spectrum.

- **Microwaves** have shorter wavelengths and are used in radars, wireless internet, kitchen ovens, etc.
- **Infrared rays** are used in short-range communication and fiber-optic communication. They are quickly absorbed by the atmosphere.
- **Visible light** is the range of wavelength to which the human eye is sensitive. It ranges from red (700 nm) up to violet (400 nm).
- **Ultraviolet rays** have higher energy than visible light and may cause an ionizing impact on tissues and lead to sunburns.
- **X-rays** are also dangerous for tissues and are used in medicine and the industry because they can easily pass through matter.
- **Gamma rays** are the electromagnetic radiation with the highest energy. They come from outer space or are emitted in nuclear reactions. They are used in medicine for diagnosis and cancer treatment.

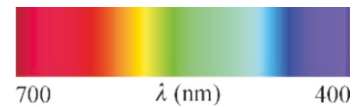


Figure 22.3: The range of visible light.

This model of electromagnetic waves was one of the great successes of classic physics. However, some observations at the start of the 20th century revived the particle model. New effects such as the photoelectric effect and Compton scattering indicated that light behaved not like a wave, but like a billiard ball.

These uncertainties were resolved in 1926 with quantum theory. According to quantum theory, not only light, but all particles have a double nature, called the wave-particle structure. They exhibited their particle nature in certain cases and their wave nature in certain other cases.

However, the classic electromagnetic wave model could explain all aspects of light, except for the phenomena of scattering, emission and absorption of light.

The Ray Model of Light

Light emitted from any source propagates into space as an electromagnetic wave. The locus of the points of this wave with the same phase is called a **wave front** (Figure 22.4). A wave front expands in space as a spherical surface centered at the source of light, if there is no obstruction. We can thus visualize light as a successions of wave fronts.

Virtual lines assumed to be perpendicular to the wave front and in the direction of propagation are called **rays**. In a homogeneous medium, rays are straight lines drawn from the source. Rays may be curved in non-homogeneous media, for example, when passing through the atmosphere. However, they shall always propagate as perpendicular to the wave front. The ray model is easier to work with than wave fronts.

Wave fronts can approximately be taken as planes at very far distances from the source of light (Figure 22.5) where the radius of the sphere is very large. Such

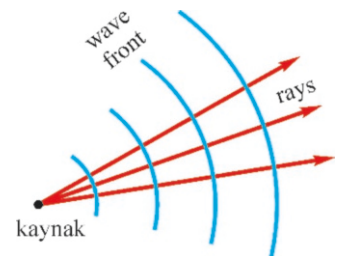


Figure 22.4: Wave fronts, and rays propagating perpendicular to them.

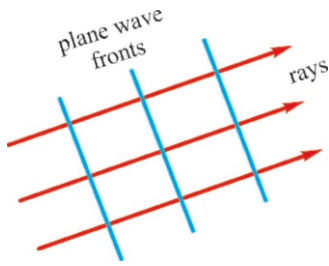


Figure 22.5: Wave fronts become planes and rays become parallel at very great distances from the source.

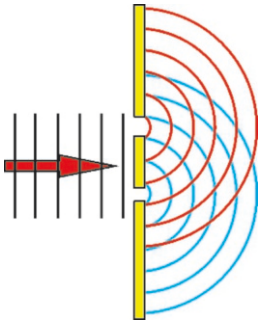


Figure 22.6: The Huygens-Fresnel principle: Each slit produces new wave fronts.

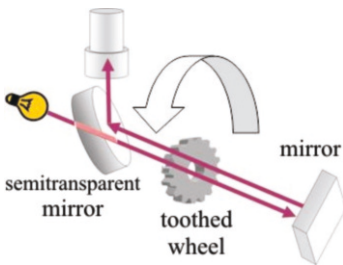


Figure 22.7: The Fizeau-Foucault apparatus for measuring the speed of light.

a wave with the source at infinity is called a **plane wave**. Accordingly, the rays become parallel to each other.

Huygens-Fresnel Principle

What will happen when a propagating light wave meets an obstacle? Huygens and Fresnel explained the propagation of light in a medium with the following principle:

Every point of a medium on which a light is incident constitutes a new source of waves, and the wave front emitted therefrom spreads evenly in every direction.

Consider two adjacent rooms. Let there be a hole in the wall between the two rooms. When someone speaks in the first room, the sound will be heard in the second room as if emitted from this hole. Even if the speaker changes position in the room, in the second room, the sound wave will always be heard as if coming from the same point. This complies with the Huygens-Fresnel principle.

Figure 22.6 shows what happens when a plane wave meets two slits in a screen. The light spreads as new spherical wave fronts are emitted from these slits.

The Speed of Light and Index of Refraction

The speed of propagation of light is a universal physical constant:

$$c = 2.9979 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s}$$

Special techniques are required to measure such a high speed. Many techniques have been developed throughout history to measure the speed of light. The most commonly known method is the apparatus developed by the French scientists Fizeau and Foucault.

In the Fizeau-Foucault apparatus, the light emitted from a very distant source passes between the teeth of a rotating wheel and is converted into light pulses (Figure 22.7). A pulse is observed after it is reflected back from a mirror and again passes between the teeth of the wheel. When a pulse is reflected from the mirror, it gets stopped if the rotation speed of the wheel is not right. By varying the rotation speed, it is ensured that a pulse passes through the teeth while both going and returning. When this is achieved, the speed of light can be calculated in terms of the distance between the mirror and the wheel, the rotation speed of the wheel and the number of teeth.

(Note: The expression above, “measuring the speed of light,” became meaningless after 1983. If we remember the modern definition of the basic unit of length (meter) as “the distance traveled by light in vacuum in $1/299\,792\,458$ seconds,” we understand that the speed of light is a given universal constant; there is no question of measuring it. What these experiments really measure is the length of the meter unit. This outlook is only correct historically: Before 1983, the meter unit was defined differently, and then the speed of light was measured.)

The speed of light decreases when the light passes through a gas or liquid environment. It is easy to understand the reason for this: Recalling the definition for the speed of light $c = 1/\sqrt{\epsilon_0\mu_0}$, as $\epsilon > \epsilon_0$ and $\mu > \mu_0$ is true for every medium, the speed of light will always be less than its value in vacuum.

This degree of slowing is expressed with a dimensionless coefficient. If we use v to indicate the speed of light in the medium and c to indicate its speed in vacuum, the ratio

$$n = \frac{c}{v} \quad (\text{index of refraction}) \quad (22.1)$$

is called the **index of refraction** of the medium. The index of refraction depends on the type of medium and the wavelength of the incident light. The index of refraction of some media are as follows:

Index of refraction of some media.			
n		n	
Vacuum	1	Acrylic glass	1.49
Air	1.0003	Flint glass	1.60
Water	1.3330	Crown glass	1.52
Ice	1.309	Silicon	3.96

Recall the relation between the velocity v , wavelength λ and frequency f (or period T) of a wave:

$$\lambda = vT = \frac{v}{f} \quad (22.2)$$

The frequency of a light wave does not change when passing from one medium into another. Frequency is a quantity determined by the source generating the wave. (When you generate a wave by vibrating a rope, the up and down vibration frequency of the rope does not vary with the propagation of the wave.) Therefore, if we write the above formula for two media and divide both sides, we get

$$\frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2} \quad (22.3)$$

22.2 REFLECTION AND REFRACTION

A light ray that reaches the interface surface between a medium with index of refraction n_1 and another medium with index of refraction n_2 will be partially reflected and be partially refracted (bent in the second medium). Let us first give the laws of reflection and refraction:

Laws of Reflection and Refraction

1. Reflected and refracted rays are in the *plane of incidence* formed by the the incident ray and the normal to the surface.

2. The angle of incidence and the angle of reflection measured from the normal to the surface are equal:

$$\theta_1 = \theta_1' \quad (22.4)$$

3. The relation between the angle of incidence and the angle of refraction is determined with Snell's law:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad (\text{Snell's law}) \quad (22.5)$$

Each of these laws can be proven using wave fronts. Let us prove the refraction law here as an example.

Consider two incident light rays with the same angle of incidence θ_1 (Figure 22.9). When the first ray reaches point A on the interface, the point B , which

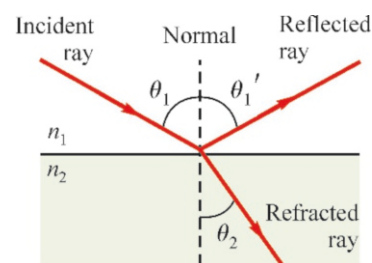


Figure 22.8: Reflection and refraction.

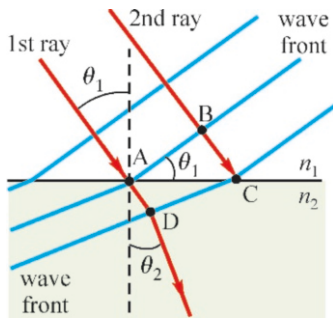


Figure 22.9: Successive wave fronts travel slower in the second medium.

is on the same wave front still has to cover distance BC . While this second ray travels distance BC at velocity v_1 , the ray at point A will start to travel at velocity v_2 in the second medium. The distances traveled in the same time interval Δt will be as follows:

$$BC = v_1 \Delta t \quad AD = v_2 \Delta t$$

The new wave front thus becomes DC . Let us write the distances BC and AD in terms of the common side AC and the angles:

$$BC = v_1 \Delta t = AC \sin \theta_1 \quad AD = v_2 \Delta t = AC \sin \theta_2$$

We now divide both sides:

$$\frac{v_1}{v_2} = \frac{\sin \theta_1}{\sin \theta_2}$$

Lastly, we replace velocities v using the formula $n = c/v$:

$$\frac{v_1}{v_2} = \frac{n_2}{n_1} = \frac{\sin \theta_1}{\sin \theta_2}$$

This is the Snell's law in Eq. (22.5).

Let us emphasize the important points of reflection and refraction:

- If the incident ray is perpendicular ($\theta_1=0$), then $\theta_2=0$ and the ray goes into the second medium without refraction.
- If the rays fall on an interface where $n_1 < n_2$ (e.g., air-water), there is always refraction in the second medium. In this case, we have $\theta_2 < \theta_1$ and the refracted ray approaches the surface normal. This is what causes a rod submerged in water to seem as if it is broken (Figure 22.10).
- **Total internal reflection.** An interesting thing happens when rays try to exit a denser medium, e.g., from water into the air ($n_1 > n_2$). In this situation, the exiting rays bend away from the normal. As the angle of incidence θ_1 increases, the angle of refraction θ_2 approaches 90° (Figure 22.11). When θ_1 is greater than this value, no refraction occurs and all of the light is reflected internally. Let us write Eq. (22.5) for θ_2 :

$$\frac{n_1}{n_2} \sin \theta_1 = \sin \theta_2$$

As $n_1/n_2 > 1$ when leaving a denser medium, this equation has no solutions for angles that make the right-hand side greater than 1. Therefore, for a solution to exist, we must have

$$\frac{n_1}{n_2} \sin \theta_1 \leq 1$$

In particular, the maximum incident angle that makes the right-hand side equal to 1 is called the **critical angle** and is indicated with θ_c :

$$\sin \theta_c = \frac{n_2}{n_1} \quad (\text{Critical angle}) \quad (22.6)$$

In conclusion, it is possible to exit from a denser medium only with angles $\theta_1 < \theta_c$. The light cannot exit the medium and is fully reflected back at angles greater than this. This interesting phenomenon is called **total internal reflection**.

The most important application of total internal reflection is fiber optic cables. Both visible light and other electromagnetic waves can be transmitted through fiber optic cables with very slight losses.



Figure 22.10: Light coming out of water shows a pencil as if broken.

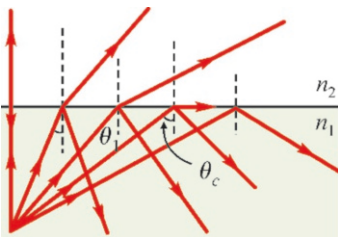


Figure 22.11: Total internal reflection and critical angle θ_c .

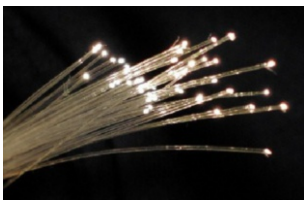


Figure 22.12: Light entering one end of a fiber optic cable is transmitted to the other side by total internal reflection.

Dispersion of Light

White light incident on a glass prism is dispersed into all of the colors of visible light after passing through the glass medium. As seen in the adjacent figure, all of the components of visible light from red to violet are refracted at different degrees. Red light, which has a greater wavelength, is refracted less and violet light, which has a lesser wavelength, is refracted more.

This property, which shows that the index of refraction is different at each wavelength, is called the **dispersion** of light:

$$n = n(\lambda) \quad (\text{Dispersion}) \quad (22.7)$$

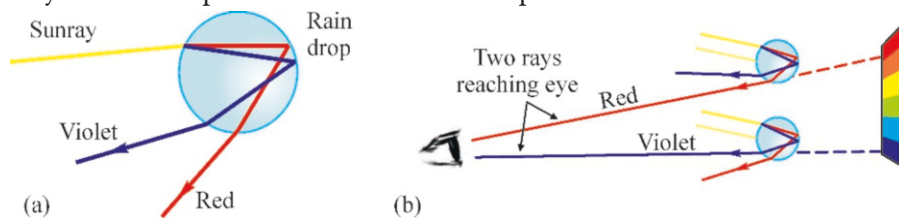
As a rule, *index of refraction decreases with wavelength*.

Recall that the index of refraction depends on the speed of light with the formula $n = c/v$. The speed of light c in vacuum is the same in all wavelengths, however, its velocity v is different at each wavelength. According to the figure above, the speed of light in a medium with a larger wavelength (red) is greater than that of violet light.

Rainbow

The dispersion property of light combined with the *total internal reflection* effect that we discussed above produces one of the most spectacular scenes in nature. A **rainbow** that contains all of the colors and that seems to be suspended in the air is observed when the sun comes out after rain.

Let us explain in a figure how a rainbow is formed. Consider the paths traveled by the red and violet components inside of a ray of sunlight incident on a spherical droplet of water, as shown in Figure 22.15a. According to dispersion property, violet light will be refracted more than red light in the droplet. When these two rays reach the back surface of the sphere, in certain droplets, the two rays incident at angles higher than the critical angle cannot exit the droplet, and thus reflect inside due to total internal reflection. After a few reflections, these two rays will finally exit the droplet and continue on their paths.



Let us now examine the effect generated when the red and violet rays from two such droplets reach our eyes. As shown in Figure 22.15b, from two droplets located at different locations, we observe the light from the top one as red and the light from the bottom one as violet. The droplets in the area between these two ends are lined up so as to reflect all of the other colors of visible light.

These particular reflections take place only within a certain range of degree. Other reflections do not reach our eyes. Also, we observe the same phenomenon within the same degree range from different directions. As a result, a colorful image appears in an arch-shaped region.

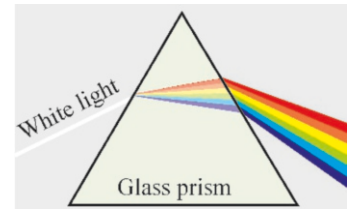


Figure 22.13: Dispersion of light.



Figure 22.14: Rainbow.

Figure 22.15: (a) Total internal reflection inside of a spherical water droplet. (b) The image of a rainbow formed in the eye by reflections from droplets at various angles.

Example 22.1

The yellow light of sodium, which has a 598 nm wavelength in vacuum, enters a glass medium with an index of refraction of 1.6. (a) What is the frequency of this light in vacuum? (b) What is its speed in a glass medium? (c) What is its wavelength in a glass medium?

Answer

(a) The relation between the speed, frequency and wavelength of a wave is found using Eq. (22.2):

$$f = \frac{v}{\lambda}$$

We calculate frequency by taking the speed of light in vacuum as c :

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{598 \times 10^{-9}} = 5 \times 10^{14} \text{ Hz}$$

(b) Frequency remains constant when light passes into another medium, however, its speed and wavelength changes. We use Eq. (22.1), in which we defined the index of refraction:

$$n = \frac{c}{v} \rightarrow v = \frac{c}{n}$$

From here, we calculate the speed of light in a glass medium:

$$v = \frac{c}{n} = \frac{3 \times 10^8}{1.6} = 1.9 \times 10^8 \text{ m/s}$$

(c) The wavelength in a glass medium can either be calculated using $\lambda = v/f$ in terms of speed and frequency or by taking the ratio of these values in the two media. There is less calculational error in taking the ratio:

$$\frac{\lambda_2}{\lambda_1} = \frac{v_2}{v_1}$$

$$\lambda_2 = \frac{v_2}{v_1} \lambda_1 = \frac{1.9}{3} \times 598 = 379 \text{ nm.}$$

Example 22.2

A light ray is sent from air to glass ($n = 1.60$) with a 30° angle of incidence. (a) What is the angle of refraction in the glass medium? (b) The light is sent at such an angle that its angle of refraction is half its angle of incidence: $\theta_2 = \theta_1/2$. What is the angle of incidence?

Answer

(a) We use Snell's law ($n_1 = 1$, $n_2 = n$):

$$\sin \theta_1 = n \sin \theta_2$$

We substitute the values and solve for θ_2 :

$$\sin \theta_2 = \frac{1}{n} \sin \theta_1 = \frac{1}{1.6} \times \sin 30 = 0.31$$

We use tables or a calculator to find the angle with a known sinus value:

$$\theta_2 = 18^\circ.$$

(b) In Snell's law, we take $\theta_2 = \theta_1/2$ and solve for θ_1 :

$$\sin \theta_1 = n \sin \theta_1/2$$

Using the half angle formula $\sin \theta_1 = 2 \sin \theta_1/2 \cos \theta_1/2$ from trigonometry and simplifying, we get

$$2 \cos \theta_1/2 = n \rightarrow \cos \frac{\theta_1}{2} = \frac{n}{2}$$

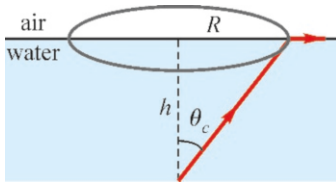
From here, we calculate the cosine of the half angle:

$$\cos \frac{\theta_1}{2} = \frac{1.6}{2} = 0.8 \rightarrow \frac{\theta_1}{2} = 37^\circ.$$

From here, we find that $\theta_1 = 74^\circ$.

Example 22.3

The light emitted from a lamp located at the bottom of a 10 m-deep-pool produces an illuminated circle at the surface of the water. What is the radius of this circle? (The index of refraction of water is 1.33.)

**Answer**

As shown in the figure, only the rays up to the critical angle

θ_c among those sent from inside of the water can pass into the air, while the rest undergo total internal reflection. We therefore first find the critical angle.

Here, the first medium is water ($n_1=n$) and the second medium is air ($n_2=1$). Let us write Snell's law:

$$n \sin \theta_1 = \sin \theta_2$$

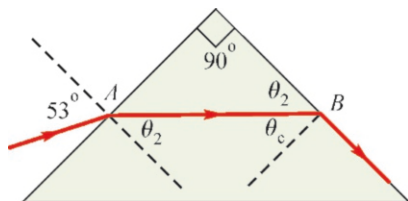
We find the critical angle by taking the maximum value of the angle of refraction in air as $\theta_2 = 90^\circ$:

$$n \sin \theta_c = 1 \times \sin 90^\circ = 1$$

$$\sin \theta_c = \frac{1}{n} = \frac{1}{1.33} = 0.75 \rightarrow \theta_c = 49^\circ$$

We write the expression for the radius R by examining the right triangle in the figure:

$$R = h \tan \theta_c = 10 \times \tan 49^\circ = 10 \times 1.15 = 11.5 \text{ m.}$$

Example 22.4

The apex angle of a prism is 90° . A ray that enters the prism with an angle of incidence of 53° from point A on one side exits from the other side at point B as parallel to the surface. Determine the index of refraction of the glass.

Answer

We write Snell's law for the first surface:

$$\sin 53^\circ = n \sin \theta_2 \quad (1)$$

We need a second equation, as both n and θ_2 are unknown. For this purpose, let us look at the second surface. As the refracted rays go parallel to the surface at point B, there is total internal reflection, in other words, the rays are incident on the surface at the critical angle θ_c . As seen in the drawing in the figure, we have $\theta_c = 90^\circ - \theta_2$. Let us write the critical angle formula:

$$\sin \theta_c = \frac{1}{n} \rightarrow n \sin(90^\circ - \theta_2) = n \cos \theta_2 = 1 \quad (2)$$

We divide equations (1) and (2) on both sides:

$$\frac{n \sin \theta_2}{n \cos \theta_2} = \frac{\sin 53^\circ}{1} \rightarrow \tan \theta_2 = \sin 53^\circ = 0.8$$

From here, we find $\theta_2 = 39^\circ$ and calculate n by substituting in the second equation:

$$n = \frac{1}{\cos 39^\circ} = 1.28$$

22.3 IMAGE BY REFLECTION – MIRRORS

Formation of an Image by Reflection

In the optics sense, an **object** is a body from which light rays originate. It can be a lamp, in other words, generate its own light. Other objects may reflect back the rays that they receive from the environment. When light rays diverging from an object reach the intersection of two media, they are partially reflected and partially transmitted to the second medium through refraction. An **image** is formed at the position where either reflected or refracted rays converge. It is a **real image** if the rays really intersect and a **virtual image** if only the extensions of the rays intersect.

(A practical method for determining whether the image is real or virtual is to place a screen at the location of the image. A real image will form on the screen, but a virtual image will not.)

In this section, we shall examine the properties of images formed through reflection. Mirrors are used to obtain images through reflection. Mirrors are surfaces that are polished or that have their reflection properties increased through coating by another material and reflect almost all of the incident light. The most commonly used types are plane and spherical mirrors.

Plane Mirror

When we look at a plane mirror, what we see makes it look as if an exact copy of whatever object is standing in front of the mirror is behind the mirror (Figure 22.16). According to our definition above, this is a *virtual image*, because the two rays are not really intersecting; rather it is as if their extensions are intersecting at a point behind the mirror. In the case of a plane mirror, the image is always virtual. However, the image has the same length, direction and distance from mirror as the object.

In Figure 22.16, the distance of the object from the mirror is indicated with p and the distance of the image to the mirror is indicated with q . The absolute value of these two distances is equal. However, the object and image positions are taken as algebraic numbers in the calculations, as we shall see later.

Sign rule 1: According to the assumption made in optics, *the positions of real objects and images are taken as positive and the positions of virtual images and objects are taken as negative.*

Accordingly, the relation between the positions of the object and the image in a plane mirror should be expressed as follows:

$$q = -p$$

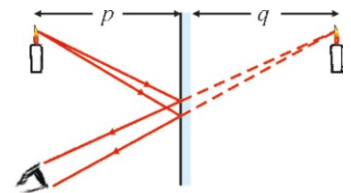


Figure 22.16: The image is virtual in a plane mirror because the extensions of the rays intersect behind the mirror.

Spherical Concave Mirror

In a concave mirror, the rays are reflected from the inner surface of a sphere. If we use C to indicate the center of the sphere with radius R , the line extending from the middle point O on the mirror to C and beyond is called the **principal axis**: $CO = R$.

At least two rays from the object are sufficient to obtain an image in a concave mirror. We know two such rays from our laws of reflection:

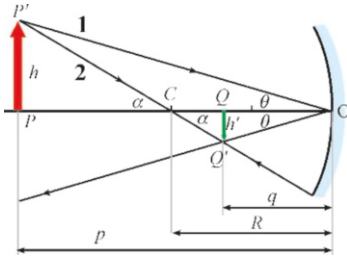


Figure 22.17: Coordinates in a concave mirror.

1. The ray drawn from the object to the middle point O will be reflected back at an equal angle with the principal axis (Figure 22.17, ray 1).
2. The ray passing through the center C of the sphere will be reflected back on itself, since it falls perpendicularly on the mirror (Figure 22.17, ray 2).

A real image forms at the intersection of these two rays. If they do not intersect, their extension beyond the mirror will form a virtual image, to be examined later.

Let us use the coordinates shown in Figure 22.17 to find the position and height of the image. Let p be the position of the object at point P as measured from the mirror and q be the position of the image at point Q .

Also, let us use h to indicate the height of the object and h' to indicate the height of the image. We make a new assumption here for the heights:

Sign rule 2: *The height of an inverted image is expressed with a negative sign.* This means that the h' shown in the figure has a negative sign.

As the right triangles OPP' and OQQ' facing the same angle θ are similar triangles, the ratios of their sides are equal:

$$\frac{h}{-h'} = \frac{p}{q} \quad (22.8)$$

Note that, as the height of the inverted image h' is negative, an extra negative sign was placed before it to make it positive.

This time, let us use the similarity of the triangles CPP' and CQQ' facing the same angle α . Remember to take the height of the inverted image as $(-h')$. When writing the ratios of the sides, we get

$$\frac{h}{-h'} = \frac{p - R}{R - q}$$

Eliminating h/h' from the last two formulas, we get

$$\frac{p}{q} = \frac{p - R}{R - q}$$

From here, we obtain the formula that gives us the position of the image q :

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R} \quad (22.9)$$

Focal point: When the object is very far away, in particular, when $p \rightarrow \infty$ in the equation above, we have $1/p = 0$ and the image position is:

$$q = \frac{R}{2}$$

This special point is called the **focal point** F and its distance to the mirror is called the **focal length** $f=R/2$. The focal point is the midpoint between the mirror and the center of the mirror.

Likewise, when the object is placed at the focal point, $p=f=R/2$ in the equation above, we have $1/q=0$ and $q\rightarrow\infty$, in other words, the image is at infinity.

In conclusion, the concave mirror equation in terms of the focal length f is:

$$\boxed{\frac{1}{p} + \frac{1}{q} = \frac{1}{f}} \quad (\text{Concave mirror equation}) \quad (22.10)$$

Magnification: The ratio of the image height h' to the object height h is called the *magnification* and is indicated with $M = h'/h$. Eq. (22.8) gives us the magnification in terms of positions:

$$M = \frac{h'}{h} = -\frac{p}{q} \quad (\text{Magnification}) \quad (22.11)$$

Also, solving the equation (22.10) for q , we get

$$q = \frac{pf}{p-f}$$

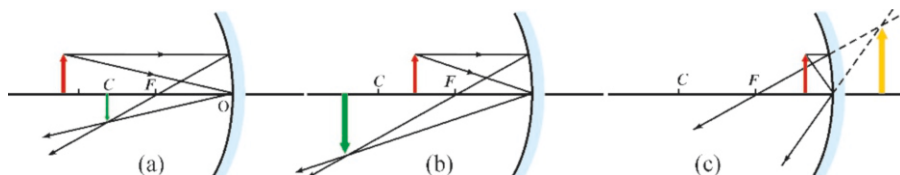


Figure 22.18: Images in an concave mirror: (a) The image is real and smaller if the object is beyond the center, (b) The image is real and larger if the object is between the center and the focal point, (c) The image is virtual if the the object is closer than the focal point.

These two equations show us the possibilities for the image position and the magnification:

- If $p > f$, we have $q > 0$ and $M < 0$. In other words, beyond the focal point, we have a real, inverted image (Figure 22.18a and b).
- If $p < f$, we have $q < 0$ and $M > 0$. In other words, inside of the focal point, we have a virtual, upright image (Figure 22.18c).

Concave mirrors are used to collect light or to send light to faraway points. They are used to focus light in telescopes, lasers and solar panels. On the other hand, light emitted from the focal point in automobile headlights and flashlights can be sent as parallel beams over very great distances. Also, they provide a bigger virtual image in dentist's and shaving mirrors.

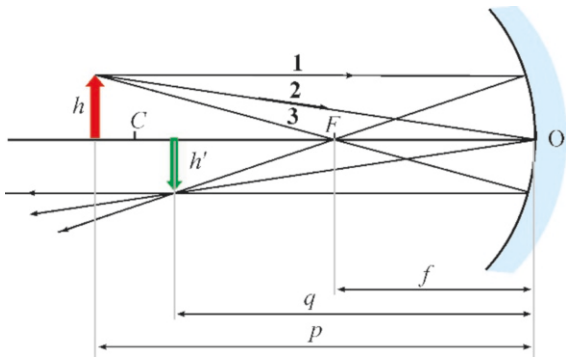


Figure 22.19: The concave mirror used in the Hubble space telescope.

Example 22.5

The focal length of a concave mirror is $f=30$ cm. (a) Where will the image of an object with 1 m in size placed 120 cm away from the mirror form and what will its size be? (b) Answer the same question for an object with 1 m in size placed 10 cm away from the mirror.

Answer A diagram should always be drawn in mirror calculations to get an idea about the image. Certain particular rays are useful for obtaining the image. Three particular rays are shown in the following figure:



1st ray: Each ray coming from infinity will be reflected from the focal point.

2nd ray: The ray incident at the central point (O) of the mirror will be reflected at an equal angle.

3rd ray: The ray passing through the focal point (F) will be reflected as parallel.

Only two of these three rays are sufficient, because the image will form at the intersection of two rays.

(a) We write the concave mirror equation (22.10):

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \rightarrow q = \frac{pf}{p-f}$$

The focal length is taken as positive if the mirror is concave: $f = 30$ cm. Accordingly, we calculate the image of an object $p = 120$ cm away:

$$q = \frac{120 \times 30}{120 - 30} = 40 \text{ cm}$$

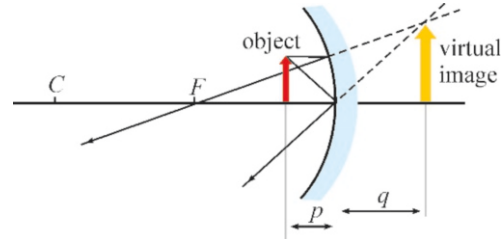
The image is real, because q is positive.

We use the magnification M that we defined with formula (22.11) to find the size of the image:

$$M = \frac{h'}{h} = -\frac{q}{p} = -\frac{40}{120} = -\frac{1}{3}$$

The negative sign indicates that the image is inverted. Therefore, a triply diminished inverted image is formed.

(b) This time, the object is placed between the focal point and the mirror. The image is formed as follows, if we take two particular rays, one from the center and one from the focal point:



The image position is calculated using the same formula:

$$q = \frac{10 \times 30}{10 - 30} = -15 \text{ cm}$$

The image is virtual, because q is negative.

We then calculate the magnification:

$$M = \frac{h'}{h} = -\frac{q}{p} = -\frac{(-15)}{10} = +1.5$$

The positive sign means that the image is upright, in other words, in the same direction as the object. Therefore, an image enlarged by 1.5 times is formed in the same direction.

Convex Mirror

A convex mirror reflects rays from the outer surface of a sphere. We again take two rays to obtain the image of a convex mirror. As shown in Figure 22.20, since one of the rays is directed towards the center C of the sphere, it is perpendicularly incident to the surface of the mirror and is reflected back in the same direction. When the other ray is drawn towards the point O on the principal axis, it will be reflected at an equal angle with the principal axis.

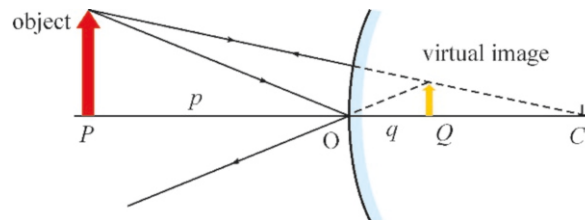


Figure 22.20: The image is always virtual in a convex mirror.

Reflected rays will never intersect at any point, regardless of which other rays are taken, but their extensions intersect behind the mirror. Therefore, the image is always virtual in a convex mirror.

There is no need to derive a new equation for the convex mirror. In a concave mirror, the distance p of the virtual image behind the mirror was taken with a negative sign. Likewise, the same formula will be valid again if we take the radius R , and therefore the focal point f , as negative, since the center of a concave

mirror is behind the mirror:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \quad (\text{Convex mirror equation, } f \text{ negative}) \quad (22.12)$$

It can easily be proven that the image is always virtual, in the same direction as the object and smaller in convex mirrors. Solving the same equation for q and writing the magnification as well, we get

$$q = \frac{pf}{p-f} \quad M = \frac{h'}{h} = -\frac{q}{p} = \frac{-f}{p-f}$$

Considering that f is always negative in the formula on the left, the numerator will always be negative and the denominator will always be positive. Therefore, q will always be negative, in other words, the image will always be virtual. Also, since the denominator is always larger, the magnification will be less than 1, in other words, the image will always be smaller than the object.

Convex mirrors can contain the image of a broader area, because the image is smaller than the object. Due to this property, they are used in the wing mirrors and rear-view mirrors of automobiles.



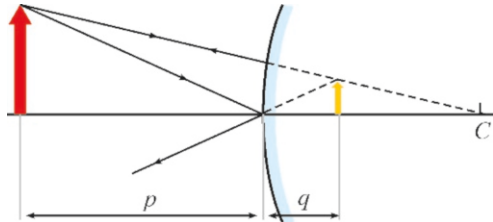
Figure 22.21: A wing mirror on a car.

Example 22.6

The focal length of a convex mirror is 20 cm. Where will the image of an object with 1 m in size placed 60 cm away from the mirror form and what will its size be?

Answer

The image will form as follows if we take two particular rays, one reflecting from the midpoint at an equal angle, and the other directed towards the center and reflected back on itself:



(a) We write the formula (22.10) that we found for the convex mirror:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \rightarrow q = \frac{pf}{p-f}$$

The focal length should be taken as *negative*, because the mirror is convex: $f = -20$ cm. We calculate the image position accordingly:

$$q = \frac{60 \times (-20)}{60 - (-20)} = -\frac{120}{80} = -15 \text{ cm}$$

The image is virtual, because q is negative.

We then calculate the magnification:

$$M = \frac{h'}{h} = -\frac{q}{p} = -\frac{(-15)}{60} = +\frac{1}{4}$$

The positive sign means that the image is upright, in other words, in the same direction as the object. Therefore, an image diminished by 3/4 is formed in the same direction.

22.4 IMAGE BY REFRACTION—LENSES

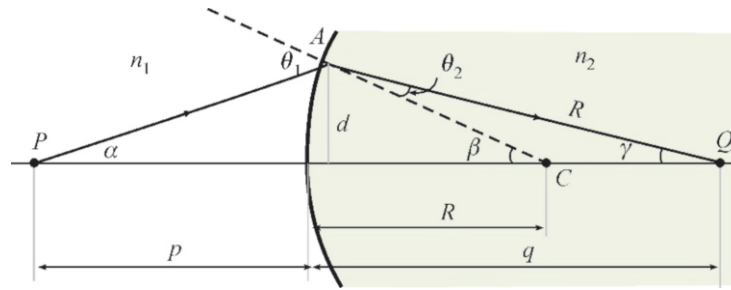
Light rays exiting from an object may again intersect to form an image after refracting into another medium. In glass lenses, rays first enter the glass medium and then get refracted from there to continue on their paths, subsequently going back into the first medium. Therefore, they get refracted twice.

Refraction from a Spherical Surface

In order to understand the image formed by a lens, let us first solve half of the problem and consider the image in a semi-infinite glass medium.

In Figure 22.22, rays coming from a medium with index of refraction n_1 are incident on a spherical surface with radius R and then pass into the second medium with index of refraction n_2 .

Figure 22.22: Coordinates in a spherical surface.



This time, let us take the object as a point P on the principal axis. Let one of the rays emitted from that point be the principal axis itself. This ray directly passes into the second medium when it is incident on the surface at a right angle. Let the other ray start at angle α with the principle axis and be incident to the surface of the glass at point A at angle θ_1 with the normal.

According to Snell's law, this ray will get refracted by a certain angle θ_2 . The image will form at the point Q where this ray intersects the principal axis.

If a ray with a very small angle α is chosen, the ray will be incident to the surface almost perpendicularly, in other words, θ_1 , and therefore θ_2 , will be very small. Accordingly, Snell's law is simplified at very small angles where we can take $\sin \theta \approx \theta$:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad \implies \quad n_1 \theta_1 \approx n_2 \theta_2$$

In a triangle, an exterior angle is equal to the sum of the two opposite interior angles. Applying this property to the triangles PAC and QAC , we get

$$\theta_1 = \alpha + \beta \quad \text{and} \quad \beta = \theta_2 + \gamma$$

Eliminating θ_1 and θ_2 from these three formulas, we get

$$n_1 \alpha + n_2 \gamma = (n_2 - n_1) \beta \quad (22.13)$$

As angles α, β, γ are very small, using the approximate expression $\tan \theta \approx \theta$, we can calculate the tangents of these angles from Figure 22.22. Using the perpendicular d facing all three angles, we get

$$\alpha \approx \tan \alpha \approx \frac{d}{p} \quad \beta \approx \tan \beta \approx \frac{d}{R} \quad \gamma \approx \tan \gamma \approx \frac{d}{q}$$

We substitute the values of these three angles in the equation (22.13) and simplify:

$$\boxed{\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}} \quad (22.14)$$

This formula is valid for all n_1, n_2 values. Also, this proof was made for a convex spherical surface. But it can also be shown to be valid for a concave spherical surface. In order to do this, we agree on the sign of the radius R of the sphere as follows: R is taken as positive if the center C of the sphere surface is inside of the medium n_2 and R is taken as negative if it is outside. Then, this formula will also be valid for refraction from a concave surface.

This expression shall be used later to obtain the lens formula.

Refraction at a Plane Surface

Let us apply the above formula to a special case in which the interface is a plane. If we consider a plane as a sphere with infinite radius, when we take $R \rightarrow \infty$ in the formula (22.14), we get

$$\frac{n_1}{p} + \frac{n_2}{q} = 0$$

From here, we find the position of the image:

$$q = -\left(\frac{n_2}{n_1}\right)p \quad (22.15)$$

According to this result, the image will always be virtual on a plane surface. For example, a pool seems to be shallower when looked at from the outside ($n_1 > n_2$). Likewise, a fish looking at us from under the water ($n_1 < n_2$) sees us as further away.

Thin Lenses

Glass lenses with two spherical surfaces are used in all kinds of optical instruments. The image may be real or virtual, depending on whether these surfaces are convex or concave. Lenses with convex surfaces are called **converging lenses** and those with concave surfaces are called **diverging lenses**. This designation will be understood later.

We have to use the formula (22.14) twice in order to obtain a formula that is valid for all lenses. To simplify the calculations, let us take the first medium as air and the index of refraction as $n \approx 1$.

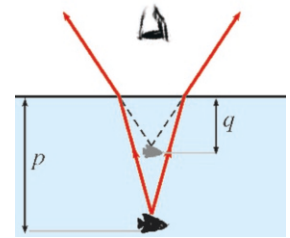
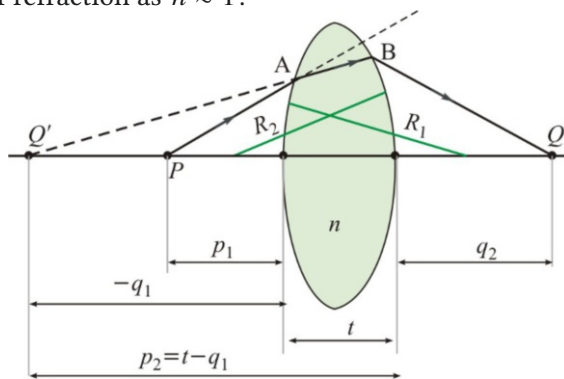


Figure 22.23: A fish under water seems nearer when looked at from above.

Figure 22.24: Coordinates for a lens with thickness t .

Let us consider a lens with radius of the first surface R_1 , radius of the second surface R_2 and thickness t (Figure 22.24). Let p_1 be the distance between an object at point P outside of this lens. Let q_1 be the position of the intermediate virtual image (Q') formed on the same side as P (q_1 is negative). Accordingly, if we apply Eq. (22.14) to a surface with radius R_1 at the interface between air ($n_1 = 1$) and glass ($n_2 = n$), we get

$$\frac{1}{p_1} + \frac{n}{q_1} = \frac{n-1}{R_1}$$

The rays intersecting at this virtual image q_1 then continue on their paths and arrive at the second surface located at distance t . If the thickness of the glass is t , this intermediate image is considered as an object at position $p_2 = t - q_1$ for the second surface. We apply the equation (22.14), this time to the surface with

radius R_2 at the interface of glass ($n_1 = n$) and air ($n_2 = 1$):

$$\frac{n}{t - q_1} + \frac{1}{q_2} = \frac{1 - n}{R_2}$$

Note that R_2 is *negative*, because it is inside of the glass.

Now, we take $t \approx 0$, assuming that the lens is thin and we add these last two equations on both sides and eliminate q_1 . Also, if we remove the subscripts and rename them as the first object ($p_1 = p$) and the final image ($q_2 = q$), we get the following result:

$$\frac{1}{p} + \frac{1}{q} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad (\text{Lens makers' equation}) \quad (22.16)$$

This result is known as the **lens makers' equation**. We can write it in terms of the focal length f to make it more practical. As $q = f$ when $p \rightarrow \infty$, substituting these values, we get

$$0 + \frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

and f is positive for a converging (convex) lens because we take $R_1 > 0$ and $R_2 < 0$. f is negative for a diverging (concave) because $R_1 < 0$ and $R_2 > 0$. Every lens has two focal points, because the rays can go in both directions.

In conclusion, the thin lens equation for both converging and diverging lenses is as follows:

$$\boxed{\frac{1}{p} + \frac{1}{q} = \frac{1}{f}} \quad (\text{Thin lens equation}) \quad (22.17)$$

f is positive for a converging lens and f is negative for a diverging lens. The image is real if q is positive and virtual if it is negative.

Magnification (M)

As in mirrors, the ratio of the height h' of the image to the height h of the object is the magnification, and its relation with the positions is as follows:

$$M = \frac{h'}{h} = -\frac{q}{p} \quad (22.18)$$

If M is positive, the image is in the same direction as the object and virtual; if M is negative, the image is inverted and real.

Power of a Lens (D).

Opticians use the optical power of a lens, rather than its focal length. The power of a lens is just the inverse of the focal length f , which is expressed in meter units:

$$D = \frac{1}{f \text{ (meters)}}$$

The unit of lens power is the **dioptr**. For example, a lens with $D = 3$ dioptr will focus parallel incident rays to $f = 1/D = 1/3 \text{ m} = 33 \text{ cm}$.

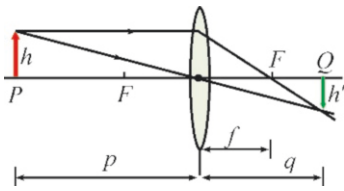


Figure 22.25: Converging lens.

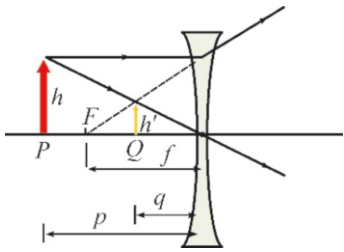


Figure 22.26: Diverging lens.

Example 22.7

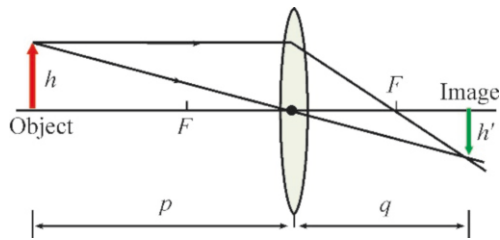
The focal length of a thin converging lens is $f = 40$ cm.

(a) Where will the image of an object with a size of 1 m and placed 60 cm away from the lens form? And what will its size be?

(b) Answer the same question for the same object placed 30 cm away from the lens.

Answer

(a) To form the image, let us consider two special rays, one coming from infinity in parallel to the axis and going through the focal point and the other passing through the center of the lens without being refracted. Accordingly, the image is formed as follows:



We write Eq. (22.17), which we found for lenses:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \rightarrow q = \frac{pf}{p-f}$$

The focal length is taken as positive, as the lens is converging: $f = 40$ cm. Accordingly, we calculate the image of an object $p = 60$ cm away:

$$q = \frac{60 \times 40}{60 - 40} = 120 \text{ cm}$$

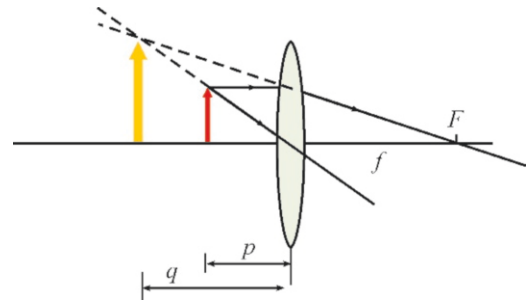
The image is real, because q is positive.

We use the magnification M that we defined with formula (22.11) to find the size of the image:

$$M = -\frac{q}{p} = -\frac{120}{60} = -2$$

The negative sign indicates that the image is inverted. Therefore, a doubly magnified inverted image is formed.

(b) This time, the object is placed between the focal point and the lens. The image is formed as follows if we take two particular rays, one of which passes through the center of the lens without being refracted and one coming from infinity as parallel to the axis and passing through the focal point.



The image position is again calculated using the same formula:

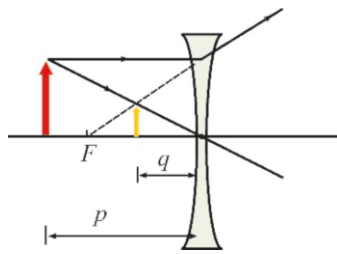
$$q = \frac{30 \times 40}{30 - 40} = -120 \text{ cm}$$

The image is virtual, because q is negative.

We then calculate the magnification:

$$M = -\frac{q}{p} = -\frac{(-120)}{30} = +4$$

The positive sign means that the image is upright, in other words, in the same direction as the object. Therefore, a quadruply enlarged image is formed in the same direction.

Example 22.8

The focal length of a diverging lens is 50 cm. Where will the image of an object with a size of 1 m placed 75 cm away from the lens form and what will its size be?

Answer

The image will form as follows if we take two particular rays, one coming from infinity and reflecting from the focal point and the other directed towards the center and passing without being refracted:

We write the thin lens equation:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \rightarrow q = \frac{pf}{p-f}$$

The focal length should be taken as *negative*, because the lens is diverging: $f = -50$ cm. We calculate the image position accordingly:

$$q = \frac{75 \times (-50)}{75 - (-50)} = -\frac{3750}{125} = -30 \text{ cm}$$

The image is virtual, because q is negative.

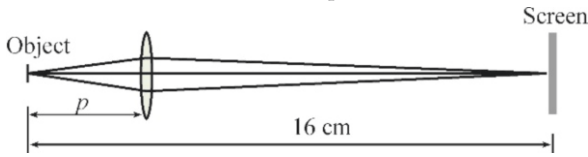
We then calculate the magnification:

$$M = -\frac{q}{p} = -\frac{(-30)}{75} = +\frac{2}{5} = 0.4$$

The positive sign means that the image is upright, in other words, in the same direction as the object. Therefore, an image diminished by $2/5$ is formed in the same direction.

Example 22.9

An object is placed 16 cm away from a screen. It is required that we obtain the image of the object on the screen using a converging lens with a focal length of 3 cm. How far away from the object should the lens be placed?

**Answer**

The image should be real (q positive), because it must appear on the screen. Therefore, the first relation between the

unknowns p and q will be as follows:

$$p + q = 16$$

Second, these p, q values fulfill the lens formula:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} = \frac{1}{3}$$

If we get $q = 16 - p$ from the first formula and substitute it in this equation, we obtain an equation for p :

$$\frac{1}{p} + \frac{1}{16 - p} = \frac{1}{3} \rightarrow p^2 - 16p + 48 = 0$$

We find the roots of this 2nd degree equation and calculate q starting from p . There are two solutions:

$$p = \begin{cases} 4 \text{ cm} \\ 12 \text{ cm} \end{cases} \rightarrow q = \begin{cases} 12 \text{ cm} \\ 4 \text{ cm} \end{cases}$$

Example 22.10

An object placed in front of a converging lens with a focal length of 24 cm has an image that is inverted and 3 times magnified. Find the position of the object.

Answer

We establish and solve two equations for the two unknowns (p, q) in this problem. The first one of these is the lens equation:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} = \frac{1}{24}$$

The other equation is obtained from the magnification:

$$M = -\frac{q}{p}$$

The magnification should be taken as negative, because the image is inverted:

$$-3 = -\frac{q}{p} \rightarrow q = 3p$$

We substitute this expression for q in the lens equation and solve for p :

$$\frac{1}{p} + \frac{1}{3p} = \frac{1}{24} \rightarrow p = \frac{4 \times 24}{3} = 32 \text{ cm}$$

And the position of the image is found as $q = 3p = 96 \text{ cm}$.

22.5 OPTICAL INSTRUMENTS**Eye**

The human eye (Figure 22.27), which is the most perfect optical instrument, can see objects and distinguish colors within a very broad range. The eye is shaped like a sphere with a 2.5 cm radius, and the *cornea* is located at the front of its external surface, called the *sclera*, and acts as a window. A liquid-filled *anterior chamber* is located behind the cornea. Then, there is the transparent *pupil* located at the center of a diaphragm with an open center, called the *iris*.

The pupil contracts and expands to adjust the amount of light entering the eye. Behind it is located the eye lens with an index of refraction of 1.396. Ciliary muscles around it hold the lens. Then, there is the liquid-filled *vitreous chamber*.

The rear of the eye is the *retina*, which is covered with many nerve ends (neurons). The *fovea*, where the nerve ends are densest, allows us to see clearly. The *optic disk* region, located at the back, where the nerve ends are gathered, is a blind spot, and rays incident here are not sent to the brain. The **optic nerve**, to which the nerve ends are connected, transmits the signals that it receives to the brain.

The eye allows us to see by forming the image of an object on the retina. For this purpose, the muscles that hold the lens vary the focal length of the lens by relaxing and contracting, and ensure that the rays intersect on the retina. The state in which the muscles are relaxed is when the eye is focused on infinity. The muscles contract more as the object approaches the eye and it ultimately cannot

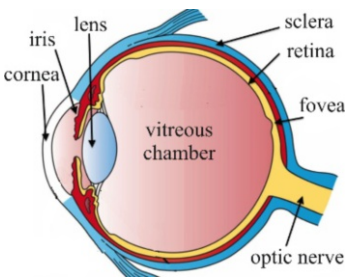


Figure 22.27: Human eye.

see clearly after a certain point, called the *near point*. The near point is around 25 cm in adults. The muscles lose their flexibility with age and the near point may recede up to 2-3 meters.

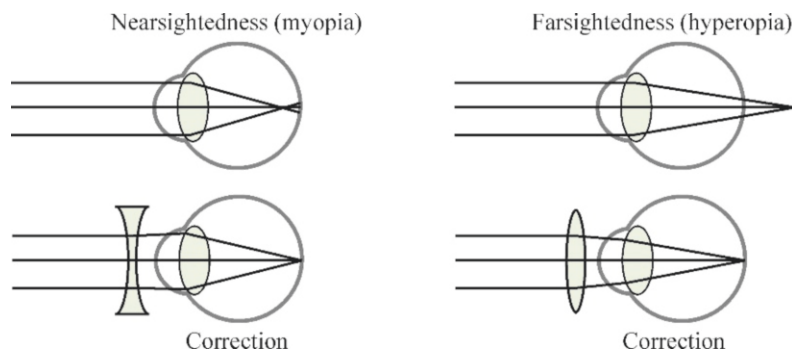


Figure 22.28: In a nearsighted eye, the image forms before the retina and is corrected with a diverging lens. In a farsighted eye, the image forms behind the retina and is corrected with a converging lens.

A net image is not formed on the retina in defective eyes (Figure 22.28). The eye has **nearsightedness** (myopia), if the image forms before the retina and **farsightedness** (hyperopia) if it forms behind the retina. The defect is corrected by using a diverging lens for nearsightedness and a converging lens for farsightedness.

Astigmatism, another eye defect, is caused by the difference in curvature in horizontal and vertical planes. An astigmatic eye does not see horizontal and vertical lines with the same clarity (Figure 22.29).

Microscope

We bring very small objects close to our eyes in order to see them. However, we cannot bring them closer than the near point of the eye, because the eye cannot focus on the object after the near point. In such a case, we use a simple magnifying glass or a more complex microscope.

Microscopes consist of at least two converging lenses. The one called the **objective lens** is located near the object and has a focal length of about 2–40 mm. The one on the side of the eye is called the **eyepiece lens** and has a larger focal length. The operating principle of the microscope is roughly as follows: If we can obtain a larger and real image of the small object near the focal point of the eyepiece, we can produce the final image as much larger and at the near point of the eye.

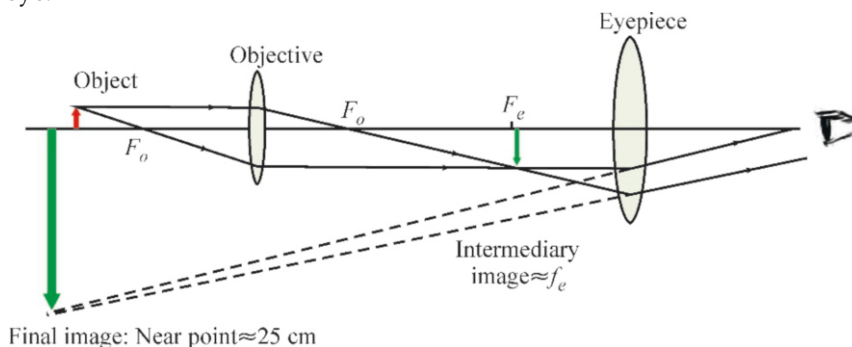


Figure 22.29: You have astigmatism if you do not see these lines with equal thickness when you look with one eye.

Figure 22.30: Microscope. The object is placed just beyond the focal point of the objective lens. The intermediate image is formed near the focal point of the eyepiece. The final image is at the near point.

In the microscope diagram in Figure 22.30, the object is placed just in front of the focal point of the objective lens. A larger and real intermediate image is thus formed. The diverging rays from this image are considered as a new object for the eyepiece. Then, the eyepiece's distance is varied to ensure that this real image is closer to the focal point of the eyepiece.

Recall the formula for magnification: If we use h to indicate the height and p to indicate the position of the object, and h' for the height and q for the position of the image, we get

$$M = \frac{h'}{h} = -\frac{q}{p}$$

First, let us calculate M_o , the magnification at the objective lens: We can take $p \approx f_o$, because the object is placed near the focal point of the objective lens. If we ignore the negative size, the magnification of the objective is as follows:

$$M_o = \frac{q_1}{f_o}$$

We take the following into consideration when calculating the magnification for the eyepiece: The angular magnification is what is important for the eye. Even if the size of the object is small, the angle at which the eye sees it becomes large. For example, a needle close to the near point of the eye is perceived as larger than a distant pole. For this reason, it is required that the final image given by the eyepiece be close to the near point of the eye, in other words, at approximately 25 cm in distance. Accordingly, the magnification of the eyepiece is (for $p_2 = f_e$ and $q_2 = 25$ cm),

$$M_e = \frac{25 \text{ cm}}{f_e}$$

Combining these results, the total magnification of the microscope is the product of these two magnifications:

$$M = M_o M_e = \frac{(25 \text{ cm}) q_1}{f_o f_e} \quad (22.19)$$

Microscopes operating with visible light have limited magnification. Microscopes using electron beams are much more powerful.

Telescope

Telescopes are used to see celestial bodies and other distant objects much larger and in detail. There are two types of telescopes: Refracting telescopes, which only use lenses and reflecting telescopes, which use a combination of lenses and mirrors.

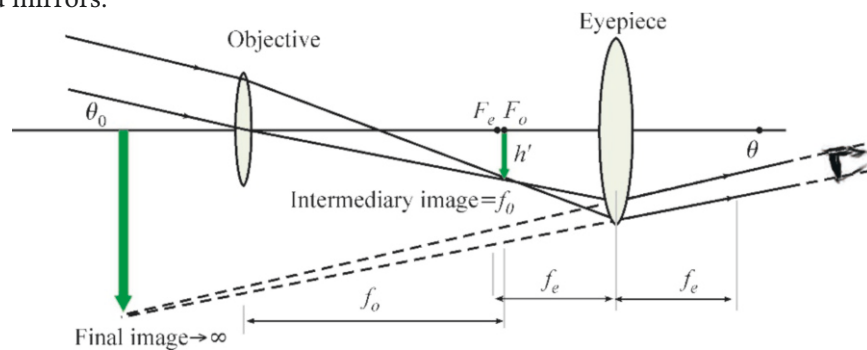


Figure 22.32: A telescope forms a larger image of an object located again at infinity.

The operating principle of the telescope is to form an image of an object located at infinity that is again located at infinity. The real image given by the objective lens is taken by the eyepiece and a final image is formed at infinity which is magnified further. In the diagram shown in Figure 22.32, incident rays from infinity form an inverted and real image at the focal point of the objective.



Figure 22.31: A microscope made in 1879.

If the focal point of the eyepiece is located right at this point, it will form a virtual image at infinity with the rays that it receives from this intermediate image.

The angular magnification of the telescope is defined as the angular ratio of the angle θ covered by the image to the angle θ_0 of the object observed with a naked eye:

$$m = \frac{\theta}{\theta_0}$$

As seen in the figure, the angle θ_0 covered by the object in the objective lens is equal to the angle covered by the intermediate image in the objective lens. The tangent of this angle is taken approximately as $\tan \theta \approx \theta$ for small angles. If we use f_o to indicate the focal length of the objective lens, we get

$$\theta_0 \approx \tan \theta_0 = \frac{h'}{f_o}$$

Likewise, the angle θ covered by the final image in our eye is equal to the angle covered in the eyepiece by the ray drawn as parallel from the intermediate image to final image. Its tangent is also taken as an approximation for small angles. If we use f_e to show the focal length of the eyepiece, we get

$$\theta \approx \tan \theta = \frac{h'}{f_e}$$

We use these two formulas to find the angular magnification of the telescope as follows:

$$m = \frac{\theta}{\theta_0} = \frac{f_o}{f_e} \quad (22.20)$$

It is necessary to use very large lenses, as the magnification of the telescope is proportional to the focal length of the objective lens. However, reflecting telescopes that utilize mirrors are preferred, because large lenses are both expensive to produce and difficult to install. Likewise, radio wave, X-ray or gamma ray telescopes are used in astronomy, rather than visible light.



Figure 22.33: The Mount Graham telescope in Arizona is the largest reflecting telescope in the world with a diameter of 2×8.4 meter.

Multiple-choice Questions

- Which of the following is correct for electromagnetic waves?
 - The electric field is perpendicular and the magnetic field is parallel to the direction of propagation.
 - The magnetic field is perpendicular and the electric field is parallel to the direction of propagation.
 - The electric and magnetic fields are perpendicular to the direction of propagation.
 - The electric and magnetic fields are parallel to the direction of propagation.
- What is the speed of light in a medium with index of refraction $n=1.5$?
 - $c/2$
 - $2c/3$
 - $3c/2$
 - $3c/4$
- Which of the following orderings of electromagnetic waves is done according to increasing frequency?
 - Microwave-X-ray-infrared-ultraviolet
 - Infrared-ultraviolet-X-ray-microwave
 - Microwave-infrared-ultraviolet-X-ray
 - Infrared-visible light-gamma rays-microwave
- Which of the following are correct?
 - Wavelength decreases as the speed of light decreases.
 - The speed of light decreases as the index of refraction of the medium increases.
 - Wavelength decreases as the index of refraction of the medium increases.
 - I & II
 - I & III
 - II & III
 - All
- Which of the following is the Huygens-Fresnel principle?
 - Microwave-X-ray-infrared-ultraviolet
 - Infrared-ultraviolet-X-ray-microwave
 - Microwave-infrared-ultraviolet-X-ray
 - Infrared-visible light-gamma rays-microwave

- (a) Every point of a medium on which light is incident reflects the wave.
 (b) Every point of a medium on which light is incident refracts the wave.
 (c) Every point of a medium on which light is incident constitutes a new source of waves.
 (d) Two waves, one passing and one reflected, are formed at every point of a medium on which light is incident.
6. Which of the following is Snell's law?
 (a) $n_1 \cos \theta_1 = n_2 \cos \theta_2$
 (b) $n_1 \sin \theta_1 = n_2 \sin \theta_2$
 (c) $n_1 \sin \theta_2 = n_2 \sin \theta_1$
 (d) $n_1 / \sin \theta_1 = n_2 / \sin \theta_2$
7. Which of the following is correct for a light ray sent from the water medium into the air?
 (a) It never passes into the air.
 (b) It always passes into the air.
 (c) It passes at angles larger than a critical angle.
 (d) It passes at angles smaller than a critical angle.
8. Which of the following is correct for a ray incident on the interface between water ($n=1.3$) and glass ($n=1.5$)?
 (a) The rays always pass from glass to water.
 (b) The rays always pass from water to glass.
 (c) The rays always pass in both ways.
 (d) All of the above.
9. In which of the following cases will the image of a real object on a mirror be real?
 I. If it is observed on a screen,
 II. If the reflected rays intersect at a point,
 III. If the image is inverted.
 (a) I & II (b) I & III (c) II & III (d) All
10. When will the image of a real object on a concave mirror be virtual?
 (a) When the object is between the center and the focal point,
 (b) When the object is between the focal point and the mirror,
 (c) When the object is between infinity and the focal point.
 (d) When the object is between infinity and the center.
11. When will the image of a real object on a convex mirror be real?
 (a) When the object is between the center and the focal point,
 (b) When the object is between the focal point and the mirror,
 (c) When the object is between infinity and the focal point,
 (d) Never.
12. What is the focal length of a concave mirror with a 40 cm radius of curvature?
 (a) 10 cm (b) 20 cm (c) 40 cm (d) 80 cm
13. What is the focal length of a convex mirror with a 20 cm radius of curvature?
 (a) -20 cm (b) -10 cm (c) 10 cm (d) 40 cm
14. When will the image of a real object on a converging lens be virtual?
 (a) Object is between the center and the focal point,
 (b) Object is between the focal point and the lens,
 (c) Object is between infinity and the focal point.
 (d) Object is between infinity and the center.
15. When will the image of a real object on a diverging lens be real?
 (a) Object is between the center and the focal point,
 (b) Object is between infinity and the center,
 (c) Object is between the focal point and the lens,
 (d) Never.
16. Which of the following are correct?
 I. The image is always virtual on a convex mirror.
 II. The image is always upright on a convex mirror.
 III. The image is always virtual on a concave mirror.
 IV. The image is always inverted on a concave mirror.
 (a) I & II (b) I & III (c) II & III (d) II & IV
17. Which of the following is correct for the magnification M ?
 (a) The image is inverted and real if $M > 0$.
 (b) The image is upright and real if $M > 0$.
 (c) The image is inverted and virtual if $M > 0$.
 (d) The image is upright and virtual if $M > 0$.
18. Which of the following is correct for the magnification M ?
 (a) The image is inverted and real if $M < 0$.
 (b) The image is upright and real if $M < 0$.
 (c) The image is inverted and virtual if $M < 0$.
 (d) The image is upright and virtual if $M < 0$.
19. Which of the following is correct?
 (a) The image is formed before the retina in a near-sighted eye.
 (b) The image is formed behind the retina in a near-sighted eye.
 (c) The image is formed before the retina in a farsighted eye.
 (d) The image is formed behind the retina in an astigmatic eye.
20. By what factor will the magnification of a microscope increase if its objective lens magnifies by 5 times and its eyepiece magnifies by 20 times?
 (a) 25 (b) 30 (c) 50 (d) 100

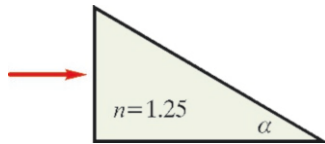
Problems

22.2 Reflection and refraction

22.1 A light with a 600 nm wavelength in vacuum enters a glass medium with an index of refraction 1.5. (a) What is the frequency of this light in vacuum? (b) What is its speed in a glass medium? (c) What is its wavelength in a glass medium?
[A: (a) 5×10^{14} Hz, (b) 2×10^8 m/s, (d) 400 nm.]

22.2 What is the wavelength in glass of a light with a 500 nm wavelength in water? ($n_{\text{water}} = 1.33$, $n_{\text{glass}} = 1.5$.)
[A: 440 nm.]

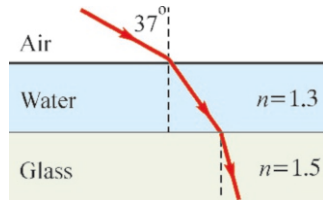
22.3 A light ray has a angle of refraction of 37° when sent from air towards glass at a 60° angle of incidence. Determine the index of refraction of the glass.
[A: 1.45.]



Problem 22.4

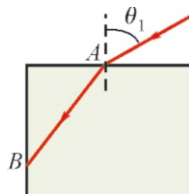
22.4 A prism with an apex angle of 90° is made of a transparent plastic material with $n=1.25$. A ray enters perpendicularly from one face of the prism, as shown in the figure above. What is the minimum value of the angle α such that the ray cannot exit from the hypotenuse?
[A: 37° .]

22.5 Water and glass layers with parallel surfaces are superimposed in the figure below. What will the angle of refraction at the glass layer be for a ray that enters the water from the air with an angle of incidence of 37° ?
[A: 24° .]



Problem 22.5 and Problem 22.6

22.6 At what maximum angle can rays sent upward from the glass layer in the figure above pass on to the water layer? (Hint: At the limit, the ray emerges into the air parallel to the surface.)
[A: 42° .]



Problem 22.7

22.7 What should the angle of incidence θ_1 be such that the light entering from the top face of the glass block shown in the figure with index of refraction $n = 1.4$ does not exit from

point B on the lateral surface? (Hint: What is the relation between the critical angle at point B and the angle of incidence at point A ?)
[A: 78° .]

22.8 A fish at the bottom of a 4 m deep lake looks upward toward the surface of the water. What will be the radius of the surface area from which this fish can see incident rays? (The index of refraction of water is 1.33.)
[A: 4.6 m.]

22.3 Image by Reflection - Mirrors

22.9 The focal length of a concave mirror is $f=30$ cm. (a) Where will the image of an object 1 m in size placed 40 cm away from the mirror form and what will its size and type be? (b) Answer the same question for an object 1 m in size placed 15 cm away.

[A: (a) $q = 120$ cm, real, inverted and 3 times larger. (b) $q = -30$ cm, virtual, upright and 2 times larger.]

22.10 The focal length of a convex mirror is 10 cm. Where will the image of an object 1 m in size placed 40 cm away from the mirror form and what will its size and type be?

[A: $q = -8$ cm, virtual, upright and 5 times smaller.]

22.11 When an object is placed 25 cm in front of a concave mirror, its image is inverted and 4 times enlarged. What is the focal length of the mirror?
[A: $f = 20$ cm.]

22.12 When an image is placed before a convex mirror with a focal length of 40 cm, its image is upright and 4 times smaller. What is the position of the object?
[A: $p = 120$ cm.]

22.13 There is 90 cm in distance between an object and a screen. At what distance from the object should a concave mirror with a 24 cm focal length be placed such that a clear image is obtained on the screen?
[A: $p = 30$ cm.]

22.14 The image of an object placed before a convex mirror is 15 cm behind the mirror, upright and 4 times smaller. What is the focal length of the mirror?
[A: $f = -20$ cm.]

22.4 Image by Refraction - Lenses

22.15 The focal length of a thin converging lens is $f = 30$ cm. (a) Where will the image of an object 1 m in size placed 90 cm away from the lens form and what will its size and type be? (b) Answer the same question for an object placed 10 cm away.

[A: (a) $q = 45$ cm, real, inverted and triply diminished, (b) $q = -15$ cm, virtual, upright and magnified 1.5 times.]

22.16 The focal length of a diverging lens is 20 cm. Where will the image of an object 1 m in size placed 60 cm away from the lens form and what will its size and type be?

[A: $q = -15$ cm, virtual, upright and 4 times diminished.]

22.17 An object is placed 125 cm away from a screen. We want to obtain the image of the object on the screen using a converging lens with a focal length of 20 cm. How far away from the object should the lens be placed?

[A: $p = 25$ cm and 100 cm.]

22.18 An object placed in front of a converging lens with a focal length of 30 cm has an image that is inverted and twice as large. Find the position of the object. [A: $p = 45$ cm.]

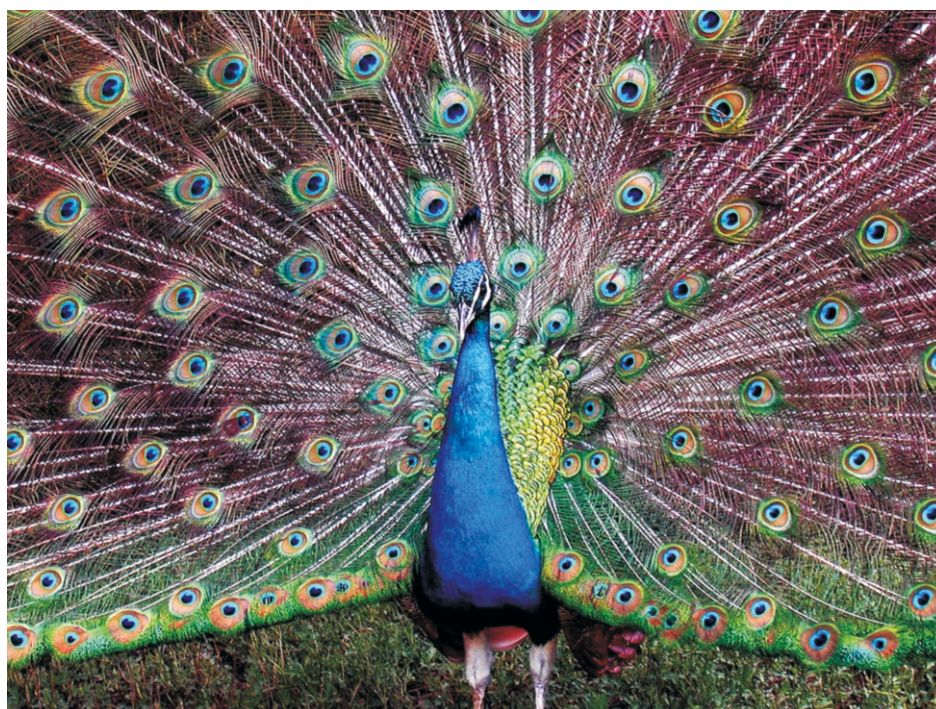
22.19 The image of an object placed before a converging lens is 200 cm away, inverted and 4 times larger. What is the focal length of the lens? [A: 40 cm.]

22.20 An object is placed 45 cm away from a screen. The image obtained on the screen by a converging lens placed in between is inverted and twice larger. What is the focal length of the lens? [A: $f = 10$ cm.]

?

23

WAVE OPTICS



The beautiful colors of this peacock are actually not the colors of its feathers. They result from an effect called *interference* of the light that results from the structure of the feathers.

In geometric optics, light was considered as a ray propagating along a line. This approach can be sufficient to obtain images with mirrors and lenses, but it fails to explain many other properties of light. We can see all of the colors of the rainbow on a soap bubble, on the surface of a compact disk or on the feathers of a peacock. This is caused by the interference effects of the light wave.

Therefore, we need to dwell further on the wave structure of light. The two most notable properties that distinguish a wave from a particle are its ability to display interference and diffraction effects. These effects constitute the basis of many techniques used in medicine and industry: X-rays, thin films, holography, etc. In this chapter, we will discuss the interference and diffraction effects that result from the wave properties of light.

23.1 YOUNG'S DOUBLE-SLIT EXPERIMENT

In chapter 11, we discussed the fact that waves display interference effects. When two waves superpose at a point at the same phase, they undergo *constructive interference* and their amplitudes are added. If they superpose at opposite phases, they undergo *destructive interference*, in other words, their amplitudes cancel each other out.

Light, which is an electromagnetic wave, should also display interference effects. However, it is very difficult to observe the interference effect, because the wavelength of light is very small ($\sim 10^{-7}$ m) and very precise experiments must be conducted.

In 1801, the English scientist Thomas Young (1773-1829) was able to demonstrate the interference of light experimentally. This setup, called **Young's double-slit experiment**, needs two sources emitting identical waves at the same phase (Figure 23.1). For this purpose, an obstacle with two slits is placed in front of a monochromatic (i.e., single wavelength) light source S . According to Huygens' principle, after the light reaches these slits, two waves with the same phase will be emitted from sources S_1 and S_2 .

When these two spherical waves arrive at a point on the screen, they will produce constructive interference if they arrive with the same phase and cause maximum illumination at that point. If they reach the screen at opposite phases, they will produce destructive interference and that point will remain dark. Therefore, dark and light **interference fringes** are observed on the screen.

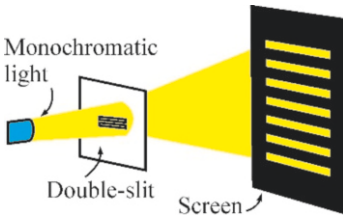
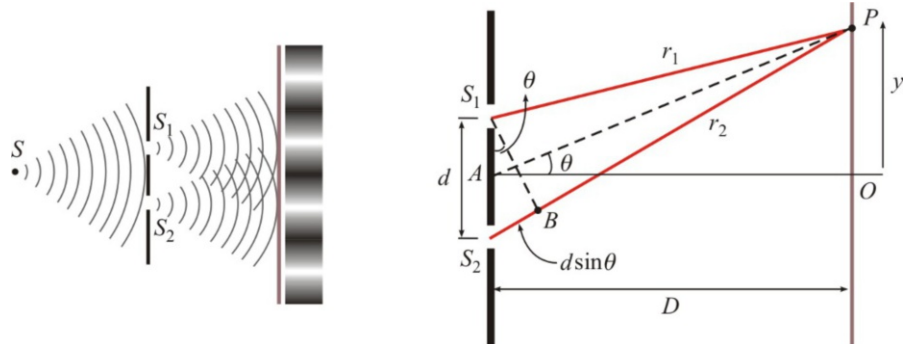


Figure 23.1: Young's double-slit experiment setup.

Figure 23.2: Young's experiment. (a) The wave from a single source generates two waves with the same phase after the slits S_1 and S_2 . (b) Coordinates.



Now let us calculate the positions of the interference fringes in Young's experiment. In the diagram in Figure 23.2, the distance between the slits is indicated with d and the distance of the screen is indicated with D . Let the distances traveled by two waves to a point P on the screen at distance y from the center O of the screen be r_1 and r_2 .

Let us draw a perpendicular line from the source S_1 to the path r_2 and form the right triangle S_1S_2B . If the screen is very far, in other words, if the paths r_1 and r_2 are very large with respect to d , the side S_2B will be the difference between the two paths. Considering the right triangle S_1S_2B , we get

$$S_2B = d \sin \theta$$

If this path difference is equal to integer multiples of the wavelength, in other words, equal to any one of the values $\lambda, 2\lambda, 3\lambda \dots$, there is constructive interference and a bright fringe appears at point P . Therefore, the angle θ_m of the m^{th} bright fringe will satisfy the following relation:

$$d \sin \theta_m = m\lambda \quad (m = 0, \pm 1, \pm 2, \pm 3 \dots) \quad (\text{Bright fringes}) \quad (23.1)$$

However, if the path difference is equal to odd multiples of a half wavelength, in other words, equal to any one of the values $\lambda/2, 3\lambda/2, 5\lambda/2 \dots$, there is destructive interference and P is in a dark fringe:

$$d \sin \theta_m = \left(m + \frac{1}{2}\right) \lambda \quad (m = 0, \pm 1, \pm 2 \dots) \quad (\text{Dark fringes}) \quad (23.2)$$

In order to simplify this formula, let us consider the right triangle AOP formed by drawing a line from the midpoint A of the slits in the figure to P . As the two sides of this triangle are perpendicular to two sides of the triangle S_1S_2B , they are similar triangles and their angle θ is equal. For small angles, both the sine and the tangent can be taken as equal to the radian value of the angle:

$$\theta_m \approx \sin \theta_m \approx \tan \theta_m = \frac{y_m}{D} \quad (23.3)$$

This angle is used in the above expressions for dark and light fringes. We can combine both cases into a single formula:

$$y_m = \begin{cases} m \frac{\lambda D}{d} & (\text{Bright}) \\ \left(m + \frac{1}{2}\right) \frac{\lambda D}{d} & (\text{Dark}) \end{cases} \quad (m = 0, \pm 1, \pm 2 \dots) \quad (23.4)$$

Therefore, if the distance D of the screen, the distance d between the slits and the wavelength λ of the used light are known, the type and number of fringes at distance y_m can be calculated. This experiment is actually mostly used to measure the wavelength of light.

Example 23.1

In a Young's experiment conducted with an Argon laser using green light with a wavelength of 514 nm, the distance between the slits is 0.1 mm and the screen is placed 3 m away.

- (a) What is the distance between two bright fringes?
 (b) What is the distance between the 3rd bright fringe and the 7th dark fringe?

Answer

(a) The positions of the bright fringes are given with Eq. (23.4):

$$y_m = m \frac{\lambda D}{d}$$

Accordingly, we find the distance between two successive fringes:

$$y_2 - y_1 = \frac{\lambda D}{d}$$

$$y_2 - y_1 = \frac{514 \times 10^{-9} \times 3}{0.1 \times 10^{-3}} = 0.015 \text{ m} = 1.5 \text{ cm}$$

(b) We write Eq. (23.4) for the 7th dark and 3rd bright fringes and calculate the difference:

$$y_7 - y_3 = (7.5 - 3) \times \frac{\lambda D}{d} = 4.5 \times 1.5 = 6.8 \text{ cm}$$

Example 23.2

In a Young's experiment conducted with a ruby laser using red light with a wavelength of 695 nm, the distance between the slits is 0.1 mm.

- (a) What is the angular difference between two bright fringes?
 (b) How many bright fringes are there in an angular area of $\pm 10^\circ$ from the center?

Answer

(a) We use Eq. (23.1), which we developed for angular positions:

$$d \sin \theta_m = m \lambda$$

We use the value $\sin \theta_m \approx \theta_m$ for small angles and calculate the angular difference $\Delta\theta$ between two successive maximums:

$$\Delta\theta = \theta_2 - \theta_1 = \frac{\lambda}{d}$$

$$\Delta\theta = \frac{695 \times 10^{-9}}{0.1 \times 10^{-3}} = 0.007 \text{ radians}$$

We convert this value into degrees by multiplying by $180/\pi$:

$$\Delta\theta = 0.007 \times 180/\pi = 0.4^\circ$$

(b) After finding the angle covered between two maximums, we calculate the number of maximums within the given range:

$$N = 2 \times \frac{10}{0.4} = 50$$

Example 23.3

In a Young's experiment, two sources generating two different lights with wavelengths $\lambda_1 = 440 \text{ nm}$ and $\lambda_2 = 695 \text{ nm}$ are used. The distance between the slits is 0.1 mm and the distance from the screen is 5 m . What is the distance between the 6th bright fringes of both types of light?

Answer

We use the formula (23.4) that gives us the positions of the

bright fringes:

$$y_m = m \frac{\lambda D}{d}$$

We write this expression for both wavelengths at the same maximum $m=6$ and calculate the difference in distance:

$$\Delta y_6 = 6(\lambda_2 - \lambda_1) \frac{D}{d}$$

$$\Delta y_6 = 6(695 - 440) \times 10^{-9} \times \frac{5}{0.1 \times 10^{-3}}$$

$$\Delta y_6 = 0.077 \text{ m} = 7.7 \text{ cm}$$

23.2 INTERFERENCE IN THIN FILMS

You can see all of the colors of the rainbow on soap bubbles and in oil slicks on puddles of rainwater. This effect is caused by the interference of the light reflected from the front and rear surfaces of a thin film layer.

Let us consider a thin film layer with thickness t . As shown in Figure 23.3, some of the light incident on this surface will get reflected from the front surface and the rest will get refracted and pass on to the second medium. When the ray inside of the film reaches the back surface of the film, some of it will get reflected again and return to the front surface. This part of the returning ray that shoots out into the air generates interference with the first reflected ray.

Let us calculate this interference for a normal, in other words, perpendicular, incident light on a thin film with refractive index n .

Let us emphasize a very important point before starting the calculation: We had discussed the following property when examining the reflection and transfer of waves in Chapter 10: *When a wave is incident from a less dense medium to a denser medium ($n_1 < n_2$), the wave is reflected with a phase difference of 180° , in other words, as an inverted wave. This phase difference is observed only in the reflection of a wave going from a less dense medium into a denser medium, and not in the reverse case.*

According to this property, the first ray coming from the air and reflected from the first surface will get reflected with a phase difference of 180° . However, two different cases can occur when the ray that passes inside of the film reaches the back surface:

- **If the film layer is denser than the rear medium (Figure 23.4):**

In this case, no phase difference occurs in the second ray reflected from this surface and it gets reflected as it is. Therefore, the phase difference between the first and second rays is caused by two factors:

- ▶ The 180° phase difference from the first reflection,
- ▶ The distance $2t$ taken when going back and forth in the film medium.

The interference would have been destructive if there was only the 180° phase difference from the reflection. However, the path difference from going back and forth in the film medium shall be added to this. If the wavelength of the light in air is λ , its wavelength in a medium with refractive index n is $\lambda_n = \lambda/n$. Therefore, we should compare the path difference inside of the film with this wavelength.

We reach the following conclusions considering that the phase difference is caused by two effects:

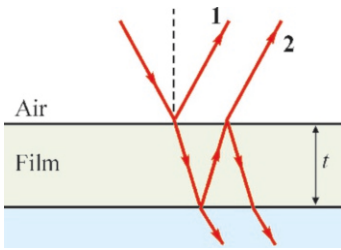


Figure 23.3: Two lights reflected from a thin film layer.

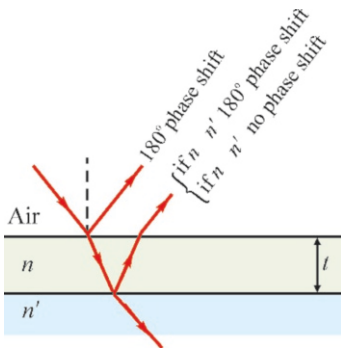


Figure 23.4: The phase difference of the second light will vary depending on whether the film layer is more or less denser than the rear medium.

- ▶ The interference is constructive if the path difference $2t$ is equal to half multiples of λ_n and the light at that wavelength is observed at maximum magnitude:

$$2t = (m + \frac{1}{2}) \lambda_n \quad (m = 0, 1, 2 \dots) \quad (\text{Maximum}) \quad (23.5)$$

- ▶ The interference is destructive if the path difference $2t$ is equal to full multiples of λ_n and the light disappears at that wavelength and is not observed:

$$2t = m \lambda_n \quad (m = 1, 2, 3 \dots) \quad (\text{Minimum}) \quad (23.6)$$

Note that these formulas are opposite to those in Young's experiment. The reason for this is that a phase difference of 180° from reflection is added here to the phase difference with the second wave.

- **If the film layer is less dense than the rear medium (Figure 23.4):**

In this case, a 180° phase difference will arise in the rays reflected from both the front and the rear surfaces. Therefore, the phase differences caused by the surface effects will cancel each other out and only the phase difference from the path difference in the film medium will be taken into consideration. In this case, we reverse Eqs. (23.5) and (23.6) that we found above for constructive and destructive interference. In other words, we use the *maximum (constructive interference) formula for the minimum (destructive interference)*.

This is the reason why various colors are observed on soap bubbles and on oil slicks. The fact that, for example, the color blue is reflected from a point of a thin film reflecting white light, is caused by the fact that all of the other colors are canceled due to destructive interference.



Figure 23.5: The colors reflected from a thin oil slick are a result of interference.

Example 23.4

An oil slick with a refractive index of 1.5 is spread over a water puddle ($n = 1.33$). What is the minimum value of the oil slick thickness such that the yellow light with a wavelength of 600 nm within the white light coming from air passes through without getting reflected?

Answer Only a phase difference of 180° will occur in the first surface, because the refractive index of the oil slick is greater than the water underneath. Also, passing through

without reflection means destructive interference. Therefore, the formula (23.6) should be used:

$$2t = m \lambda_n$$

If the path difference is equal to one wavelength for $m=1$, it shall be multiples of the wavelength for the other m values. Also, in the oil medium, we take the wavelength as $\lambda_m = \lambda/n$:

$$2t = 1 \times \frac{\lambda}{n} \rightarrow t = \frac{\lambda}{2n} = \frac{600}{2 \times 1.5} = 200 \text{ nm}$$

Yellow light will not get reflected at multiples of this thickness.

Example 23.5

A thin transparent film with refractive index $n=1.3$ is spread on a window glass with a refractive index of 1.5. The thickness of the film should be multiples of which value such that the reflected light does not contain red? (Take the wavelength of red light as 700 nm.)

Answer Two 180° phase differences will occur this time, as

the refractive index of the film layer is less than the refractive index of the rear glass.

In this case, we must invert Eqs. (23.5-23.6) for destructive interference of the red light:

$$2t = (m + \frac{1}{2}) \lambda_n \quad (\text{Minimum})$$

We take $m = 0$ and $\lambda_n = \lambda/n$ and calculate the thickness:

$$2t = \frac{1}{2} \times \frac{\lambda}{n} \rightarrow t = \frac{\lambda}{4n} = \frac{700}{4 \times 1.3} = 135 \text{ nm}$$



Figure 23.6: Diffraction in the shadow of a razor blade (Lecture Demonstration Services, Harvard Science Center).

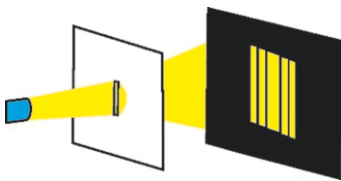


Figure 23.7: Diffraction in a single slit.

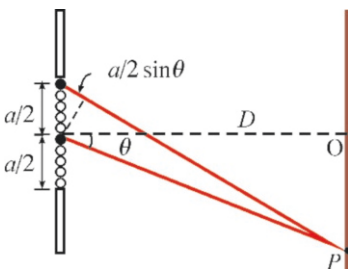


Figure 23.8: Two matching (black) points in the top and bottom half.



Figure 23.9: Diffraction fringes in a single slit.

23.3 DIFFRACTION FROM A SINGLE SLIT

In geometric optics, light was considered as rays propagating in straight lines. If this was correct, an obstacle placed in front of a source of light would have a shadow with sharp edges. But this is not the case. If you closely examine the shadow of a razor blade in Figure 23.6, you will notice that there are second, third, etc., shadows on the sides. These are called **diffraction fringes**.

Diffraction is the general name given to the effects caused by the deviation of light from its linear path. In addition to the double-slit experiment that we discussed earlier, the wave nature of light can be observed even in the shadow of a single slit.

Let us consider the single-slit experiment so as to understand diffraction at the simplest level. In the setup shown in Figure 23.7, a monochromatic light emitted from a source passes through an obstacle with only a single slit and is then projected onto a screen. On the screen, in addition to an image of the slit at the center, we also observe weaker second, third, etc., slit images on both sides of the center.

To understand the diffraction effect in this experiment, the width of the slit should be taken into consideration. (The widths of the slits were ignored in Young's experiment.) Let us use Figure 23.8 to explain the diffraction fringes formed by rays from a single slit with width a on a very distant screen.

According to Huygens's principle, secondary waves with the same phase are generated at each point of a slit on which light is incident. It is clear that the center of the screen will be bright, as all of these waves will reach the center in the same phase.

Let us consider the first dark fringe right next to the center. As all light rays reaching this dark fringe from various parts of the slit are very distant from the screen, they will all have started with the same angle θ . Let us match every point at the top half of the slit with a point at the bottom half.

As shown in Figure 23.8, the wave from the top point and the wave from just below the central point (the two black circles in the figure) should have a phase difference of $\lambda/2$, because they generate destructive interference on the dark fringe on the screen. After these two points, the two points following them in the top and bottom halves of the slit should also have a phase difference of $\lambda/2$. Therefore, all matching pairs of points will have a phase difference of half a wavelength. As shown in Figure 23.8, as the path difference between two matching waves is $(a/2) \sin \theta$, the condition for the first dark fringe is written as follows:

$$\frac{a}{2} \sin \theta = \frac{\lambda}{2} \quad \longrightarrow \quad \sin \theta = \frac{\lambda}{a}$$

This method can be repeated by considering that the slit is divided into 4 parts, 6 parts, 8 parts, etc. It can be shown that the dark fringes will have the value $\sin \theta = 2\lambda/a, 3\lambda/a$, etc., each time. In conclusion, the position of dark fringes in single-slit diffraction will fulfill the following condition:

$$\sin \theta = m \frac{\lambda}{a} \quad (m = 1, 2, 3 \dots) \quad (\text{Condition for dark fringes}) \quad (23.7)$$

Likewise, we use the small angle approach to find the distances y_m of the fringes from the center on a screen located at distance D :

$$\sin \theta \approx \tan \theta = \frac{y}{D}$$

$$y_m = m \frac{\lambda D}{a} \quad (m = 1, 2, 3 \dots) \quad (\text{Positions of dark fringes}) \quad (23.8)$$

We can determine the positions of the bright fringes using a similar method:

$$y_m = \left(m + \frac{1}{2}\right) \frac{\lambda D}{a} \quad (m = 1, 2, 3 \dots) \quad (\text{Positions of bright fringes}) \quad (23.9)$$

Intensity of Light in Diffraction

The intensity of light varies gradually between dark and light areas in diffraction fringes. The calculation for the distribution of the resultant intensity of two waves at a certain point is long and complex, and we shall therefore present it here without proof. If we use I_0 to indicate the intensity of light at the central point $\theta = 0$, the expression for the light intensity at any angle is as follows:

$$I = I_0 \left[\frac{\sin(\pi a \sin \theta / \lambda)}{\pi a \sin \theta / \lambda} \right]^2 \quad (23.10)$$

Notice that this function also includes the condition for dark fringes. This curve is shown in Figure 23.10.

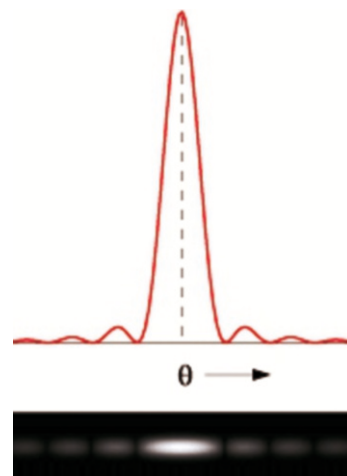


Figure 23.10: Distribution of the intensity of light in diffraction fringes.

Example 23.6

A red light with a wavelength of 700 nm is sent through a 0.1 mm wide slit. A screen is located 5 m away.

- What is the position of the first dark fringe?
- What is the width of the bright area at the center?

Answer

(a) We use Eq. (23.8), which we found for dark fringes:

$$y_m = m \lambda D / a \quad (m = 1, 2, 3 \dots)$$

We calculate for $m = 1$:

$$y_1 = \frac{\lambda D}{a} = \frac{700 \times 10^{-9} \times 5}{0.1 \times 10^{-3}} = 0.035 \text{ m} = 35 \text{ mm}.$$

(b) The bright fringe at the center is limited by the first dark fringes at both sides. Therefore, its width is 2 times the value that we found in item (a):

$$\Delta y = 2 \times y_1 = 70 \text{ mm}.$$

Example 23.7

A red light with wavelength $\lambda_1 = 700 \text{ nm}$ and a green light with wavelength $\lambda_2 = 650 \text{ nm}$ are simultaneously sent through a slit. On a screen that is 2 m away, the distance between the 5th dark fringe of the red light and the 3rd bright fringe of the green light is observed to be 2 mm. Determine the width of the slit.

Answer

Eq. (23.8) for dark fringes gives us, for $m=5$ of the red light:

$$y_1(m=5) = 5 \frac{\lambda_1 D}{a}$$

Eq. (23.9) for bright fringes gives us, for $m=3$ of the green light:

$$y_2(m=3) = \left(3 + \frac{1}{2}\right) \frac{\lambda_2 D}{a}$$

We equate the difference between these two positions to 2 mm and solve for the unknown a :

$$y_1(m=5) - y_2(m=3) = 0.002$$

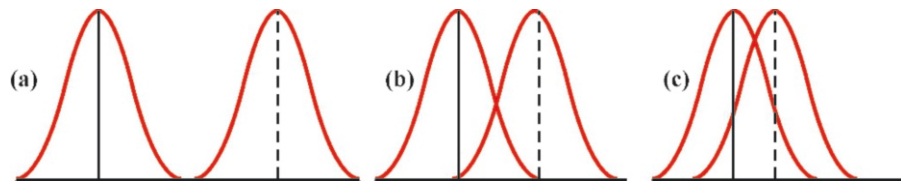
$$a = \frac{(5\lambda_1 - 3.5\lambda_2) D}{0.002} = 0.0012 \text{ m} = 1.2 \text{ mm}$$

Resolution

The image formed in our eye or on photographic film by optical instruments such as microscopes, telescopes or spectrosopes must be very sharp. However, there is a limit to the ability of an instrument to distinguish between objects that are close together, regardless of how perfectly it is manufactured. This ability, limited by the diffraction of light, is called the **resolution** or the resolving power of the instrument.

The limit of the resolution is caused by diffraction. Let us consider two light sources S_1 and S_2 located near each other (e.g., two neighboring stars in the sky). Let the rays from these two sources pass through a slit with width a and be

Figure 23.11: (a) Two resolvable images. (b) Two images on the limit of resolution, (c) Two unresolvable images. Note that the central maximum of one coincides with the first minimum of the other.



projected onto a screen. As we discussed in the topic of single-slit diffraction, the image of each on the screen will show a distribution around a central maximum (Figure 23.11). The centers of these two images become unresolvable if their central maximums overlap.

The limit of resolution is accepted as *the limit at which the central maximum of one of the images coincides with the first minimum of the other image* (Figure 23.11b). We obtain the minimum angle θ at this limit using Eq. (23.8), which we obtained for single-slit diffraction. The expression giving the first minimum was found to be the following:

$$\sin \theta = \frac{\lambda}{a}$$

We can take the angle (in radian) instead of the sine for small angles. Accordingly, the smallest angle θ_{\min} that can be resolved by the optical instrument is:

$$\theta_{\min} = \frac{\lambda}{a} \quad (\text{for rectangular slit}) \quad (23.11)$$

This formula is valid for a rectangular slit. The formula changes slightly when calculated for the circular objective lenses of optical instruments:

$$\theta_{\min} = 1.22 \frac{\lambda}{a} \quad (\text{For circular slit}) \quad (23.12)$$

a is the diameter of the circle here.

The smaller the value of θ_m , the higher resolution of the instrument. Accordingly, to increase resolution, we should either use light with a smaller wavelength or increase the diameter of the objective lens.

Example 23.8

A camera is required to resolve two points separated by 1 cm at a distance of 100 m in the photo that it shoots. As the average wavelength is 500 nm within the visible light range, what is the minimum value of the diameter of the objective lens?

Answer

We use Eq. (23.12) given for circular slits:

$$\theta_{\min} = 1.22\lambda/a$$

The angle θ_{\min} here should be equal to the angle covered by the width of 1 cm located 100 meters away:

$$\theta_{\min} = 0.01/100 = 0.0001 \text{ radians}$$

We use these values to calculate the diameter a of the circular objective lens:

$$a = 1.22 \frac{\lambda}{\theta_{\min}} = 1.22 \times \frac{500 \times 10^{-9}}{0.0001}$$

$$a = 0,006 \text{ m} = 6 \text{ mm}$$

Problems

23.1 Young's Double-Slit Experiment

23.1 In a Young's experiment conducted with a ruby laser using red light with a wavelength of 700 nm, the distance between the slits is 0.1 mm and the screen is placed 5 m away.

(a) What is the distance between two bright fringes? (b) What is the distance between the 3rd bright fringe and the 5th dark fringe? [A: (a) 3.5 mm, (b) 8.8 mm.]

23.2 In a Young's experiment conducted with a He-Cd laser using red light with a wavelength of 450 nm, the distance between the slits is 0.1 mm. (a) What is the angular difference between the fringes? (b) How many bright fringes are there in an angular area of $\pm 1^\circ$ from the center?

[A: (a) 0.27° , (b) $7.8 \approx 8$ fringes.]

23.3 In a Young's experiment, sources generating two lights with wavelengths $\lambda_1 = 600$ nm and $\lambda_2 = 650$ nm are used. The distance between the slits is 0.1 mm and the distance from the screen is 3 m. What is the distance between the 5th bright fringes of both types of light?

[A: 7.5 mm.]

23.4 A Young's experiment is designed with a light with a 700 nm wavelength such that the angle between the bright fringes is 0.01° . What should the distance between the two slits be?

[A: 4 mm.]

23.5 The angle between the bright fringes is 2° in a Young's experiment. What will the angle be when this experimental setup is placed under water with refractive index $n = 1.33$? (Hint: Take the ratios and work with the wavelength in water.)

[A: 1.5° .]

23.2 Interference in Thin Films

23.6 An oil slick with a refractive index of 1.4 is located on a puddle of water ($n = 1.33$). What is the minimum

thickness of the oil slick such that red light with a wavelength of 700 nm within the white light coming from air is not reflected?

[A: 250 nm.]

23.7 A coat of thin film with refractive index $n=1.3$ is applied to an objective lens with a refractive index of 1.6. The thickness of the film should be multiples of which value such that the reflected light does not contain blue? (Take the wavelength of blue light as 400 nm.)

[A: 77 nm.]

23.3 Diffraction from a Single Slit

23.8 A light with a wavelength of 600 nm is sent through a 0.1 mm wide slit. (a) On a screen 4 m away, what is the width of the central area at the center? (b) What is the position of the 7th dark fringe?

[A:(a) 4.8 cm, (b) 17 cm.]

23.9 Two lights with wavelengths $\lambda_1=500$ nm and $\lambda_2=600$ nm are simultaneously sent through a slit. On a screen that is 3 m away, the distance between the 7th bright fringe of the first light and the 5th dark fringe of the second light is observed to be 3 mm. Determine the width of the slit.

[A: 2 mm.]

23.10 A telescope is required to resolve two points at a distance of 100 km and separated by 1 cm. As the average wavelength is 500 nm within the visible light range, what is the minimum value of the diameter of the telescope?

[A: 6.1 m.]

APPENDIX A. PHYSICAL CONSTANTS

Quantity	Symbol	Value
Speed of light in vacuum	c	$2.997\,92 \times 10^8$ m/s
Charge of electron	e	$1.602\,19 \times 10^{-19}$ C
Gravitational constant	G	6.6726×10^{-11} N m ² /kg ²
Avogadro's number	N_A	$6.022\,05 \times 10^{23}$
Universal gas constant	R	$\left\{ \begin{array}{l} 8.31451 \text{ J}/(\text{mol K}) \\ 0.0820578 \text{ litre atm}/(\text{mol K}) \end{array} \right.$
Permittivity of vacuum	ϵ_0	$8.854\,187 \times 10^{-12}$ C ² /(N m ²)
Coulomb force constant	$k = 1/(4\pi\epsilon_0)$	$8.987\,551\,78 \times 10^9$ N m ² /C ²
Permeability of vacuum	μ_0	$4\pi \times 10^{-7}$ N/A ²
Electron mass	m_e	$9.109\,390 \times 10^{-31}$ kg
Proton mass	m_p	$1.672\,623 \times 10^{-27}$ kg
Neutron mass	m_n	$1.674\,929 \times 10^{-27}$ kg
Planck constant	h	$6.626\,18 \times 10^{-34}$ J s
Bohr radius	$a_B = \hbar^2/(ke^2m_e)$	$5.291\,773 \times 10^{-11}$ m

APPENDIX B. USEFUL MATHEMATICAL RELATIONS

Derivative	Integral	Trigonometry
$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$	$\int (f+g) dx = \int f dx + \int g dx$	$\sin^2 \alpha + \cos^2 \alpha = 1$
$\frac{d}{dx}(fg) = f \frac{dg}{dx} + g \frac{df}{dx}$	$\int f g' dx = fg - \int g f' dx$	$\sin(-\alpha) = -\sin \alpha$
$\frac{dx}{dx} = 1$	$\int dx = x$	$\cos(-\alpha) = \cos \alpha$
$\frac{d}{dx} x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1}$	$\sin 2\alpha = 2 \sin \alpha \cos \alpha$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\int \frac{dx}{x} = \ln x$	$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1$
$\frac{d}{dx} e^x = e^x$	$\int e^x dx = e^x$	$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$
$\frac{d}{dx} \sin x = \cos x$	$\int \sin x dx = -\cos x$	$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$
$\frac{d}{dx} \cos x = -\sin x$	$\int \cos x dx = \sin x$	$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$	$\int \tan x dx = -\ln \cos x $	Law of Cosines: $a^2 = b^2 + c^2 - 2bc \cos A$
$\frac{d}{dx} \cot x = -\frac{1}{\sin^2 x}$	$\int \cot x dx = \ln \sin x $	Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

C. TRIGONOMETRIC TABLE

For angles greater than 45°, use the right-hand column together with the bottom labels.

Degree	Radian	Sin	Cos	Tan	Cot	←	
00	0.000	0.000	1.000	0.000	∞	1.571	90
01	0.018	0.018	1.000	0.018	57.290	1.553	89
02	0.035	0.035	0.999	0.035	28.636	1.536	88
03	0.052	0.052	0.999	0.052	19.081	1.518	87
04	0.070	0.070	0.998	0.070	14.301	1.501	86
05	0.087	0.087	0.996	0.088	11.430	1.484	85
06	0.105	0.105	0.995	0.105	9.514	1.466	84
07	0.122	0.122	0.993	0.123	8.144	1.449	83
08	0.140	0.139	0.990	0.141	7.115	1.431	82
09	0.157	0.156	0.988	0.158	6.314	1.414	81
10	0.175	0.174	0.985	0.176	5.671	1.395	80
11	0.192	0.191	0.982	0.194	5.145	1.379	79
12	0.209	0.208	0.978	0.213	4.705	1.361	78
13	0.227	0.225	0.974	0.231	4.332	1.344	77
14	0.244	0.242	0.970	0.249	4.011	1.327	76
15	0.262	0.259	0.966	0.268	3.732	1.309	75
16	0.279	0.276	0.961	0.287	3.487	1.292	74
17	0.297	0.292	0.956	0.306	3.271	1.274	73
18	0.314	0.309	0.951	0.325	3.078	1.257	72
19	0.332	0.326	0.946	0.344	2.904	1.239	71
20	0.349	0.342	0.940	0.364	2.748	1.222	70
21	0.367	0.358	0.934	0.384	2.605	1.204	69
22	0.384	0.375	0.927	0.404	2.475	1.187	68
23	0.401	0.391	0.921	0.425	2.356	1.169	67
24	0.419	0.407	0.914	0.445	2.246	1.152	66
25	0.436	0.423	0.906	0.466	2.145	1.135	65
26	0.454	0.438	0.899	0.488	2.050	1.117	64
27	0.471	0.454	0.891	0.510	1.963	1.100	63
28	0.489	0.470	0.883	0.532	1.881	1.082	62
29	0.506	0.485	0.875	0.554	1.804	1.065	61
30	0.524	0.500	0.866	0.577	1.732	1.047	60
31	0.541	0.515	0.857	0.601	1.664	1.030	59
32	0.559	0.530	0.848	0.625	1.600	1.012	58
33	0.576	0.545	0.839	0.649	1.540	0.995	57
34	0.593	0.559	0.829	0.675	1.483	0.977	56
35	0.611	0.574	0.819	0.700	1.428	0.960	55
36	0.628	0.588	0.809	0.727	1.376	0.943	54
37	0.646	0.602	0.799	0.754	1.327	0.925	53
38	0.663	0.616	0.788	0.781	1.280	0.908	52
39	0.681	0.629	0.777	0.810	1.235	0.890	51
40	0.698	0.643	0.766	0.839	1.192	0.873	50
41	0.716	0.656	0.755	0.869	1.150	0.855	49
42	0.733	0.669	0.743	0.900	1.111	0.838	48
43	0.751	0.682	0.731	0.933	1.072	0.820	47
44	0.768	0.695	0.719	0.966	1.036	0.803	46
45	0.785	0.707	0.707	1.000	1.000	0.785	45
		Cos	Sin	Cot	Tan	Radian	Degree

D. PERIODIC TABLE

Lanthanides that start at 58 and Actinides that start at 90 are shown separately.

1 H	2 He											10 Ne	18 Ar				
1.01	4.00											20.18	39.95				
0.00009	0.00018											0.0017	0.0018				
3 Li	4 Be	5 B	6 C	7 N	8 O	9 F	10 Ne	11 Na	12 Mg	13 Al	14 Si	15 P	16 S	17 Cl	18 Ar		
6.94	9.01	10.81	12.01	14.01	16.00	19.00	20.18	22.99	24.31	26.98	28.09	30.97	32.06	35.45	39.95		
0.534	1.85	2.34	2.267	0.0013	0.0014	0.0017	0.0009	0.971	1.738	2.698	2.330	1.82	2.067	0.0032	0.0018		
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
39.10	40.08	44.96	47.90	50.94	52.00	54.94	55.85	58.93	58.70	63.55	65.38	69.72	72.59	74.92	78.96	79.90	83.80
0.862	1.54	2.989	4.54	6.11	7.15	7.44	7.874	8.86	8.912	8.96	7.134	5.907	5.323	5.776	4.809	3.122	0.0037
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
85.47	87.62	88.91	91.22	92.91	95.94	98	101.07	102.91	106.04	107.87	112.41	114.82	118.69	121.75	127.60	126.90	131.30
1.532	2.64	4.469	6.506	8.57	10.22	11.5	12.37	12.41	12.02	10.501	8.69	7.31	7.287	6.685	6.232	4.93	0.0059
55 Cs	56 Ba	57 La	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
132.91	137.33	138.91	178.49	180.95	183.85	186.21	190.2	192.22	195.09	196.97	200.59	204.37	207.2	208.98	209	210	222
1.873	3.594	6.145	13.31	16.654	19.25	21.02	22.61	22.56	21.46	19.282	13.534	11.85	11.342	9.807	9.32	7	0.0097
87 Fr	88 Ra	89 Ac											69 Tm	70 Yb	71 Lu		
223	226.03	227.03											168.93	173.04	174.97		
1.87	5.5	10.07											6.965	9.84	9.84		
			58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
			140.12	140.91	144.24	145	150.4	151.96	157.25	158.93	162.50	164.93	167.26	168.93	173.04	174.97	
			6.77	6.773	7.007	7.26	7.52	5.243	7.895	8.229	8.55	8.795	9.066	9.321	9.665	9.84	
			90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	
			232.04	231.04	238.03	237.05	244	243	247	247	251	252	257	258	259	260	
			11.72	15.37	18.95	20.45	19.84	13.69	13.51	14.79	15.1	13.5	-	-	-	-	

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